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INTRINSIC GEOMETRY OF IDEAL SPACE

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INTRINSIC GEOMETRY OF IDEAL SPACE

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SURFACES IN A REGION: GEODESICS

Orientation-variables.

193. The preceding investigations have been concerned with properties of a region depending upon any line-direction through a point; we now pass to the consideration of some of its properties depending upon any superficial orientation at a point. An indication has already (§ 159) been given as to surface-variables at a point; they can be taken, either as new variables ξ, η, ζ, which satisfy a permanent relation

$$\sum a\xi^2 = 1$$

when in their canonical form; or they can be composed from two non-coincident directions, with line-variables p_1', q_1', r_1', and p_2', q_2', r_2', at an inclination ϵ, such that

$$\frac{\xi}{q_1'r_2' - r_1'q_2'} = \frac{\eta}{r_1'p_2' - p_1'r_2'} = \frac{\zeta}{p_1'q_2' - q_1'p_2'} = \frac{1}{\sin \epsilon}.$$

Any other direction P', Q', R', lying in the plane through these two directions, that is, lying in the orientation, must be such that

$$P' = \lambda p_1' + \mu p_2', \quad Q' = \lambda q_1' + \mu q_2', \quad R' = \lambda r_1' + \mu r_2',$$

for parametric quantities λ, μ; and therefore the variables of the direction satisfy the relation

$$\xi P' + \eta Q' + \zeta R' = 0.$$

We also may have a surface, passing through the point and lying wholly within the region; and then, when the tangent plane to the surface at the point is taken, the orientation-variables of the plane are postulated as the orientation-variables of the surface. Such a surface can be given, without any implied restriction, by an equation

$$\theta(p, q, r) = 0,$$

between the representative parameters of the region: if the surface be one of a family, the equation would involve some magnitude, constant for a surface, and changing from surface to surface within the family. If the two directions p_1', q_1', r_1', and p_2', q_2', r_2', lie in the tangent plane, we have

$$\theta_1 p_1' + \theta_2 q_1' + \theta_3 r_1' = 0, \quad \theta_1 p_2' + \theta_2 q_2' + \theta_3 r_2' = 0,$$

so that

$$\frac{\theta_1}{\xi}=\frac{\theta_2}{\eta}=\frac{\theta_3}{\zeta}.$$

It has appeared (*l.c.*) that there is a quantity, denoted by θ_n, and defined by the relation

$$\Omega\theta_n^2=\sum a\theta_1^2;$$

and therefore, with the foregoing definitions of ξ, η, ζ, we have

$$\frac{\theta_1}{\xi}=\frac{\theta_2}{\eta}=\frac{\theta_3}{\zeta}=\Omega^{\frac{1}{2}}\theta_n.$$

Within the region there exists a unique direction normal to the surface; if its direction-variables be denoted by $\frac{dp}{dn}, \frac{dq}{dn}, \frac{dr}{dn}$, so that

$$\sum A\left(\frac{dp}{dn}\right)^2=1,$$

being the permanent arc-relation of the region for this direction, then (*l.c.*) we have the equivalent equations

$$\left.\begin{aligned}\Omega\theta_n\frac{dp}{dn}&=a\theta_1+h\theta_2+g\theta_3\\ \Omega\theta_n\frac{dq}{dn}&=h\theta_1+b\theta_2+f\theta_3\\ \Omega\theta_n\frac{dr}{dn}&=g\theta_1+f\theta_2+c\theta_3\end{aligned}\right\},\qquad \left.\begin{aligned}\theta_1&=\theta_n\left(A\frac{dp}{dn}+H\frac{dq}{dn}+G\frac{dr}{dn}\right)\\ \theta_2&=\theta_n\left(H\frac{dp}{dn}+B\frac{dq}{dn}+F\frac{dr}{dn}\right)\\ \theta_3&=\theta_n\left(G\frac{dp}{dn}+F\frac{dq}{dn}+C\frac{dr}{dn}\right)\end{aligned}\right\}.$$

The prime normal of any regional geodesic, and all the other non-gremial principal lines of the geodesic, are at right angles to every direction within the region: and therefore they are at right angles to every direction on the surface. But a regional geodesic is not a surface geodesic in general—the surface must belong to the particular class of surfaces hereafter discussed under the title of geodesic surfaces; and the surface geodesics, the prime normals of which are necessarily (§ 94) orthogonal to the surface, must be investigated independently.

Accordingly, we begin with the geometrical relation between a regional surface $\theta(p, q, r)=0$ and its tangent plane.

Tangent plane of a parametric surface.

194. Any spatial direction in the region, given by the typical equation

$$dy=y_1\,dp+y_2\,dq+y_3\,dr,$$

is tangential to the surface $\theta(p, q, r)=0$ when the directing increments dp, dq, dr, satisfy the equation

$$\theta_1\,dp+\theta_2\,dq+\theta_3\,dr=0.$$

Hence the tangent plane of the surface is given by the typical equation

$$\begin{aligned}\bar{y}-y&=\lambda\,dy\\&=\lambda y_1\,dp+\lambda y_2\,dq+\lambda y_3\,dq\\&=\alpha y_1+\beta y_2+\gamma y_3,\end{aligned}$$

where λ is a parameter, and $\alpha=\lambda\,dp$, $\beta=\lambda\,dq$, $\gamma=\lambda\,dr$, provided the parametric quantities α, β, γ, satisfy the condition

$$\theta_1\alpha+\theta_2\beta+\theta_3\gamma=0.$$

The equations of the plane can also be taken in the form

$$\left\|\begin{matrix}\bar{y}-y, & y_1, & y_2, & y_3\\ 0, & \theta_1, & \theta_2, & \theta_3\end{matrix}\right\|=0\,;$$

but the earlier form, explicitly involving the three parameters subject to the single explicit condition, is the more convenient.

From a point S on the surface near O, having η_0 as its typical space-coordinate, let a perpendicular of length P and with direction-cosines typified by Y_θ be drawn upon the tangent plane at O, the small arc OS along the surface being denoted by δ, and let the foot of the perpendicular on the plane be the foregoing point given by

$$\bar{y}-y=\alpha y_1+\beta y_2+\gamma y_3.$$

Then we have

$$Y_\theta P=\eta_0-\bar{y}=\eta_\theta-(y+\alpha y_1+\beta y_2+\gamma y_3).$$

Because P is the perpendicular, the magnitude

$$\sum\{\eta_0-(y+\alpha y_1+\beta y_2+\gamma y_3)\}^2$$

must be a minimum among all the magnitudes which can arise for all permissible values of parameters α, β, γ, that is, for all quantities α, β, γ, satisfying the condition

$$\theta_1\alpha+\theta_2\beta+\theta_3\gamma=0.$$

The critical equations for such a minimum are

$$\begin{aligned}\mu\theta_1&=\sum y_1\{\eta_\theta-(y+\alpha y_1+\beta y_2+\gamma y_3)\},\\ \mu\theta_2&=\sum y_2\{\eta_\theta-(y+\alpha y_1+\beta y_2+\gamma y_3)\},\\ \mu\theta_3&=\sum y_3\{\eta_\theta-(y+\alpha y_1+\beta y_2+\gamma y_3)\},\end{aligned}$$

where μ is a multiplier left undetermined in the formation of the critical equations; and they can be simplified to the forms

$$\left.\begin{aligned}\mu\theta_1+A\alpha+H\beta+G\gamma&=\sum y_1(\eta_\theta-y)\\ \mu\theta_2+H\alpha+B\beta+F\gamma&=\sum y_2(\eta_\theta-y)\\ \mu\theta_3+G\alpha+F\beta+C\gamma&=\sum y_3(\eta_\theta-y)\end{aligned}\right\}.$$

For the small arc OS on the surface, we use p', q', r', to denote the direction-variables; and for second derivatives along that arc, whatever it may be ultimately, we use p_θ'', q_θ'', r_θ''. Then, accurately up to the second power of δ inclusive, we have

$$\begin{aligned} \eta_\theta - y &= y'\delta + \tfrac{1}{2}y_\theta''\delta^2 \\ &= (y_1 p' + y_2 q' + y_3 q')\,\delta + \tfrac{1}{2}y_\theta''\delta^2. \end{aligned}$$

Also

$$\begin{aligned} y_\theta'' &= y_1 p_\theta'' + y_2 q_\theta'' + y_3 r_\theta'' + y_{11}p'^2 + 2y_{12}p'q' + y_{22}q'^2 + 2y_{13}p'r' + 2y_{23}q'r' + y_{33}r'^2 \\ &= y_1(p_\theta'' + \textstyle\sum \Gamma_{11}p'^2) + y_2(q_\theta'' + \sum \Delta_{11}p'^2) + y_3(r_\theta'' + \sum \Theta_{11}p'^2) \\ &\qquad + \eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2 + 2\eta_{13}p'r' + 2\eta_{23}q'r' + \eta_{33}r'^2 \\ &= y_1(p_\theta'' + \textstyle\sum \Gamma_{11}p'^2) + y_2(q_\theta'' + \sum \Delta_{11}p'^2) + y_3(r_\theta'' + \sum \Theta_{11}p'^2) + \dfrac{Y}{\rho}, \end{aligned}$$

where $1/\rho$ is the circular curvature of the regional geodesic in the direction p', q', r', manifestly a regional geodesic touching the surface along the direction of the small arc OS, and where Y is the typical direction-cosine of the prime normal of that geodesic. Hence

$$\begin{aligned} \textstyle\sum y_1(\eta_\theta - y) = A\{p'\delta + \tfrac{1}{2}(p_\theta'' + \sum \Gamma_{11}p'^2)\,\delta^2\} \\ + H\{q'\delta + \tfrac{1}{2}(q_\theta'' + \textstyle\sum \Delta_{11}p'^2)\,\delta^2\} \\ + G\{r'\delta + \tfrac{1}{2}(r_\theta'' + \textstyle\sum \Theta_{11}p'^2)\,\delta^2\}, \end{aligned}$$

the term in $1/\rho$ vanishing because $\sum Y y_1 = 0$; and the right-hand side is accurate up to δ^2 inclusive. Similarly for the quantities $\sum y_2(\eta_\theta - y)$ and $\sum y_3(\eta_\theta - y)$; thus the critical equations become

$$\begin{aligned} -\mu\theta_1 = A\{\alpha - p'\delta - \tfrac{1}{2}(p_\theta'' + \textstyle\sum \Gamma_{11}p'^2)\,\delta^2\} \\ + H\{\beta - q'\delta - \tfrac{1}{2}(q_\theta'' + \textstyle\sum \Delta_{11}p'^2)\,\delta^2\} \\ + G\{\gamma - r'\delta - \tfrac{1}{2}(r_\theta'' + \textstyle\sum \Theta_{11}p'^2)\,\delta^2\}, \end{aligned}$$

with two others of like form, all accurately up to δ^2. Hence, also up to that order inclusive, we have

$$\begin{aligned} \Omega\{\alpha - p'\delta - \tfrac{1}{2}(p_\theta'' + \textstyle\sum \Gamma_{11}p'^2)\,\delta^2\} &= -\mu(a\theta_1 + h\theta_2 + g\theta_3) \\ \Omega\{\beta - q'\delta - \tfrac{1}{2}(q_\theta'' + \textstyle\sum \Delta_{11}p'^2)\,\delta^2\} &= -\mu(h\theta_1 + b\theta_2 + f\theta_3) \\ \Omega\{\gamma - r'\delta - \tfrac{1}{2}(r_\theta'' + \textstyle\sum \Theta_{11}p'^2)\,\delta^2\} &= -\mu(g\theta_1 + f\theta_2 + c\theta_3). \end{aligned}$$

Multiply these equations by θ_1, θ_2, θ_3, respectively, and add the products. Because

$$\textstyle\sum a\theta_1^2 = \Omega\theta_n^2,$$

$$\theta_1\alpha + \theta_2\beta + \theta_3\gamma = 0, \quad \theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

the result can be taken in the form

$$\mu\theta_n^2 = \tfrac{1}{2}\{\theta_1(p_\theta'' + \textstyle\sum \Gamma_{11}p'^2) + \theta_2(q_\theta'' + \sum \Delta_{11}p'^2) + \theta_3(r_\theta'' + \sum \Theta_{11}p'^2)\}\,\delta^2.$$

Now the second differentiation represented by p_θ'', q_θ'', r_θ'', has been effected in the surface ; consequently

$$\theta_1 p_\theta'' + \theta_2 q_\theta'' + \theta_3 r_\theta'' + \theta_{11} p'^2 + 2\theta_{12} p'q' + \theta_{22} q'^2 + 2\theta_{13} p'r' + 2\theta_{23} q'r' + \theta_{33} r'^2 = 0.$$

We introduce new symbols ϑ_{ij}, according to the definitions

$$\vartheta_{ij} = \theta_{ij} - \theta_1 \Gamma_{ij} - \theta_2 \Delta_{ij} - \theta_3 \Theta_{ij},$$

for all the combinations $i, j, = 1, 2, 3$; and now the last equation can be written

$$\theta_1 (p_\theta'' + \sum \Gamma_{11} p'^2) + \theta_2 (q_\theta'' + \sum \Delta_{11} p'^2) + \theta_3 (r_\theta'' + \sum \Theta_{11} p'^2) + \sum \vartheta_{11} p'^2 = 0.$$

We thus have

$$\mu \theta_n^2 = -\tfrac{1}{2} (\sum \vartheta_{11} p'^2) \delta^2,$$

so that

$$\mu = -\frac{\delta^2}{2\theta_n^2} (\sum \vartheta_{11} p'^2),$$

being a value of the undetermined multiplier μ ; and therefore

$$\alpha = p'\delta + \tfrac{1}{2} (p_\theta'' + \sum \Gamma_{11} p'^2) \delta^2 + \tfrac{1}{2} \frac{\delta^2}{\Omega \theta_n^2} (a\theta_1 + h\theta_2 + g\theta_3)(\sum \vartheta_{11} p'^2),$$

$$\beta = q'\delta + \tfrac{1}{2} (q_\theta'' + \sum \Delta_{11} p'^2) \delta^2 + \tfrac{1}{2} \frac{\delta^2}{\Omega \theta_n^2} (h\theta_1 + b\theta_2 + f\theta_3)(\sum \vartheta_{11} p'^2),$$

$$\gamma = r'\delta + \tfrac{1}{2} (r_\theta'' + \sum \Theta_{11} p'^2) \delta^2 + \tfrac{1}{2} \frac{\delta^2}{\Omega \theta_n^2} (g\theta_1 + f\theta_2 + c\theta_3)(\sum \vartheta_{11} p'^2).$$

Returning to the equations for the length and the direction of the perpendicular on the tangent plane, in the typical form

$$\begin{aligned} Y_\theta P &= \eta_\theta - (y + \alpha y_1 + \beta y_2 + \gamma y_3) \\ &= y'\delta + \tfrac{1}{2} y_\theta'' \delta^2 - (\alpha y_1 + \beta y_2 + \gamma y_3) \end{aligned}$$

accurate up to the second order, and substituting the foregoing values of α, β, γ, as well as the earlier value (p. 1) of y_θ'', we have

$$Y_\theta P = \tfrac{1}{2} \frac{Y}{\rho} \delta^2 - \tfrac{1}{2} \frac{\delta^2}{\Omega \theta_n^2} (\sum \vartheta_{11} p'^2) W,$$

where

$$\begin{aligned} W &= y_1 (a\theta_1 + h\theta_2 + g\theta_3) + y_2 (h\theta_1 + b\theta_2 + f\theta_3) + y_3 (g\theta_1 + f\theta_2 + c\theta_3) \\ &= \Omega \theta_n \left(y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn} \right) = \Omega \theta_n \frac{dy}{dn}, \end{aligned}$$

and the magnitude $\dfrac{dy}{dn}$ denotes a typical spatial direction-cosine of the regional normal to the surface. Hence

$$Y_\theta P = \tfrac{1}{2} \left\{ \frac{Y}{\rho} - \left(\frac{\sum \vartheta_{11} p'^2}{\theta_n} \right) \frac{dy}{dn} \right\} \delta^2.$$

Let $1/\rho_0$ denote the circular curvature of the section of the surface by the plane, through the direction p', q', r', and the perpendicular, so that

$$\frac{1}{\rho_0}=\operatorname*{Lim}_{\delta\to 0}\frac{2P}{\delta^2};$$

and also (with a permanent and new use of the symbol γ) write

$$\frac{1}{\gamma}=-\frac{\sum \vartheta_{11}p'^2}{\theta_n},$$

a magnitude the significance of which will be obtained later: then we have

$$\frac{Y_\theta}{\rho_0}=\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn},$$

as a relation connecting the length and the direction of the perpendicular on the tangent plane.

Intrinsic equations of superficial geodesics: regional flexure.

195. The intrinsic equations of superficial geodesics are obtained by assigning the conditions that the integral

$$\int\left\{\sum A\left(\frac{dp}{dt}\right)^2\right\}^{\frac{1}{2}}dt$$

shall be a minimum for values of p, q, r, that obey the relation $\theta(p, q, r)=0$. These critical conditions, by use of the frequently recurring analysis of the same type for different problems, are found to be

$$Ap_\theta''+Hq_\theta''+Gr_\theta''+\sum[ij,1]p_i'p_j'=M\theta_1,$$
$$Hp_\theta''+Bq_\theta''+Fr_\theta''+\sum[ij,2]p_i'p_j'=M\theta_2,$$
$$Gp_\theta''+Fq_\theta''+Cr_\theta''+\sum[ij,3]p_i'p_j'=M\theta_3,$$

where the summations are for the combinations $i, j, =1, 2, 3$; where $p_i=p, q, r$, according as $i=1, 2, 3$; where the symbols $[ij, k]$ have the significance given in § 12; and where M is a multiplier undetermined by the critical equations. Hence, changing this multiplier to λ where $\lambda=M\Omega^{-1}$, we have

$$p_\theta''+\sum\Gamma_{11}p'^2=\lambda(a\theta_1+h\theta_2+g\theta_3),$$
$$q_\theta''+\sum\Delta_{11}p'^2=\lambda(h\theta_1+b\theta_2+f\theta_3),$$
$$r_\theta''+\sum\Theta_{11}p'^2=\lambda(g\theta_1+f\theta_2+c\theta_3);$$

and the multiplier λ still has to be determined.

As in § 194, we have

$$\theta_1p_\theta''+\theta_2q_\theta''+\theta_3r_\theta''+\theta_{11}p'^2+2\theta_{12}p'q'+\theta_{22}q'^2+2\theta_{13}p'r'+2\theta_{23}q'r'+\theta_{33}r'^2=0$$

along the surface. Hence multiplying the three equations by θ_1, θ_2, θ_3, respectively, adding, and using this surface-relation, we have

$$-\sum \vartheta_{11} p'^2 = \lambda \sum a\theta_1{}^2 = \lambda \theta_n{}^2 \Omega,$$

so that

$$\lambda = -\frac{\sum \vartheta_{11} p'^2}{\theta_n{}^2 \Omega}.$$

Now (§ 193)

$$\Omega \theta_n \frac{dp}{dn} = a\theta_1 + h\theta_2 + g\theta_3,$$

so that

$$\lambda (a\theta_1 + h\theta_2 + g\theta_3) = -\frac{\sum \vartheta_{11} p'^2}{\theta_n} \frac{dp}{dn},$$

and similarly for the other right-hand sides Hence the intrinsic equations of a geodesic on the parametric surface $\theta(p, q, r) = 0$ are

$$\left.\begin{aligned} p_\theta'' + \sum \Gamma_{11} p'^2 &= -\frac{\sum \vartheta_{11} p'^2}{\theta_n} \frac{dp}{dn} \\ q_\theta'' + \sum \Delta_{11} p'^2 &= -\frac{\sum \vartheta_{11} p'^2}{\theta_n} \frac{dq}{dn} \\ r_\theta'' + \sum \Theta_{11} p'^2 &= -\frac{\sum \vartheta_{11} p'^2}{\theta_n} \frac{dr}{dn} \end{aligned}\right\}.$$

The term *regional flexure* of a superficial geodesic is used to denote the arc-rate of angular deviation of the superficial geodesic from its regional geodesic tangent (the regional geodesic through O in the direction p', q', r', on the surface). A point S along the superficial geodesic at a small arc-distance δ from O has its set of space-coordinates which may be typified by η_θ as in the preceding investigation: let a point T, taken along the regional geodesic in the same direction and at the same arc-distance δ from O, have its typical space-coordinate typified by η, so that

$$\eta_\theta = y + y'\delta + \tfrac{1}{2} y_\theta'' \delta^2, \quad \eta = y + y'\delta + \tfrac{1}{2} y'' \delta^2,$$

accurately up to the second order in δ inclusive. Thus

$$\eta_\theta - \eta = \tfrac{1}{2}(y_\theta'' - y'')\delta^2.$$

But

$$\begin{aligned} y_\theta'' &= y_1 p_\theta'' + y_2 q_\theta'' + y_3 r_\theta'' + \sum y_{11} p'^2, \\ y'' &= y_1 p'' + y_2 q'' + y_3 r'' + \sum y_{11} p'^2, \end{aligned}$$

and therefore

$$\begin{aligned} y_\theta'' - y'' &= y_1(p_\theta'' - p'') + y_2(q_\theta'' - q'') + y_3(r_\theta'' - r'') \\ &= y_1(p_\theta'' + \sum \Gamma_{11} p'^2) + y_2(q_\theta'' + \sum \Delta_{11} p'^2) + y_3(r_\theta'' + \sum \Theta_{11} p'^2) \\ &= -\frac{\sum \vartheta_{11} p'^2}{\theta_n} \left(y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn} \right) \\ &= -\frac{\sum \vartheta_{11} p'^2}{\theta_n} \frac{dy}{dn}. \end{aligned}$$

Let the magnitude of the radius of regional flexure be denoted by γ, and let Z temporarily denote the typical direction-cosine of that radius; then, by considering projections along the typical spatial axis, we have, to the second order,

$$\eta_\theta - \eta = Z\frac{\delta^2}{2\gamma}.$$

When the results are combined, they yield the typical relation

$$\frac{Z}{\gamma} = -\frac{\sum \vartheta_{11}p'^2}{\theta_n}\frac{dy}{dn}.$$

Measuring this flexure by the deviation along the positively-drawn regional normal to the surface, and having regard to the fact that Z and $\frac{dy}{dn}$ denote direction-cosines, we have

$$Z = \frac{dy}{dn}, \quad \frac{1}{\gamma} = -\frac{\sum \vartheta_{11}p'^2}{\theta_n}.$$

The first result shews that the direction of the radius of regional flexure of the superficial geodesic is along the positively-drawn regional normal to the surface; and the second result shews that the magnitude γ of that radius of regional flexure is expressible by the formula

$$-\frac{\theta_n}{\gamma} = \vartheta_{11}p'^2 + 2\vartheta_{12}p'q' + \vartheta_{22}q'^2 + 2\vartheta_{13}p'r' + 2\vartheta_{23}q'r' + \vartheta_{33}r'^2,$$

where

$$\vartheta_{ij} = \theta_{ij} - \theta_1\Gamma_{ij} - \theta_2\Delta_{ij} - \theta_3\Theta_{ij},$$

for all the combinations $i, j, =1, 2, 3$.

Moreover, with this significance for γ, the intrinsic equations of a superficial geodesic are

$$p_\theta'' - p'' = \frac{1}{\gamma}\frac{dp}{dn}, \quad q_\theta'' - q'' = \frac{1}{\gamma}\frac{dq}{dn}, \quad r_\theta'' - r'' = \frac{1}{\gamma}\frac{dr}{dn}.$$

Also, if now we use Y_0 (instead of Y_θ) to denote the typical direction-cosine of the prime normal of the superficial geodesic and retain ρ_0 to denote the radius of circular curvature of that geodesic, then there exist the equations typified by

$$\frac{Y_0}{\rho_0} = \frac{Y}{\rho} + \frac{1}{\gamma}\frac{dy}{dn},$$

where $\frac{dy}{dn}$ is the typical direction-cosine of the regional normal to the surface.

The last set of equations, on the elimination of ρ_0, ρ, γ, leads to the group of other equations

$$\left\| Y_0, \quad Y, \quad \frac{dy}{dn} \right\| = 0,$$

shewing that the prime normals of the regional geodesic and of the superficial geodesic and the flexural radius of the superficial geodesic lie in one plane. Also, because $\sum Y \frac{dy}{dn} = 0$, the flexural normal is at right angles to the prime normal of the regional geodesic: indeed, its direction lies in the tangent flat of the region *.

Within their plane, let ψ denote the inclination of the two prime normals, so that

$$\sum YY_0 = \cos\psi;$$

and then we have the results

$$\frac{\cos\psi}{\rho_0} = \frac{1}{\rho}, \quad \frac{\sin\psi}{\rho_0} = \frac{1}{\gamma}, \quad \frac{1}{\rho_0^2} = \frac{1}{\rho^2} + \frac{1}{\gamma^2}, \quad \frac{1}{\rho_0} = \frac{\cos\psi}{\rho} + \frac{\sin\psi}{\gamma},$$

$$Y_0 = Y\cos\psi + \frac{dy}{dn}\sin\psi.$$

In this plane, there is a direction at right angles to the prime normal of the geodesic: denoting its typical direction-cosine by $\overline{Y}_0$, we have

$$\overline{Y}_0 = -Y\sin\psi + \frac{dy}{dn}\cos\psi.$$

Moreover, the tangent of the superficial geodesic (being the tangent also of the regional geodesic) is at right angles to this plane. It is at right angles to the prime normal of the regional geodesic because $\sum Yy' = 0$. Also

$$\begin{aligned}\sum y' \frac{dy}{dn} &= \sum \left\{ (y_1 p' + y_2 q' + y_3 r') \left(y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn} \right) \right\} \\ &= p' \left(A\frac{dp}{dn} + H\frac{dq}{dn} + G\frac{dr}{dn} \right) + q' \left(H\frac{dp}{dn} + B\frac{dq}{dn} + F\frac{dr}{dn} \right) \\ &\qquad + r' \left(G\frac{dp}{dn} + F\frac{dq}{dn} + C\frac{dr}{dn} \right) \\ &= \frac{1}{\theta_n} (\theta_1 p' + \theta_2 q' + \theta_3 r') = 0,\end{aligned}$$

so that the tangent of the geodesic is at right angles to the regional normal of the surface. Consequently it is at right angles to the plane in question. It is therefore at right angles to every line in this plane; and thus, as is to be expected, it is at right angles to the prime normal of the superficial geodesic.

* When the region is homaloidal, so that it is the plenary homaloidal space of the surface, as in the Gauss theory of surfaces, then $1/\rho = 0$, and

$$\frac{Y_0}{\rho_0} = \frac{1}{\gamma}\frac{dy}{dn}:$$

that is, the regional flexure of a superficial geodesic becomes the circular curvature of the geodesic, and the direction of the flexural radius is the unique normal to the surface. The tangent flat of the region has become the region itself.

We have an immediate geometrical construction for this configuration. Let OF represent, both in magnitude and direction, the radius γ of regional flexure, the direction being unique for all geodesics through O: let OT be the tangent of any geodesic: and let OR represent, both in magnitude and direction, the radius ρ of circular curvature of the regional geodesic drawn in the direction OT. Then OF, OT, OR, are a set of three lines perpendicular to one another. Let OS be the perpendicular upon RF; then OS represents, in magnitude and in direction, the radius of circular curvature of the superficial geodesic drawn in the direction OT.

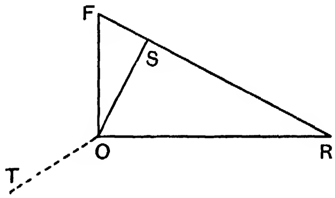

FIG. 20.

Also the locus of the centre of regional flexure of the surface (that is, of all the superficial geodesics) is a range of the line OF whose extremities are the two principal centres of such flexure, one for its maximum value, one for its minimum value; the range is the direct interval between the extremities if they are on the same side of O, but it is the complement of that interval if O lies between the extremities.

Principal regional flexures of superficial geodesics: surface-measures.

196. Now, in the region, the normal to the parametric surface is independent of all directions in the surface; and the quantity θ_n is independent of p', q', r'. Hence to find the principal values (that is, the maximum and the minimum values) of the regional flexure at any point of a surface, it is sufficient to determine the maximum and minimum values of the right-hand side in the equation

$$-\frac{\theta_n}{\gamma}=\vartheta_{11}p'^2+2\vartheta_{12}p'q'+\vartheta_{22}q'^2+2\vartheta_{12}p'r'+2\vartheta_{22}q'r'+\vartheta_{33}r'^2,$$

for all the admissible values of p', q', r', satisfying the conditions

$$\sum Ap'^2=1,\quad \theta_1p'+\theta_2q'+\theta_3r'=0.$$

The critical equations, in their initial form, are

$$\begin{aligned}\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r'&=\epsilon\theta_1+\eta(Ap'+Hq'+Gr'),\\ \vartheta_{21}p'+\vartheta_{22}q'+\vartheta_{23}r'&=\epsilon\theta_2+\eta(Hp'+Bq'+Fr'),\\ \vartheta_{31}p'+\vartheta_{32}q'+\vartheta_{33}r'&=\epsilon\theta_3+\eta(Gp'+Fq'+Cr'),\end{aligned}$$

ϵ and η being two quantities undetermined in the formation of these critical equations.

Let the equations be multiplied by p', q', r', respectively, and the products be added: then, when the conditions are used,

$$-\frac{\theta_n}{\gamma}=\eta,$$

determining the value of η. Thus the critical equations become

$$\left(\vartheta_{11}+\frac{\theta_n}{\gamma}A\right)p'+\left(\vartheta_{12}+\frac{\theta_n}{\gamma}H\right)q'+\left(\vartheta_{13}+\frac{\theta_n}{\gamma}G\right)r'=\epsilon\theta_1,$$

$$\left(\vartheta_{21}+\frac{\theta_n}{\gamma}H\right)p'+\left(\vartheta_{22}+\frac{\theta_n}{\gamma}B\right)q'+\left(\vartheta_{23}+\frac{\theta_n}{\gamma}F\right)r'=\epsilon\theta_2,$$

$$\left(\vartheta_{31}+\frac{\theta_n}{\gamma}G\right)p'+\left(\vartheta_{32}+\frac{\theta_n}{\gamma}F\right)q'+\left(\vartheta_{33}+\frac{\theta_n}{\gamma}C\right)r'=\epsilon\theta_3.$$

Also we have
$$\theta_1 p'+\theta_2 q'+\theta_3 r'=0.$$

The determinantal elimination of the quantities p', q', r', ϵ, which occur linearly in these four equations, leads to the single equation

$$\begin{vmatrix} \vartheta_{11}+\frac{1}{\gamma}\theta_n A, & \vartheta_{12}+\frac{1}{\gamma}\theta_n H, & \vartheta_{13}+\frac{1}{\gamma}\theta_n G, & \theta_1 \\ \vartheta_{21}+\frac{1}{\gamma}\theta_n H, & \vartheta_{22}+\frac{1}{\gamma}\theta_n B, & \vartheta_{23}+\frac{1}{\gamma}\theta_n F, & \theta_2 \\ \vartheta_{31}+\frac{1}{\gamma}\theta_n G, & \vartheta_{32}+\frac{1}{\gamma}\theta_n F, & \vartheta_{33}+\frac{1}{\gamma}\theta_n C, & \theta_3 \\ \theta_1, & \theta_2, & \theta_3, & 0 \end{vmatrix}=0,$$

a quadratic in γ, determining one maximum value and one minimum value. Let two quantities M and μ be taken such that

$$M\theta_n{}^4\Omega=\sum\{(\vartheta_{22}\vartheta_{33}-\vartheta_{23}{}^2)\theta_1{}^2\},$$
$$\mu\theta_n{}^3\Omega-\sum\{(C\vartheta_{22}-2F\vartheta_{23}+B\vartheta_{33})\theta_1{}^2\},$$

so that M and μ are contravariants of the two ternary quantics

$$\sum\vartheta_{11}p'^2,\quad \sum Ap'^2,$$

with θ_1, θ_2, θ_3, as the contragredient variables, similar to the contravariant $\sum a\theta_1{}^2$; then the foregoing quadratic in γ becomes

$$M\theta_4{}^n\Omega+\frac{\theta_n}{\gamma}\mu\theta_n{}^3\Omega+\frac{1}{\gamma^2}\Omega\theta_n{}^4=0,$$

that is,

$$\frac{1}{\gamma^2}+\frac{1}{\gamma}\mu+M=0.$$

Accordingly, if the maximum and the minimum values of γ be denoted by γ_1 and γ_2, we have

$$\frac{1}{\gamma_1}+\frac{1}{\gamma_2}=-\mu,\quad \frac{1}{\gamma_1\gamma_2}=M,$$

with the foregoing values of μ and M in terms of magnitudes connected with the region and the parametric equation of the surface.

Next, the principal directions of regional flexure (that is, the respective directions of the maximum and minimum values) are given by the values of p', q', r', given in turn by the successive substitution of the two principal values of γ. In particular, the two principal directions are at right angles. To establish this result, let p_1', q_1', r_1', be the variables for the direction of γ_1, and p_2', q_2', r_2', be the variables of the direction of γ_2. When the first forms of the equations for p_1', q_1', r_1', and γ_1 are used, we multiply them by p_2', q_2', r_2', respectively, and add: then

$$\sum \vartheta_{11} p_1' p_2' = \epsilon(\theta_1 p_2' + \theta_2 q_2' + \theta_3 r_2') + \eta_1 \sum A p_1' p_2' = \eta_1 \sum A p_1' p_2',$$

where $\eta_1 = -\theta_n/\gamma_1$. When the same forms are taken for p_2', q_2', r_2', and γ_2, and are multiplied by p_1', q_1', r_1', and then added, we find

$$\sum \vartheta_{11} p_1' p_2' = \epsilon(\theta_1 p_1' + \theta_2 q_1' + \theta_3 r_1') + \eta_2 \sum A p_1' p_2' = \eta_2 \sum A p_1' p_2',$$

where $\eta_2 = -\theta_n/\gamma_2$. We assume that, for our surface $\theta = 0$, the quantities γ_1 and γ_2 are unequal; and therefore, for such a surface,

$$\sum \vartheta_{11} p_1' p_2' = 0, \quad \sum A p_1' p_2' = 0.$$

The latter result shews the two directions are at right angles.

As yet, though the value of η connected with a principal direction of regional flexure has been determined, being equal to $-\theta_n/\gamma$, the corresponding value of ϵ has not been obtained. For this purpose, we take the second forms of the equations for the principal directions; and add them, after multiplying them by $a\theta_1 + h\theta_2 + g\theta_3$, $h\theta_1 + b\theta_2 + f\theta_3$, $g\theta_1 + f\theta_2 + c\theta_3$, respectively. Then the right-hand side

$$= \epsilon \sum a\theta_1^2 = \epsilon \theta_n^2 \Omega.$$

For the left-hand side, the aggregate of the terms in θ_n/γ has for the coefficient of that magnitude

$$\begin{aligned}(Ap' + Hq' + Gr')(a\theta_1 + h\theta_2 + g\theta_3)&\\ + (Hp' + Bq' + Fr')(h\theta_1 + b\theta_2 + f\theta_3)&\\ + (Gp' + Fq' + Cr')(g\theta_1 + f\theta_2 + c\theta_3)&\\ = \Omega(\theta_1 p' + \theta_2 q' + \theta_3 r') = 0;&\end{aligned}$$

and so we have

$$\begin{aligned}\epsilon\theta_n^2\Omega = (\vartheta_{11} p' + \vartheta_{12} q' + \vartheta_{13} r')(a\theta_1 + h\theta_2 + g\theta_3)&\\ + (\vartheta_{21} p' + \vartheta_{22} q' + \vartheta_{23} r')(h\theta_1 + b\theta_2 + f\theta_3)&\\ + (\vartheta_{31} p' + \vartheta_{32} q' + \vartheta_{33} r')(g\theta_1 + f\theta_2 + c\theta_3).&\end{aligned}$$

Later, in § 207, it will be proved that the value of the quantity on the right-hand side is $\Omega\theta_n \dfrac{d\theta_n}{ds}$; and therefore

$$\epsilon = \frac{1}{\theta_n} \frac{d\theta_n}{ds},$$

where the derivation now is taken along one of the principal directions. Thus the equations of the principal directions of regional flexure are

$$\left.\begin{aligned}
\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r'&=\frac{\theta_1}{\theta_n}\frac{d\theta_n}{ds}-\frac{\theta_n}{\gamma}(Ap'+Hq'+Gr')\\
\vartheta_{12}p'+\vartheta_{22}q'+\vartheta_{23}r'&=\frac{\theta_2}{\theta_n}\frac{d\theta_n}{ds}-\frac{\theta_n}{\gamma}(Hp'+Bq'+Fr')\\
\vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r'&=\frac{\theta_3}{\theta_n}\frac{d\theta_n}{ds}-\frac{\theta_n}{\gamma}(Gp'+Fq'+Cr')
\end{aligned}\right\}.$$

It is to be noted that, along each of the principal directions, the equation

$$\begin{vmatrix}
\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r', & Ap'+Hq'+Gr', & \theta_1\\
\vartheta_{12}p'+\vartheta_{22}q'+\vartheta_{23}r', & Hp'+Bq'+Fr', & \theta_2\\
\vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r', & Gp'+Fq'+Cr', & \theta_3
\end{vmatrix}=0$$

is satisfied. It will appear later (§ 200) that this relation implies a property of the regional torsion of a superficial geodesic: viz. this regional torsion vanishes at a point where the superficial geodesic touches a principal direction of regional flexure *.

Again, the directions on the surface $\theta=0$, which allow a vanishing flexure for the superficial geodesic, satisfy the equation

$$\vartheta_{11}p'^2+2\vartheta_{12}p'q'+\vartheta_{22}q'^2+2\vartheta_{13}p'r'+2\vartheta_{23}q'r'+\vartheta_{33}r'^2=0,$$

as well as the equation

$$\theta_1p'+\theta_2q'+\theta_3r'=0,$$

characteristic of all directions on the surface. Thus there are two such directions at each point of the surface, real or imaginary: for each of these two superficial geodesics, there is secondary contact with the tangential regional geodesic. Also, the angles between these directions of regional inflexions are bisected by directions (§ 159, *Ex.* 1) which satisfy the equations

$$\theta_1p'+\theta_2q'+\theta_3r'=0,$$

$$\begin{vmatrix}
\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r', & Ap'+Hq'+Gr', & \theta_1\\
\vartheta_{12}p'+\vartheta_{22}q'+\vartheta_{23}r', & Hp'+Bq'+Fr', & \theta_2\\
\vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r', & Gp'+Fq'+Cr', & \theta_3
\end{vmatrix}=0:$$

that is, the bisectors are the two principal directions of regional flexure.

Thus there is an indicatrix conic for the regional flexure of each surface within the region. The conic is the intersection of a quadric by a plane: the axes of the

* It is the analogue of the property that the torsion of any geodesic of an amplitude, existing freely in any plenary homaloidal space, vanishes at a point where the geodesic touches a curve of circular curvature in the amplitude.

conic are the principal directions of regional flexure, while the asymptotes of the conic are the directions of zero flexure; and the radius of regional flexure of a superficial geodesic is proportional to the square of the central radius vector of the indicatrix conic drawn in the direction of the tangent to the geodesic.

Curves of spatial curvature on a surface.

197. The preceding investigation leads to the curves of regional flexure on a surface in a region, being the directions at any point where the flexure of a superficial geodesic is a maximum or a minimum among all the directions; and the regional geodesics, which have secondary contact with the surface, are the (real or imaginary) flexural asymptotes on the surface.

There exist also the curves of circular curvature on the surface, the directions at any point being such that the circular curvature of the superficial geodesic in that direction is a maximum or a minimum among all the directions. These curves will be called the *curves of spatial curvature* of the surface.

Further, there exist the curves on the surface, the directions at any point being such that the circular curvature of the regional geodesic in that direction touching the surface is a maximum or a minimum among all the regional geodesics touching the surface at the point. These curves will be called the *curves of regional curvature* of the surface.

The regional flexure $1/\gamma$ of a superficial geodesic in any direction, the circular curvature $1/\rho_0$ of that geodesic, and the circular curvature $1/\rho$ of the regional geodesic touching the superficial geodesic, are connected by the equation

$$\frac{1}{\rho_0^2} = \frac{1}{\rho^2} + \frac{1}{\gamma^2}.$$

It will appear that, at any point of a regional surface (as of any surface in a plenary space of more than three dimensions), there exist four curves of spatial curvature: also that there are four curves of regional curvature at any point of the surface; and it has been seen that there are two curves of regional flexure at any point of the surface, together with two asymptotic lines of regional flexure the angles between which are bisected by the curves of regional flexure. In general, no curve of any one of the three sets coincides with any curve of either of the other two sets. An asymptotic line lies between a curve of maximum regional curvature and a curve of minimum spatial curvature. A curve of regional flexure lies between a curve of minimum regional curvature and a curve of maximum spatial curvature. Each sector in the tangent plane, constituted by the acute angle between the direction of a curve of regional flexure and the direction of an asymptotic line contains one direction of maximum regional curvature, one direction of minimum regional curvature, one direction of maximum spatial curvature, and one direction of minimum spatial curvature. These properties, and other

similar descriptive properties, can easily be derived from the respective sets of equations which are characteristic of the various curves at the point.

The curves of spatial curvature on the surface are the directions giving a maximum or a minimum value γ of the circular curvature $1/\rho_0$ of a superficial geodesic. We therefore must seek the critical equations giving the directions of a maximum or a minimum value of $\frac{1}{\rho_0^2}$ (not being a zero value) where

$$\frac{1}{\rho_0^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2},$$

with the known value (§ 161) of $1/\rho$, while

$$-\frac{\theta_n}{\gamma}=\vartheta_{11}p'^2+2\vartheta_{12}p'q'+2\vartheta_{13}p'r'+\vartheta_{22}q'^2+2\vartheta_{23}q'r'+\vartheta_{23}r'^2.$$

Also, the admissible values of p', q', r', are subject to the two conditions

$$\sum Ap'^2=1,\quad \theta_1p'+\theta_2q'+\theta_3r'=0.$$

We write

$$\left.\begin{aligned}\vartheta_1&=\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r'\\ \vartheta_2&=\vartheta_{21}p'+\vartheta_{22}q'+\vartheta_{23}r'\\ \vartheta_3&=\vartheta_{31}p'+\vartheta_{32}q'+\vartheta_{33}r'\end{aligned}\right\},$$

analogous to the magnitudes u_1, u_2, u_3, derived from $\sum Ap'^2$, and to the magnitudes v_1, v_2, v_3, derived from the expression $\sum \bar{A}p'^2$ as the value of $\frac{1}{\rho}$. Then the first form of the three critical equations for the present investigation is

$$\frac{1}{\rho}v_1-\frac{1}{\gamma\theta_n}\vartheta_1=t\theta_1+Tu_1,$$

$$\frac{1}{\rho}v_2-\frac{1}{\gamma\theta_n}\vartheta_2=t\theta_2+Tu_2,$$

$$\frac{1}{\rho}v_3-\frac{1}{\gamma\theta_n}\vartheta_3=t\theta_3+Tu_3,$$

where t and T denote multipliers left undetermined in the establishment of the critical equations.

In the first place, by the elimination of t and T, we have the equation

$$\begin{vmatrix}\frac{1}{\rho}v_1-\frac{1}{\gamma\theta_n}\vartheta_1, & u_1, & \theta_1\\ \frac{1}{\rho}v_2-\frac{1}{\gamma\theta_n}\vartheta_2, & u_2, & \theta_2\\ \frac{1}{\rho}v_3-\frac{1}{\gamma\theta_n}\vartheta_3, & u_3, & \theta_3\end{vmatrix}=0.$$

a relation which can be changed into an equation involving only p', q', r', as variables. For, with the notation of § 168, we have

$$\frac{1}{\rho} v_1 = Q_1, \quad \frac{1}{\rho} v_2 = Q_2, \quad \frac{1}{\rho} v_3 = Q_3,$$

where Q_1, Q_2, Q_3, are homogeneous cubics in p', q', r'; also $1/\gamma$ is a homogeneous quadratic in p', q', r', while ϑ_1, ϑ_2, ϑ_3, are linear in those magnitudes; thus the constituents of the first column in the determinant are homogeneous cubics. The constituents in the second column are linear. Hence the foregoing determinantal equation is of the fourth degree in p', q', r', and is homogeneous in the direction-variables; and therefore, when combined with the linear relation

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

it provides four sets of values. We thus have, at any point of a regional surface, four curves of spatial curvature.

The value of T is at once obtainable in the form

$$T = \frac{1}{{\rho_0}^2};$$

but this form adds no obvious significance to the equation.

Another form can be given to the determinantal equation which appears to have a more immediately geometrical significance. It can be written

$$\frac{1}{\rho} \begin{vmatrix} v_1, & u_1, & \theta_1 \\ v_2, & u_2, & \theta_2 \\ v_3, & u_3, & \theta_3 \end{vmatrix} = \frac{1}{\gamma \theta_n} \begin{vmatrix} \vartheta_1, & u_1, & \theta_1 \\ \vartheta_2, & u_2, & \theta_2 \\ \vartheta_3, & u_3, & \theta_3 \end{vmatrix}.$$

It will appear later (§§ 200, 201) that

$$\begin{vmatrix} v_1, & u_1, & \theta_1 \\ v_2, & u_2, & \theta_2 \\ v_3, & u_3, & \theta_3 \end{vmatrix} = -\Omega^{\frac{1}{2}} \frac{\theta_n}{\tau_\theta},$$

where $1/\tau_\theta$ denotes the regional tilt of the superficial geodesic, and that

$$\begin{vmatrix} \vartheta_1, & u_1, & \theta_1 \\ \vartheta_2, & u_2, & \theta_2 \\ \vartheta_3, & u_3, & \theta_3 \end{vmatrix} = -\Omega^{\frac{1}{2}} \frac{{\theta_n}^2}{\sigma_\theta},$$

where $1/\sigma_\theta$ denotes the regional torsion of that geodesic. Consequently the foregoing equation can be changed to the form

$$\frac{1}{\gamma \sigma_\theta} - \frac{1}{\rho \tau_\theta} = 0,$$

a relation between curvatures alone, belonging to the curves of spatial curvature of the regional surface *.

Curves of regional curvature on a surface.

198. The curves of regional curvature on the surface are in such directions as to give a maximum value or a minimum value of the circular curvature of a regional geodesic touching the surface. Their directions consequently are given by the maximum and the minimum values of $1/\rho$, for all values of p', q', r', admissible under the two conditions

$$\sum Ap'^2=1, \quad \theta_1 p'+\theta_2 q'+\theta_3 r'=0.$$

Now we have (§ 168)

$$\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right)=2v_1, \quad \frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right)=2v_2, \quad \frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right)=2v_3.$$

Thus the critical equations are

$$\left.\begin{array}{l} v_1=\lambda_0 u_1+\kappa_0\theta_1 \\ v_2=\lambda_0 u_2+\kappa_0\theta_2 \\ v_3=\lambda_0 u_3+\kappa_0\theta_3 \end{array}\right\},$$

where λ_0 and κ_0 are multipliers left undetermined in the construction of the critical equations. Consequently, as

$$v_1=\rho Q_1, \quad v_2=\rho Q_2, \quad v_3=\rho Q_3,$$

where Q_1, Q_2, Q_3, are (§ 168) homogeneous cubics in p', q', r', the directions of the curves of spatial curvature satisfy the equation

$$\begin{vmatrix} Q_1, & u_1, & \theta_1 \\ Q_2, & u_2, & \theta_2 \\ Q_3, & u_3, & \theta_3 \end{vmatrix}=0,$$

manifestly homogeneous of degree four in p', q', r'. When combined with the superficial relation $\theta_1 p'+\theta_2 q'+\theta_3 r'=0$, this equation shews that there are four curves of regional curvature through a point on a regional surface.

The explicit values of λ_0 and κ_0 can be obtained. In the first place, multiply by p', q', r', and add the products: then

$$\frac{1}{\rho}=\lambda_0.$$

* The result is in accord with the result obtained when the region is a primary region, that is, existing in a plenary homaloidal space of four dimensions; see *G.F.D.*, vol. ii, §§ 365, 370.

Again, from the critical equations, we have

$$\kappa_0^2 \sum a\theta_1^2 = \sum a\left(v_1 - \frac{u_1}{\rho}\right)^2$$
$$= \sum a v_1^2 - \frac{2}{\rho}\sum a u_1 v_1 + \frac{1}{\rho^2}\sum a u_1^2$$
$$= \frac{\Omega}{\sigma^2};$$

and therefore

$$\kappa_0 = \frac{1}{\sigma\theta_n}.$$

Thus the equations for the directions of the curves of regional curvature on the surface are

$$\left.\begin{aligned} v_1 - \frac{u_1}{\rho} &= \frac{1}{\sigma}\frac{\theta_1}{\theta_n} \\ v_2 - \frac{u_2}{\rho} &= \frac{1}{\sigma}\frac{\theta_2}{\theta_n} \\ v_3 - \frac{u_3}{\rho} &= \frac{1}{\sigma}\frac{\theta_3}{\theta_n} \end{aligned}\right\}.$$

We have proved that the typical direction-cosine of the binormal of any regional geodesic (§ 172) is given by

$$l_3 = l y_1 + m y_2 + n y_3,$$

where

$$\left.\begin{aligned} \frac{1}{\sigma}(Al + Hm + Gn) &= \frac{1}{\rho}u_1 - v_1 \\ \frac{1}{\sigma}(Hl + Bm + Fn) &= \frac{1}{\rho}u_2 - v_2 \\ \frac{1}{\sigma}(Gl + Fm + Cn) &= \frac{1}{\rho}u_3 - v_3 \end{aligned}\right\}.$$

Thus along a curve of regional curvature on the surface, we have (except as to sign)

$$l = \frac{dp}{dn}, \quad m = \frac{dq}{dn}, \quad n = \frac{dr}{dn},$$

and therefore, also except as to sign,

$$l_3 = \frac{dy}{dn}:$$

in other words, along a curve of regional curvature on the surface, the binormal of the tangential regional geodesic is normal to the surface, and therefore it is at right angles to the binormal of the superficial geodesic. This result also appears

as a corollary of a later investigation (§ 202); for the inclination χ, in general, is there given by

$$\cos\chi = \frac{\sigma}{\Omega^{\frac{1}{2}}\theta_n} \begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

which vanishes along the curve. Moreover, it then also follows that the regional tilt $1/\tau_\theta$ of the superficial geodesic vanishes along a curve of regional curvature.

It is to be noted that, when we denote the equations of the curves of regional flexure, the curves of spatial curvature, and the curves of regional curvature, on the surface by $E_f=0$, $E_s=0$, $E_r=0$, respectively, in the adopted forms, then

$$E_s = E_r - \frac{1}{\gamma\theta_n} E_f,$$

so that, in general, no curve of any one set coincides with any curve of either of the other sets.

Finally, as regards the curves of regional curvature, we have, along their direction,

$$\frac{1}{\sigma\theta_n}\sum av_1\theta_1 = \sum av_1^2 - \frac{1}{\rho}\sum au_1v_1$$
$$= \frac{\Omega}{\sigma^2},$$

so that the covariant $\sum av_1\theta_1$ has the value $\Omega\theta_n/\sigma$ along such a curve.

Regional curvatures of superficial geodesics.

199. When a parametric surface is contained in a region, the surface is also a configuration in the homaloidal plenary space of the region, and it has the regular curvatures for its geodesics as determined by its relation to that space; for convenience, these regular curvatures will be called *spatial*. The surface also has its restricted relations to the containing region; its geodesics are, of course, unaltered and therefore their principal lines are unaltered; but the changes of inclination of the principal lines, relative to the region, are affected by the region. There will be consequent measures of curvature of the superficial geodesics, relative to the region; and these will be called *regional*. One such regional measure is the regional flexure of a superficial geodesic, being connected with the rate of deviation of that geodesic from the tangent regional geodesic. The others can be obtained through an orthogonal frame for the superficial geodesic, constructed with reference to the region.

The principal lines of the geodesic, gremial to the surface, are its tangent line and its binormal, the latter lying in the tangent plane of the surface at right

angles to the tangent line. Let P', Q', R', denote the regional direction-variables of this binormal, so that we have

$$\theta_1 P' + \theta_2 Q' + \theta_3 R' = 0\,;$$

then its typical spatial direction-cosine is λ_3, where

$$\lambda_3 = y_1 P' + y_2 Q' + y_3 R'.$$

It is to be at right angles to the tangent line, with direction-variables p', q', r'; and therefore

$$\sum A p' P' = 0,$$

that is,

$$u_1 P' + u_2 Q' + u_3 R' = 0.$$

Hence, for P', Q', R', we have *

$$\frac{P'}{\theta_2 u_3 - \theta_3 u_2} = \frac{Q'}{\theta_3 u_1 - \theta_1 u_3} = \frac{R'}{\theta_1 u_2 - \theta_2 u_1} = \frac{1}{\Omega^{\frac{1}{2}} \theta_n},$$

the common value of the fractions being obtained from the relation $\sum AP'^2 = 1$; and thus the direction of the binormal is analytically determinate.

Another direction in the region at O is organically connected with the surface. We have had the regional normal to the surface, being a line within the tangent flat of the region; its typical direction-cosine is $\dfrac{dy}{dn}$, where

$$\frac{dy}{dn} = y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn}.$$

As this direction is perpendicular to the tangent plane of the surface, it is at right angles to every direction in that tangent plane; and therefore it is at right angles to the binormal, a property satisfied by the foregoing magnitudes, because

$$\begin{aligned}\sum \left(\lambda_3 \frac{dy}{dn}\right) &= \sum \left\{ (y_1 P' + y_2 Q' + y_3 R') \left(y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn} \right) \right\} \\ &= \sum \left\{ P' \left(A \frac{dp}{dn} + H \frac{dq}{dn} + G \frac{dr}{dn} \right) \right\} \\ &= \frac{1}{\theta_n} (\theta_1 P' + \theta_2 Q' + \theta_3 R') = 0.\end{aligned}$$

Thus there are three lines within the tangent flat of the region at right angles to one another, two of them being principal lines of the superficial geodesic, and the third being the regional normal of the surface itself. These will be taken as

* In selecting the signs of P', Q', R', account is taken of the values of the direction-variables (§§ 91, 106) of the binormal of a superficial geodesic for the spatial direction-cosines.

the foundation of the orthogonal frame of the geodesic relative to the region. They also are leading lines of the flat, and are equivalent to the three gremial principal lines of the tangent regional geodesic. Accordingly, as the normal to the region associated with the direction of the curve is the prime normal of the regional geodesic, we shall take that line as the fourth principal line of the orthogonal frame of the geodesic in relation to the region.

Now consider the variation of this frame, so far as inclinations of its lines are concerned, for successive places Q along the superficial geodesic in the immediate vicinity of O. Let $\delta\mu_1$ denote the angle of regional flexure of the superficial geodesic; the spatial direction-cosine of the radius of regional flexure is $\frac{dy}{dn}$, and therefore, from the arc in the orbicular representation of the whole configuration, we have

$$\begin{aligned}\frac{dy}{dn}\delta\mu_1 &= (y'+ty_0''+\ldots)-(y'+ty''+\ldots)\\ &= (y_0''-y'')t = \left(\frac{Y_0}{\rho_0}-\frac{Y}{\rho}\right)t,\end{aligned}$$

when the arc QO $(=t)$ is small; if we define the flexure $1/\gamma_\theta$ by the equation

$$\frac{1}{\gamma_\theta}=\operatorname*{Lim}_{t\to 0}\frac{\delta\mu_1}{t},$$

we have, as the first relation for the frame,

$$\frac{1}{\gamma_\theta}\frac{dy}{dn}=\frac{Y_0}{\rho_0}-\frac{Y}{\rho},$$

the result already established.

Let $d\mu_2$, $d\mu_3$, $d\mu_4$, denote the small angles in the displaced position of the frame at Q, corresponding to the small angles in the displaced position of the spatial frame at Q as used in the construction of the Frenet equations *. Then, if we write

$$y'=l_1,\quad \frac{dy}{dn}=l_2,\quad \lambda_3=l_3,\quad Y=l_4,$$

these small angles, and the variations of these typical direction-cosines, are connected by the relations

$$\begin{aligned}dl_1 &= l_2\,d\mu_1,\\ dl_2 &= -l_1\,d\mu_1+l_3\,d\mu_2,\\ dl_3 &= -l_2\,d\mu_2+l_4\,d\mu_3,\end{aligned}$$

these being sufficient to express the variations of the directions within the osculating block of the region.

* *G.F.D.*, vol. i, §§ 164, 197.

Regional torsion of a superficial geodesic.

200. The significance of the first equation, with the due interpretation of dl_1, has been obtained. We shall denote by $1/\sigma_\theta$ and $1/\tau_\theta$ quantities, called the regional torsion of the superficial geodesic and the regional tilt of the superficial geodesic, and defined by the equations

$$\frac{1}{\sigma_\theta}=\operatorname*{Lim}_{t\to 0}\left(\frac{d\mu_2}{t}\right),\quad \frac{1}{\tau_\theta}=\operatorname*{Lim}_{t\to 0}\left(\frac{d\mu_3}{t}\right);$$

and now the second and the third of the equations become

$$\frac{d}{ds}\left(\frac{dy}{dn}\right)=-\frac{y'}{\gamma}+\frac{l_3}{\sigma_\theta},$$

$$\frac{dl_3}{ds}=-\frac{1}{\sigma_\theta}\frac{dy}{dn}+\frac{Y}{\tau_\theta}.$$

We proceed to deduce values of the regional torsion and the regional tilt.

From the equation

$$\frac{d}{ds}\left(\frac{dy}{dn}\right)=-\frac{y'}{\gamma}+\frac{l_3}{\sigma_\theta},$$

we have

$$\frac{1}{\sigma_\theta}=\sum\left\{l_3\frac{d}{ds}\left(\frac{dy}{dn}\right)\right\}=\sum\left[(y_1P'+y_2Q'+y_3R')\left\{\frac{d}{ds}\left(\frac{dy}{dn}\right)\right\}\right].$$

Now

$$\sum y_1\frac{dy}{dn}=A\frac{dp}{dn}+H\frac{dq}{dn}+G\frac{dr}{dn}=\frac{\theta_1}{\theta_n};$$

and therefore

$$\sum y_1\left\{\frac{d}{ds}\left(\frac{dy}{dn}\right)\right\}+\sum\left\{\frac{dy}{dn}(y_{11}p'+y_{12}q'+y_{13}r')\right\}$$
$$=\theta_1\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{1}{\theta_n}(\theta_{11}p'+\theta_{12}q'+\theta_{13}r').$$

As regards the second sum on the left-hand side, we have

$$\sum y_{11}\frac{dy}{dn}=\frac{dp}{dn}\sum y_1y_{11}+\frac{dq}{dn}\sum y_2y_{11}+\frac{dr}{dn}\sum y_3y_{11}$$
$$=\frac{1}{\Omega\theta_n}\{(a\theta_1+h\theta_2+g\theta_3)(A\Gamma_{11}+H\Delta_{11}+G\Theta_{11})$$
$$+(h\theta_1+b\theta_2+f\theta_3)(H\Gamma_{11}+B\Delta_{11}+F\Theta_{11})$$
$$+(g\theta_1+f\theta_2+c\theta_3)(G\Gamma_{11}+F\Delta_{11}+C\Theta_{11})\}$$
$$=\frac{1}{\theta_n}(\theta_1\Gamma_{11}+\theta_2\Delta_{11}+\theta_3\Theta_{11});$$

and similarly, for all values of i and j, $=1, 2, 3$,

$$\sum y_{ij}\frac{dy}{dn}=\frac{1}{\theta_n}(\theta_1\Gamma_{ij}+\theta_2\Delta_{ij}+\theta_3\Theta_{ij}).$$

Consequently we have

$$\sum y_1 \frac{d}{ds}\left(\frac{dy}{dn}\right) = \theta_1 \frac{d}{ds}\left(\frac{1}{\theta_n}\right) + \frac{1}{\theta_n}(\vartheta_{11}p' + \vartheta_{12}q' + \vartheta_{13}r'),$$

and, in the same manner,

$$\sum y_2 \frac{d}{ds}\left(\frac{dy}{dn}\right) = \theta_2 \frac{d}{ds}\left(\frac{1}{\theta_n}\right) + \frac{1}{\theta_n}(\vartheta_{21}p' + \vartheta_{22}q' + \vartheta_{23}r'),$$

$$\sum y_3 \frac{d}{ds}\left(\frac{dy}{dn}\right) = \theta_3 \frac{d}{ds}\left(\frac{1}{\theta_n}\right) + \frac{1}{\theta_n}(\vartheta_{31}p' + \vartheta_{32}q' + \vartheta_{33}r').$$

When these values are substituted in the expression for $1/\sigma_\theta$, the total coefficient of $\frac{d}{ds}\left(\frac{1}{\theta_n}\right)$

$$= \theta_1 P' + \theta_2 Q' + \theta_3 R' = 0;$$

and when the explicit values of P', Q', R', are substituted, we find

$$-\frac{\Omega^{\frac{1}{2}}\theta_n^{\,2}}{\sigma_\theta} = \begin{vmatrix} \vartheta_{11}p' + \vartheta_{12}q' + \vartheta_{13}r', & Ap' + Hq' + Gr', & \theta_1 \\ \vartheta_{21}p' + \vartheta_{22}q' + \vartheta_{23}r', & Hp' + Bq' + Fr', & \theta_2 \\ \vartheta_{31}p' + \vartheta_{32}q' + \vartheta_{33}r', & Gp' + Fq' + Cr', & \theta_3 \end{vmatrix}.$$

It has already (§ 198) appeared that the right-hand side vanishes along the principal directions of regional flexure, being those directions which provide a maximum or a minimum value of that flexure. This property can be associated with the characteristic property of a different (but ultimately equivalent) definition —connected with the intersection of consecutive radii of regional flexure along the direction. The typical equation of a radius of regional flexure is

$$\bar{y} - y = \frac{1}{\gamma}\frac{dy}{dn};$$

hence when this radius, for a superficial geodesic in the direction p', q', r', is intersected by the radius through a consecutive point on the geodesic, and when we take $\bar{y}$ as the typical coordinate of the intersection, we have relations typified by the equation

$$-y' = \frac{d}{ds}\left(\frac{1}{\gamma}\right)\frac{dy}{dn} + \frac{1}{\gamma}\frac{d}{ds}\left(\frac{dy}{dn}\right).$$

From this set of equations, two inferences can be derived. In the first place, let the typical equation be multiplied by $\frac{dy}{dn}$ and the product be summed for the set; because

$$\sum y' \frac{dy}{dn} = 0, \quad \sum \left(\frac{dy}{dn}\right)^2 = 1,$$

we have

$$\frac{d}{ds}\left(\frac{1}{\gamma}\right)=0,$$

giving a stationary value for $\frac{1}{\gamma}$, a result to be expected owing to the intersection of the consecutive radii. Next, multiply by y_1 and add: then by y_2 and add: lastly by y_3 and add; then, successively, we find

$$-(Ap'+Hq'+Br')=\frac{\theta_1}{\gamma}\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{1}{\gamma\theta_n}(\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r'),$$

$$-(Hp'+Bq'+Fr')=\frac{\theta_2}{\gamma}\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{1}{\gamma\theta_n}(\vartheta_{21}p'+\vartheta_{22}q'+\vartheta_{23}r'),$$

$$-(Gp'+Fq'+Cr')=\frac{\theta_3}{\gamma}\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{1}{\gamma\theta_n}(\vartheta_{31}p'+\vartheta_{32}q'+\vartheta_{33}r').$$

Elimination of $\frac{d}{ds}\left(\frac{1}{\theta_n}\right)$ and of $\frac{1}{\gamma}$ between these equations leads to the result

$$\frac{1}{\sigma_\theta}=0,$$

and also to the same equation as before for the principal directions of regional flexure on the altered definition.

Regional tilt of a superficial geodesic.

201. Next, for the regional tilt of the superficial geodesic, we proceed from the equation

$$\frac{d\lambda_3}{ds}=\frac{dl_3}{ds}=-\frac{1}{\sigma_0}\frac{dy}{dn}+\frac{Y}{\tau_0},$$

the arc-differentiation being taken along the superficial geodesic so that the left-hand side (with the former notation) is $\frac{dl_3}{ds_0}$. Now

$$\frac{du_1}{ds_0}-\frac{du_1}{ds}=A(p_\theta''-p'')+H(q_\theta''-q'')+G(r_\theta''-r'')$$

$$=\frac{1}{\gamma}\left(A\frac{dp}{dn}+H\frac{dq}{dn}+G\frac{dr}{dn}\right)=\frac{1}{\gamma}\frac{\theta_1}{\theta_n};$$

and similarly

$$\frac{du_2}{ds_\theta}-\frac{du_2}{ds}=\frac{1}{\gamma}\frac{\theta_2}{\theta_n},\quad \frac{du_3}{ds_\theta}-\frac{du_3}{ds}=\frac{1}{\gamma}\frac{\theta_3}{\theta_n}.$$

The value of P' is (§ 199)

$$P'=\frac{1}{\Omega^{\frac{1}{2}}\theta_n}(\theta_2u_3-\theta_3u_2),$$

and therefore

$$\frac{dP'}{ds_0}-\frac{dP'}{ds}=\frac{1}{\Omega^{\frac{1}{2}}\theta_n}\left\{\theta_2\left(\frac{1}{\gamma}\frac{\theta_3}{\theta_n}\right)-\theta_3\left(\frac{1}{\gamma}\frac{\theta_2}{\theta_n}\right)\right\}=0;$$

and similarly

$$\frac{dQ'}{ds_\theta} - \frac{dQ'}{ds} = 0, \quad \frac{dR'}{ds_\theta} - \frac{dR'}{ds} = 0.$$

Consequently

$$\frac{dl_3}{ds_\theta} = \frac{dl_3}{ds},$$

so that the first variation of l_3 is the same along the superficial geodesic as along the regional geodesic tangent; and the latter may be used in the calculations.

Multiplying the equation, which expresses $\frac{dl_3}{ds}$, first by $\frac{dy}{dn}$ and adding all the products for all the dimensions of the plenary space, we have

$$-\frac{1}{\sigma_\theta} = \sum \frac{dl_3}{ds}\frac{dy}{dn}.$$

Because

$$\sum l_3 \frac{dy}{dn} = 0,$$

we have

$$\sum \frac{dl_3}{ds}\frac{dy}{dn} = -\sum l_3 \frac{d}{ds}\left(\frac{dy}{dn}\right);$$

and therefore

$$\frac{1}{\sigma_\theta} = \sum l_3 \frac{d}{ds}\left(\frac{dy}{dn}\right),$$

as before.

Next, we have $\sum y_1 Y = 0$, and therefore

$$\sum y_1 \frac{dY}{ds_\theta} = -\sum Y \frac{dy_1}{ds} = -\sum Y(y_{11}p' + y_{12}q' + y_{13}r') = -v_1;$$

and similarly

$$\sum y_2 \frac{dY}{ds_\theta} = -v_2, \quad \sum y_3 \frac{dY}{ds_\theta} = -v_3.$$

Also, we have $\sum Y l_3 = 0$, and therefore

$$\sum Y \frac{dl_3}{ds_\theta} = -\sum l_3 \frac{dY}{ds_\theta} = -\sum (y_1 P' + y_2 Q' + y_3 R')\frac{dY}{ds_\theta} = P'v_1 + Q'v_2 + R'v_3.$$

Hence, on multiplying the same equation by Y and adding all the products as before, we have

$$\frac{1}{\tau_\theta} = \sum Y \frac{dl_3}{ds_\theta} = P'v_1 + Q'v_2 + R'v_3,$$

or

$$-\Omega^{\frac{1}{2}}\frac{\theta_n}{\tau_\theta} = \begin{vmatrix} v_1, & v_2, & v_3 \\ u_1, & u_2, & u_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}.$$

Ex. Some covariants, belonging to the complete system for a regional surface, can be expressed in terms of this quantity τ_θ.

In connection with the coil $1/\kappa$ of a regional geodesic, there are (§§ 182, 189) quantities u_5, v_5, w_5, such that

$$u_5=\frac{1}{\kappa}(A\bar{a}+H\bar{\beta}+G\bar{\gamma}),\quad v_5=\frac{1}{\kappa}(H\bar{a}+B\bar{\beta}+F\bar{\gamma}),\quad w_5=\frac{1}{\kappa}(G\bar{a}+F\bar{\beta}+C\bar{\gamma}),$$

where

$$\bar{a},\ \bar{\beta},\ \bar{\gamma}=\frac{\sigma}{\Omega^{\frac{1}{2}}}\left\Vert \begin{matrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{matrix}\right\Vert ;$$

and

$$\sum A\bar{a}^2=1,\quad \sum au_5{}^2=\frac{\Omega}{\kappa^2}.$$

Now we have the relations, for a superficial direction p', q', r',

$$\begin{aligned} u_5p'+v_5q'+w_5r'&=0,\\ \theta_1p'+\theta_2q'+\theta_3r'&=0,\\ u_1p'+u_2q'+u_3r'&=1,\\ v_1p'+v_2q'+v_3r'&=\frac{1}{\rho};\end{aligned}$$

and therefore

$$\begin{vmatrix} 0, & u_5, & v_5, & w_5 \\ 0, & \theta_1, & \theta_2, & \theta_3 \\ 1, & u_1, & u_2, & u_3 \\ \frac{1}{\rho}, & v_1, & v_2, & v_3 \end{vmatrix}=0,$$

or

$$\frac{1}{\rho}\begin{vmatrix} u_5, & v_5, & w_5 \\ \theta_1, & \theta_2, & \theta_3 \\ u_1, & u_2, & u_3 \end{vmatrix}=\begin{vmatrix} u_5, & v_5, & w_5 \\ \theta_1, & \theta_2, & \theta_3 \\ v_1, & v_2, & v_3 \end{vmatrix}.$$

Again, from the first three relations, we have

$$p'\begin{vmatrix} u_5, & v_5, & w_5 \\ \theta_1, & \theta_2, & \theta_3 \\ u_1, & u_2, & u_3 \end{vmatrix}=v_5\theta_3-w_5\theta_2,$$

with corresponding values for q' and r'. If, then, ∇ denotes the coefficient of p' on the left-hand side, we have

$$\begin{aligned}\nabla^2=\sum A(p'\nabla)^2&=\sum A(v_5\theta_3-w_5\theta_2)^2\\ &=\frac{1}{\Omega}\{(\sum au_5{}^2)(\sum a\theta_1{}^2)-(\sum a\theta_1u_5)^2\}\\ &=\Omega\frac{\theta_n{}^2}{\kappa^2}-\frac{1}{\Omega}(\sum a\theta_1u_5)^2.\end{aligned}$$

Further,

$$\sum a\theta_1 u_5 = \frac{1}{\kappa}\sum a\theta_1 (A\bar{a} + H\bar{\beta} + G\bar{\gamma})$$

$$= \frac{\Omega}{\kappa}\frac{\sigma}{\Omega^{\frac{1}{2}}}\begin{vmatrix} \theta_1, & \theta_2, & \theta_3 \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix}$$

$$= \Omega\theta_n \frac{\sigma}{\kappa\tau_\theta};$$

and therefore

$$\nabla^2 = \Omega\frac{\theta_n{}^2}{\kappa^2}\left(1 - \frac{\sigma^2}{\tau_\theta{}^2}\right),$$

implying that $1/\tau_\theta$ numerically is smaller than $1/\sigma$.

We thus obtain a geometrical significance for the covariant

$$\begin{vmatrix} u_5, & v_5, & w_5 \\ \theta_1, & \theta_2, & \theta_3 \\ u_1, & u_2, & u_3 \end{vmatrix}$$

in the form

$$\Omega^{\frac{1}{2}}\frac{\theta_n}{\kappa}\left(1 - \frac{\sigma^2}{\tau_\theta{}^2}\right)^{\frac{1}{2}}.$$

and a geometrical significance for the covariant

$$\begin{vmatrix} u_5, & v_5, & w_5 \\ \theta_1, & \theta_2, & \theta_3 \\ v_1, & v_2, & v_3 \end{vmatrix}$$

in the form

$$\Omega^{\frac{1}{2}}\frac{\theta_n}{\kappa\rho}\left(1 - \frac{\sigma^2}{\tau_\theta{}^2}\right)^{\frac{1}{2}};$$

and incidentally, we have proved the relation

$$\sum a\theta_1 u_5 - \Omega\theta_n \frac{\sigma}{\kappa\tau_\theta}.$$

Regional frame of a superficial geodesic.

202. Before discussing the spatial curvatures of a superficial geodesic which are of higher grade than the circular curvature, it is necessary to bring the regional frame of that geodesic into relation with the spatial frame of the regional geodesic in the same direction. The tangent line, being the same for the two geodesics, is common to the two frames. The inclination of the two prime normals has been obtained (§ 195); and these two prime normals and the regional normal to the surface lie in one plane. Other lines to be associated with these lines are the regional binormal of the superficial geodesic, as well as the binormal and the trinormal of the regional geodesic.

Now the tangent flat of the region contains

(i) the binormal and (ii) the trinormal of the regional geodesic, with l_3 and l_4 respectively as typical direction-cosines:

(iii) the binormal of the superficial geodesic (this direction lying also in the tangent plane of the surface), with a typical direction-cosine λ_3 such that

$$\lambda_3 = y_1 P' + y_2 Q' + y_3 R' :$$

(iv) the regional normal to the surface, with $\frac{dy}{dn}$ as its typical direction-cosine.

All these lines are at right angles to the common tangent of the two geodesics, with y' as its typical direction-cosine, this tangent line also lying in the tangent flat of the region; hence the four lines lie in one and the same plane which is perpendicular to the tangent.

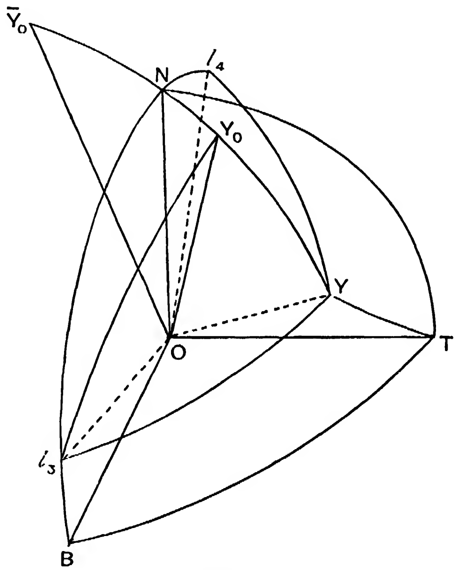

FIG. 21.

In the diagram, OT is the common tangent of the superficial geodesic and the regional geodesic; the lines OY_0 and OB are the prime normal and the regional binormal of the superficial geodesic, its regional trinormal being not yet determined; the lines OY, Ol_3, Ol_4, are respectively the prime normal, the binormal, and the trinormal of the regional geodesic; and ON is the regional normal to the surface. The tangent plane of the surface is TOB; the tangent flat of the region is TBl_3Nl_4T, the lines OB, Ol_3, ON, Ol_4, lying in a plane perpendicular to OT; and OY is orthogonal to the tangent flat of the region. The lines OY, OY_0, ON, lie in one plane; and in the continuation of this plane, there lies a line $O\bar{Y}_0$, at right angles to OY_0, so that $N\bar{Y}_0 = YY_0$.

The inclination of the prime normals has been denoted (§ 195) by ψ, so that

$$YOY_0 = \psi, \quad Y_0ON = \tfrac{1}{2}\pi - \psi ;$$

with the analytical relations

$$\sum YY_0 = \cos\psi, \quad \sum \frac{dy}{dn} Y_0 = \sin\psi,$$

$$\frac{\cos\psi}{\rho_0} = \frac{1}{\rho}, \quad \frac{\sin\psi}{\rho_0} = \frac{1}{\gamma},$$

$$\frac{Y_0}{\rho_0} = \frac{Y}{\rho} + \frac{1}{\gamma}\frac{dy}{dn}.$$

As regards the line $O\overline{Y}_0$, taken in the plane as drawn at right angles to OY_0, its typical direction-cosine $\overline{Y}_0$ is given by

$$\overline{Y}_0 = -Y\sin\psi + \frac{dy}{dn}\cos\psi,$$

while

$$Y_0 = \quad Y\cos\psi + \frac{dy}{dn}\sin\psi.$$

We denote by χ the inclination of the binormals of the two geodesics, so that

$$BOl_3 = \chi = NOl_4, \quad l_3ON = \tfrac{1}{2}\pi - \chi,$$

the lines Ol_3 and Ol_4 being the directions of the binormal and the trinormal of the regional geodesic. Also as YOl_3Nl_4Y is a flat, in which the three lines OY, Ol_3, Ol_4, are orthogonal to one another, we have

$$NYl_4 = \chi, \quad l_3YN = \tfrac{1}{2}\pi - \chi.$$

Then, with the usual formulæ of spherical trigonometry,

$$\sum \overline{Y}_0 l_3 = \cos \overline{Y}_0 Ol_3 = \cos \overline{Y}_0 N \cos Nl_3 = \sin\psi\sin\chi,$$
$$\sum \overline{Y}_0 l_4 = \cos \overline{Y}_0 Ol_4 = \cos \overline{Y}_0 N \cos Nl_4 = \sin\psi\cos\chi,$$

results which must be verified by any values of $\sin\chi$ and $\cos\chi$ that will be obtained.

Now we have

$$\begin{aligned}\cos\chi &= \sum l_3\lambda_3 \\ &= \sum (y_1l + y_2m + y_3n)(y_1P' + y_2Q' + y_3R'),\end{aligned}$$

where l, m, n, have the values obtained in § 172: that is,

$$\begin{aligned}\cos\chi &= \sum (Al + Hm + Gn)P' \\ &= \frac{\sigma}{\Omega^{\frac{1}{2}}\theta_n}\sum\left(\frac{u_1}{\rho} - v_1\right)\begin{vmatrix} u_2, & u_3 \\ \theta_2, & \theta_3\end{vmatrix} \\ &= \frac{\sigma}{\Omega^{\frac{1}{2}}\theta_n}\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ \theta_1, & \theta_2, & \theta_3\end{vmatrix} \\ &= \frac{\sigma}{\tau_\theta},\end{aligned}$$

with the former expression for the regional tilt.

Similarly, we find (for Nl_3)

$$\begin{aligned}\sin\chi &= \sum l_3\frac{dy}{dn} \\ &= \sum (y_1l + y_2m + y_3n)\left(y_1\frac{dp}{dn} + y_2\frac{dq}{dn} + y_3\frac{dr}{dn}\right)\end{aligned}$$

$$
\begin{aligned}
&-\sum\left(A\frac{dp}{dn}+H\frac{dq}{dn}+G\frac{dr}{dn}\right)l \\
&=\frac{1}{\theta_n}(\theta_1 l+\theta_2 m+\theta_3 n) \\
&\quad-\frac{\sigma}{\Omega\theta_n}\sum a v_1\theta_1,
\end{aligned}
$$

because $\theta_1 p'+\theta_2 q'+\theta_3 r'=0$, arising through the substituted values of l, m, n; and it is easy to verify that

$$\left\{\frac{1}{\Omega\theta_n}\sum a v_1\theta_1\right\}^2=\frac{1}{\sigma^2}-\frac{1}{\tau_0{}^2}.$$

Similarly for the other inclinations.

Ex. By means of the values of the direction-cosines, it is possible to verify the results, which are summarised for convenience:

$$
\left.\begin{aligned}
\textstyle\sum Y_0 Y&=\cos\psi\\
\textstyle\sum Y_0 l_3&=\sin\psi\sin\chi\\
\textstyle\sum Y_0 l_4&=\sin\psi\cos\chi
\end{aligned}\right\};\qquad
\left.\begin{aligned}
\textstyle\sum Y_0 Y&=-\sin\psi\\
\textstyle\sum \bar{Y}_0 l_3&=\cos\psi\sin\chi\\
\textstyle\sum \bar{Y}_0 l_4&=\cos\psi\cos\chi
\end{aligned}\right\};
$$

$$
\left.\begin{aligned}
\frac{dy}{dn}&=l_3\sin\chi+l_4\cos\chi\\
\lambda_3&=l_3\cos\chi-l_4\sin\chi\\
\textstyle\sum\lambda_3\frac{dy}{dn}&=0
\end{aligned}\right\};
$$

$$
\left.\begin{aligned}
\textstyle\sum l_3\frac{dy}{dn}&=\sin\chi\\
\textstyle\sum l_4\frac{dy}{dn}&=\cos\chi\\
\textstyle\sum Y\frac{dy}{dn}&=0\\
\textstyle\sum Y_0\frac{dy}{dn}&=\sin\psi\\
\textstyle\sum \bar{Y}_0\frac{dy}{dn}&=\cos\psi
\end{aligned}\right\};\qquad
\left.\begin{aligned}
\textstyle\sum l_3\lambda_3&=\cos\chi\\
\textstyle\sum l_4\lambda_3&=-\sin\chi\\
\textstyle\sum Y\lambda_3&=0\\
\textstyle\sum Y_0\lambda_3&=0\\
\textstyle\sum \bar{Y}_0\lambda_3&=0
\end{aligned}\right\}.
$$

Spatial torsion of a superficial geodesic.

203. We now proceed to find the spatial torsion of a superficial geodesic, its circular curvature being already known. The direction of that geodesic being the same as that of the regional geodesic which is its tangent, first-order differentiation with respect to the arc is the same for the two geodesics; but second-order (and

higher-order) arc-differentiations are not the same for them. As before, repeated differentiations along the arc of the superficial geodesic will be denoted by a subscript; and, in evaluating them, repeated use is made of the intrinsic equations of the superficial geodesics in the form

$$p_\theta'' - p'' = \frac{1}{\gamma}\frac{dp}{dn}, \qquad q_\theta'' - q'' = \frac{1}{\gamma}\frac{dq}{dn}, \qquad r_\theta'' - r'' = \frac{1}{\gamma}\frac{dr}{dn}.$$

We take various magnitudes in succession. It has been proved (§ 168) that

$$\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right) = 2v_1, \qquad \frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right) = 2v_2, \qquad \frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right) = 2v_3;$$

and therefore

$$\begin{aligned}\left(\frac{d}{ds_\theta} - \frac{d}{ds}\right)\frac{1}{\rho} &= 2\{v_1(p_\theta'' - p'') + v_2(q_\theta'' - q'') + v_3(r_\theta'' - r'')\}\\ &= \frac{2}{\gamma}\left(v_1\frac{dp}{dn} + v_2\frac{dq}{dn} + v_3\frac{dr}{dn}\right)\\ &= \frac{2}{\Omega\theta_n\gamma}\sum av_1\theta_1\\ &= \frac{2}{\gamma\sigma}\sin\chi,\end{aligned}$$

using the value of the covariant just obtained (§ 202).

Again, proceeding from the relation

$$\frac{Y}{\rho} = \sum \eta_{11}p'^2,$$

and using the former significance (§ 188) for η_1, η_2, η_3, given by

$$\eta_i = \eta_{i1}p' + \eta_{i2}q' + \eta_{i3}r',$$

for $i=1, 2, 3$, we have

$$\begin{aligned}\left(\frac{d}{ds_\theta} - \frac{d}{ds}\right)\frac{Y}{\rho} &= 2\{\eta_1(p_\theta'' - p'') + \eta_2(q_\theta'' - q'') + \eta_3(r_\theta'' - r'')\}\\ &= \frac{2}{\gamma}\left(\eta_1\frac{dp}{dn} + \eta_2\frac{dq}{dn} + \eta_3\frac{dr}{dn}\right);\end{aligned}$$

and therefore

$$\begin{aligned}\frac{1}{\rho}\left(\frac{dY}{ds_\theta} - \frac{dY}{ds}\right) &= \frac{2}{\gamma}\left\{(\eta_1 - Yv_1)\frac{dp}{dn} + (\eta_2 - Yv_2)\frac{dq}{dn} + (\eta_3 - Yv_3)\frac{dr}{dn}\right\}\\ &= 2\frac{l_5}{\gamma}\left(u_5\frac{dp}{dn} + v_5\frac{dq}{dn} + w_5\frac{dr}{dn}\right),\end{aligned}$$

by the formulæ (§ 184)

$$\eta_1 = Yv_1 + l_5 u_5, \quad \eta_2 = Yv_2 + l_5 v_5, \quad \eta_3 = Yv_3 + l_5 w_5.$$

Now, by § 182, we have

$$u_5 = \frac{1}{\kappa}(A\bar{a} + H\bar{\beta} + G\bar{\gamma}), \quad v_5 = \frac{1}{\kappa}(H\bar{a} + B\bar{\beta} + F\bar{\gamma}), \quad w_5 = \frac{1}{\kappa}(G\bar{a} + F\bar{\beta} + C\bar{\gamma}),$$

where

$$\bar{a}, \bar{\beta}, \bar{\gamma} = \frac{\sigma}{\Omega^{\frac{1}{2}}} \begin{Vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{Vmatrix};$$

and therefore

$$\begin{aligned} \frac{dY}{ds_\theta} - \frac{dY}{ds} &= \frac{2\rho}{\gamma\kappa} l_5 \sum (A\bar{a} + H\bar{\beta} + G\bar{\gamma}) \frac{dp}{dn} \\ &= \frac{2\rho}{\gamma\kappa\theta_n} l_5 (\bar{a}\theta_1 + \bar{\beta}\theta_2 + \bar{\gamma}\theta_3) \\ &= \frac{2\rho}{\gamma\kappa} l_5 \cos\chi. \end{aligned}$$

We thus have the values of

$$\frac{d}{ds_\theta}\left(\frac{1}{\rho}\right), \quad \frac{dY}{ds_\theta},$$

expressed in terms of magnitudes belonging to the regional geodesic in the same direction.

We require the arc-derivative of $\frac{dy}{dn}$ along the superficial geodesic. Because

$$\frac{dy}{dn} = y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn},$$

and because all the magnitudes on the right-hand side are free from the variables p', q', r', the first arc-derivative along the superficial geodesic is the same as the first arc-derivative along the regional geodesic: thus it is equal to $\frac{d}{ds}\left(\frac{dy}{dn}\right)$, and the value of this magnitude must be obtained. We have

$$\frac{d}{ds}\left(\frac{dy}{dn}\right) = y_1' \frac{dp}{dn} + y_2' \frac{dq}{dn} + y_3' \frac{dr}{dn} + y_1 \frac{d}{ds}\left(\frac{dp}{dn}\right) + y_2 \frac{d}{ds}\left(\frac{dq}{dn}\right) + y_3 \frac{d}{ds}\left(\frac{dr}{dn}\right).$$

Now

$$\begin{aligned} y_1' &= y_{11}p' + y_{12}q' + y_{13}r' \\ &= \eta_1 + y_1 a_1 + y_2 \xi_1 + y_3 \phi_1, \end{aligned}$$

with the notation of § 172 for a_1, ξ_1, ϕ_1; that is,

$$y_1' = Yv_1 + l_5 u_5 + y_1 a_1 + y_2 \xi_1 + y_3 \phi_1.$$

There are similar expressions for y_2' and for y_3'. When these are substituted, and the coefficients of y_1, y_2, y_3, are collected, we have a result of the form

$$\begin{aligned}\frac{d}{ds}\left(\frac{dy}{dn}\right)&=y_1U+y_2V+y_3W\\&\quad+Y\left(v_1\frac{dp}{dn}+v_2\frac{dq}{dn}+v_3\frac{dr}{dn}\right)\\&\quad+l_5\left(u_5\frac{dp}{dn}+v_5\frac{dq}{dn}+w_5\frac{dr}{dn}\right)\\&=y_1U+y_2V+y_3W-\frac{Y}{\sigma}\sin\chi+\frac{l_5}{\kappa}\cos\chi,\end{aligned}$$

where U, V, W, are definite magnitudes.

Instead of developing the actual expressions for U, V, W, we can obtain their values by using the results

$$\sum y_1\frac{d}{ds}\left(\frac{dy}{dn}\right)=\theta_1\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{\vartheta_1}{\theta_n},$$

$$\sum y_2\frac{d}{ds}\left(\frac{dy}{dn}\right)=\theta_2\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{\vartheta_2}{\theta_n},$$

$$\sum y_3\frac{d}{ds}\left(\frac{dy}{dn}\right)=\theta_3\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{\vartheta_3}{\theta_n},$$

obtained in evaluating $1/\sigma_0$ in § 200. In fact, we multiply by y_1 and add for all the space-coordinates: then by y_2 and add similarly: then by y_3 and add similarly. Because

$$\sum y_iY=0,\quad \sum y_il_5=0,$$

for $i=1, 2, 3$, we have successively

$$AU+HV+GW=\theta_1\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{\vartheta_1}{\theta_n},$$

$$HU+BV+FW=\theta_2\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{\vartheta_2}{\theta_n},$$

$$GU+FV+CW=\theta_3\frac{d}{ds}\left(\frac{1}{\theta_n}\right)+\frac{\vartheta_3}{\theta_n}.$$

Thus values of U, V, W, are obtained.

A further modification proves convenient. In § 178, we proved the relations

$$y_1=y'u_1+l_3\sigma\left(\frac{u_1}{\rho}-v_1\right)+l_4(A\bar{a}+H\bar{\beta}+G\bar{\gamma}),$$

$$y_2=y'u_2+l_3\sigma\left(\frac{u_2}{\rho}-v_2\right)+l_4(H\bar{a}+B\bar{\beta}+F\bar{\gamma}),$$

$$y_3=y'u_3+l_3\sigma\left(\frac{u_3}{\rho}-v_3\right)+l_4(G\bar{a}+F\bar{\beta}+C\bar{\gamma});$$

so we substitute these values of y_1, y_2, y_3, in the expression for $\frac{d}{ds}\left(\frac{dy}{dn}\right)$. The resulting coefficient of y'

$$\begin{aligned}&= Uu_1 + Vu_2 + Wu_3\\ &= \frac{1}{\Omega}\left\{\left(\sum a u_1 \theta_1\right)\frac{d}{ds}\left(\frac{1}{\theta_n}\right) + \frac{1}{\theta_n}\left(\sum a u_1 \vartheta_1\right)\right\},\end{aligned}$$

when the deduced values of U, V, W, are substituted: or, as

$$\sum a u_1 \theta_1 = \Omega(\theta_1 p' + \theta_2 q' + \theta_3 r') = 0,$$

$$\sum a u_1 \vartheta_1 = \Omega(\vartheta_1 p' + \vartheta_2 q' + \vartheta_3 r') = -\Omega\frac{\theta_n}{\gamma},$$

the resulting coefficient of y' is $-\frac{1}{\gamma}$. Thus we have an expression of the form

$$\frac{d}{ds}\left(\frac{dy}{dn}\right) = -\frac{y'}{\gamma} + l_3 D + l_4 E - \frac{Y}{\sigma}\sin\chi + \frac{l_5}{\kappa}\cos\chi.$$

Because

$$\sum l_3 \frac{dy}{dn} = \sin\chi,$$

we have

$$\begin{aligned}\sum\left\{l_3\frac{d}{ds}\left(\frac{dy}{dn}\right)\right\} - \frac{d\chi}{ds}\cos\chi &= -\sum\frac{dy}{dn}\frac{dl_3}{ds}\\ &= -\sum\frac{dy}{dn}\left(\frac{l_4}{\tau} - \frac{Y}{\sigma}\right)\\ &= -\frac{1}{\tau}\cos\chi,\end{aligned}$$

so that

$$D = \sum l_3 \frac{d}{ds}\left(\frac{dy}{dn}\right) = \left(\frac{d\chi}{ds} - \frac{1}{\tau}\right)\cos\chi.$$

Similarly, because

$$\sum l_4 \frac{dy}{dn} = \cos\chi,$$

we deduce

$$E = \sum l_4 \frac{d}{ds}\left(\frac{dy}{dn}\right) = -\left(\frac{d\chi}{ds} - \frac{1}{\tau}\right)\sin\chi.$$

Thus, finally, we have

$$\frac{d}{ds}\left(\frac{dy}{dn}\right) = -\frac{y'}{\gamma} + \left(\frac{d\chi}{ds} - \frac{1}{\tau}\right)(l_3\cos\chi - l_4\sin\chi) - \frac{Y}{\sigma}\sin\chi + \frac{l_5}{\kappa}\cos\chi.$$

We now can obtain an expression for the spatial torsion of a superficial geodesic. Let it be denoted by $1/\sigma_0$, and let the typical direction-cosine of the binormal be

denoted by $\bar{l}_3$; then, with the earlier notation for the magnitudes of the geodesic, the Frenet equation is

$$\frac{dY_0}{ds_0}=\frac{\bar{l}_3}{\sigma_0}-\frac{y'}{\rho_0}.$$

Now

$$Y_0=Y\cos\psi+\frac{dy}{dn}\sin\psi;$$

and we have had a typical direction-cosine $\overline{Y}_0$, representing a line at right angles to the superficial prime normal and lying in the plane through the prime normal of the regional geodesic and the regional normal to the surface, where

$$\overline{Y}_0=-Y\sin\psi+\frac{dy}{dn}\cos\psi.$$

Also,

$$\frac{d}{ds_0}\left(\frac{dy}{dn}\right)=\frac{d}{ds}\left(\frac{dy}{dn}\right);$$

consequently

$$\begin{aligned}\frac{dY_0}{ds_0}&=\overline{Y}_0\frac{d\psi}{ds_0}+\frac{dY}{ds_0}\cos\psi+\frac{d}{ds}\left(\frac{dy}{dn}\right)\sin\psi\\&=\overline{Y}_0\frac{d\psi}{ds_0}+\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}+2l_5\frac{\rho}{\gamma\kappa}\cos\chi\right)\cos\psi+\frac{d}{ds}\left(\frac{dy}{dn}\right)\sin\psi,\end{aligned}$$

on using the relation

$$\frac{dY}{ds_0}=\frac{dY}{ds}+2l_5\frac{\rho}{\gamma\kappa}\cos\chi.$$

Let the value of $\frac{d}{ds}\left(\frac{dy}{dn}\right)$ be inserted: the two terms on the right-hand side involving l_5 combine, because $\rho\cos\psi=\gamma\sin\psi$; hence

$$\begin{aligned}\frac{\bar{l}_3}{\sigma_0}-\frac{y'}{\rho_0}&=\frac{dY_0}{ds_0}\\&=\overline{Y}_0\frac{d\psi}{ds_0}+\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)\cos\psi+3\frac{l_5}{\kappa}\cos\chi\sin\psi\\&\quad+\left\{-\frac{y'}{\gamma}+\left(\frac{d\chi}{ds}-\frac{1}{\tau}\right)(l_3\cos\chi-l_4\sin\chi)-Y\frac{\sin\chi}{\sigma}\right\}\sin\psi.\end{aligned}$$

The direction of the binormal of the superficial geodesic, typified by $\bar{l}_3$, is at right angles to the tangent and to the prime normal of the superficial geodesic, with typical direction-cosines y' and Y_0 respectively. Two such lines, lying within the osculating block of the regional geodesic in the direction p', q', r', have typical direction-cosines $\overline{Y}_0$ and λ_3, the former being at right angles to the prime normal of the superficial geodesic, and the latter lying in the tangent plane of the surface at right angles to the tangent line of the two geodesics. The lines, of higher rank

in the orthogonal frame of the regional geodesic, can be combined with the lines having y', Y_0, $\overline{Y}_0$, λ_3, as typical direction-cosines, to constitute an orthogonal frame of reference for the plenary space; and we therefore, for the present purpose, can propound an expression

$$\bar{l}_3 = \epsilon \overline{Y}_0 + \eta\lambda_3 + \eta_5 l_5 + \eta_6 l_6 + \dots,$$

where the coefficients ϵ, η, η_5, η_6, ..., have to be determined, always subject to the permanent relation

$$1 = \sum \bar{l}_3^2 = \epsilon^2 + \eta^2 + \eta_5^2 + \sum_{i=6} \eta_i^2.$$

Let this propounded value of $\bar{l}_3$ be substituted in the developed Frenet equation for the spatial torsion of the superficial geodesic. That equation then becomes typical of the aggregate of such relations for the plenary homaloidal space, there being one such equation for each of the space-coordinates; and it is an equation which is linear and homogeneous in the typical direction-cosines of various organic lines. We consider the successive direction-cosines in turn, after the substitution is made.

The terms in the typical direction-cosine y' cancel, on account of the equation

$$\frac{1}{\rho_0} = \frac{\cos\psi}{\rho} + \frac{\sin\psi}{\gamma}.$$

Next, the terms in l_6, l_7, ..., occur only in the propounded expression for $\bar{l}_3$; hence multiplying the equation by l_6 and adding the products for the set, we have

$$\eta_6 = 0;$$

and similarly $\eta_7 = 0 = \eta_8 = \dots$, so that we now have

$$\bar{l}_3 = \epsilon \overline{Y}_0 + \eta\lambda_3 + \eta_5 l_5,$$

where $\epsilon^2 + \eta^2 + \eta_5^2 = 1$. The typical relation becomes

$$\begin{aligned} &\frac{1}{\sigma_0}(\epsilon \overline{Y}_0 + \eta\lambda_3 + \eta_5 l_5) \\ &\quad = \overline{Y}_0\left(\frac{d\psi}{ds_\theta} + \frac{\sin\chi}{\sigma}\right) + \lambda_3\left\{\left(\frac{d\chi}{ds_\theta} - \frac{1}{\tau}\right)\sin\psi + \frac{1}{\sigma}\cos\chi\cos\psi\right\} + 3\frac{l_5}{\kappa}\cos\chi\sin\psi, \end{aligned}$$

on using the relations

$$\lambda_3 = l_3\cos\chi - l_4\sin\chi, \quad l_3\cos\psi = (\overline{Y}_0 + Y\sin\psi)\sin\chi + \lambda_3\cos\chi\cos\psi.$$

Hence

$$\frac{\epsilon}{\sigma_0} = \frac{d\psi}{ds_\theta} + \frac{\sin\chi}{\sigma},$$

$$\frac{\eta}{\sigma_0} = \left(\frac{d\chi}{ds_\theta} - \frac{1}{\tau}\right)\sin\psi + \frac{1}{\sigma}\cos\chi\cos\psi,$$

$$\frac{\eta_5}{\sigma_0} = \frac{3}{\kappa}\cos\chi\sin\psi.$$

Manifestly the spatial binormal lies in a flat containing the lines with typical direction-cosines $\bar{Y}_0$, λ_3, l_5; and the magnitude of the spatial torsion of the superficial geodesic is obtained by substituting the foregoing values of ϵ, η, η_5, in the relation

$$\epsilon^2+\eta^2+\eta_5^2=1.$$

Spatial torsion of a superficial geodesic: another determination.

204. The spatial torsion of a geodesic on a regional surface can be discussed in a different manner.

When the surface is $\theta(p, q, r)=0$, we have

$$-r'=c_1 p'+c_2 q',$$

where

$$c_1\theta_3=\theta_1, \quad c_2\theta_3=\theta_2;$$

and therefore the superficial arc, being also an arc in the region, is given by

$$A_0 p'^2+2H_0 p'q'+B_0 q'^2=1,$$

where

$$A_0=A-2c_1G+c_1^2C, \quad H_0=H-c_1F-c_2G+c_1c_2C, \quad B_0=B-2c_2F+c_2^2C,$$

so that

$$V_0^2=A_0B_0-H_0^2=\frac{1}{\theta_3^2}\sum a\theta_1^2=\Omega\frac{\theta_n^2}{\theta_3^2}.$$

Again, the circular curvature of the superficial geodesic is given by the equation

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn}.$$

When p and q are maintained as the parameters of the surface, let

$$\frac{Y_0}{\rho_0}=\zeta_{11}p'^2+2\zeta_{12}p'q'+\zeta_{22}q'^2.$$

Now

$$\frac{Y}{\rho}=\sum\eta_{11}p'^2=\bar{\eta}_{11}p'^2+2\bar{\eta}_{12}p'q'+\bar{\eta}_{22}q'^2,$$

and

$$\frac{\theta_n}{\gamma}=\sum\vartheta_{11}p'^2=\bar{\theta}_{11}p'^2+2\bar{\theta}_{12}p'q'+\bar{\theta}_{22}q'^2,$$

where

$$\left.\begin{aligned}\bar{\eta}_{11}&=\eta_{11}-2c_1\eta_{13}+c_1^2\eta_{33}\\ \bar{\eta}_{12}&=\eta_{12}-c_1\eta_{23}-c_2\eta_{13}+c_1c_2\eta_{33}\\ \bar{\eta}_{22}&=\eta_{22}-2c_2\eta_{23}+c_2^2\eta_{33}\end{aligned}\right\}, \quad \left.\begin{aligned}\bar{\theta}_{11}&=\vartheta_{11}-2c_1\vartheta_{13}+c_1^2\vartheta_{33}\\ \bar{\theta}_{12}&=\vartheta_{12}-c_1\vartheta_{23}-c_2\vartheta_{31}+c_1c_2\vartheta_{33}\\ \bar{\theta}_{22}&=\vartheta_{22}-2c_2\vartheta_{23}+c_2^2\vartheta_{33}\end{aligned}\right\};$$

and therefore

$$\left.\begin{aligned}\zeta_{11}&=\bar{\eta}_{11}-\frac{1}{\theta_n}\bar{\theta}_{11}\frac{dy}{dn}\\ \zeta_{12}&=\bar{\eta}_{12}-\frac{1}{\theta_n}\bar{\theta}_{12}\frac{dy}{dn}\\ \zeta_{22}&=\bar{\eta}_{22}-\frac{1}{\theta_n}\bar{\theta}_{22}\frac{dy}{dn}\end{aligned}\right\}.$$

The secondary magnitudes $\bar{A}_0$, $\bar{H}_0$, $\bar{B}_0$, are such that

$$\bar{A}_0=\sum Y_0\zeta_{11},\quad \bar{H}_0=\sum Y_0\zeta_{12},\quad \bar{B}_0=\sum Y_0\zeta_{22};$$

and therefore

$$\frac{\bar{A}_0}{\rho_0}=\sum\frac{Y_0}{\rho_0}\zeta_{11}=\sum\left(\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn}\right)\left(\bar{\eta}_{11}-\frac{1}{\theta_n}\bar{\theta}_{11}\frac{dy}{dn}\right).$$

But, for all values of i, j, k, $=1, 2, 3$, we have

$$\sum\eta_{ij}y_k=0,\quad \sum Yy_k=0,$$

and therefore

$$\sum\frac{dy}{dn}Y=\sum\left\{\left(y_1\frac{dp}{dn}+y_2\frac{dq}{dn}+y_3\frac{dr}{dn}\right)Y\right\}=0,$$

$$\sum\bar{\eta}_{11}y_k=\sum\{(\eta_{11}-2c_1\eta_{13}+c_1^2\eta_{33})y_k\}=0;$$

also

$$\sum Y\bar{\eta}_{11}=\sum\{Y(\eta_{11}-2c_1\eta_{13}+c_1^2\eta_{33})\}=\bar{A}-2c_1\bar{G}+c_1^2\bar{C};$$

consequently

$$\frac{\bar{A}_0}{\rho_0}=\frac{1}{\rho}(\bar{A}-2c_1\bar{G}+c_1^2\bar{C})-\frac{1}{\gamma\theta_n}\bar{\theta}_{11}.$$

Similarly

$$\frac{\bar{H}_0}{\rho_0}=\frac{1}{\rho}(\bar{H}-c_1\bar{F}-c_2\bar{G}+c_1c_2\bar{C})-\frac{1}{\gamma\theta_n}\bar{\theta}_{12},$$

$$\frac{\bar{B}_0}{\rho_0}=\frac{1}{\rho}(\bar{B}-2c_2\bar{F}+c_2^2\bar{C})-\frac{1}{\gamma\theta_n}\bar{\theta}_{22}.$$

In accordance with the usual notation (§ 107), we write

$$\bar{v}_1=\bar{A}_0p'+\bar{H}_0q',\quad \bar{v}_2=\bar{H}_0p'+\bar{B}_0q',$$

$$\bar{u}_1=A_0p'+H_0q',\quad \bar{u}_2=H_0p'+B_0q';$$

also we wrote (§ 197)

$$\vartheta_1=\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r',$$

$$\vartheta_2=\vartheta_{12}p'+\vartheta_{22}q'+\vartheta_{23}r',$$

$$\vartheta_3=\vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r'.$$

Then we find

$$\bar{u}_1=u_1-c_1u_3,\quad \bar{u}_2=u_2-c_2u_3,$$

for the primary quantities; and, for the secondary quantities,

$$\left.\begin{aligned}\frac{\bar{v}_1}{\rho_0}&=\frac{1}{\rho}(v_1-c_1v_3)-\frac{1}{\gamma\theta_n}(\vartheta_1-c_1\vartheta_3)\\ \frac{\bar{v}_2}{\rho_0}&=\frac{1}{\rho}(v_2-c_2v_3)-\frac{1}{\gamma\theta_n}(\vartheta_2-c_2\vartheta_3)\end{aligned}\right\}.$$

The spatial torsion of a superficial geodesic is (§ 106) such that

$$\frac{V_0}{\sigma_0}=\begin{vmatrix}\bar{v}_1, & \bar{v}_2\\ \bar{u}_1, & \bar{u}_2\end{vmatrix};$$

and therefore

$$\frac{1}{\rho_0\sigma_0}\frac{\theta_n}{\theta_3}\Omega^{\frac{1}{2}}=\frac{1}{\rho}\begin{vmatrix}v_1-c_1v_3, & v_2-c_2v_3\\ u_1-c_1u_3, & u_2-c_2u_3\end{vmatrix}-\frac{1}{\gamma\theta_n}\begin{vmatrix}\vartheta_1-c_1\vartheta_3, & \vartheta_2-c_2\vartheta_3\\ u_1-c_1u_2, & u_2-c_2u_3\end{vmatrix}$$

$$=\frac{1}{\rho\theta_3}\begin{vmatrix}v_1, & v_2, & v_3\\ u_1, & u_2, & u_3\\ \theta_1, & \theta_2, & \theta_3\end{vmatrix}-\frac{1}{\gamma\theta_n\theta_3}\begin{vmatrix}\vartheta_1, & \vartheta_2, & \vartheta_3\\ u_1, & u_2, & u_3\\ \theta_1, & \theta_2, & \theta_3\end{vmatrix}$$

$$=-\Omega^{\frac{1}{2}}\frac{1}{\rho\tau_\theta}\frac{\theta_n}{\theta_3}+\Omega^{\frac{1}{2}}\frac{1}{\gamma\sigma_\theta}\frac{\theta_n}{\theta_3},$$

where $1/\sigma_\theta$ is the regional torsion (§ 200) of the superficial geodesic and $1/\tau_\theta$ is the regional tilt (§ 201). Hence we have the relation

$$\frac{1}{\rho_0\sigma_0}=\frac{1}{\gamma\sigma_\theta}-\frac{1}{\rho\tau_\theta}.$$

The result accords with the property of the curves of curvature on the surface which, always, are characterised by the vanishing torsion of their geodesic tangents, and, for a regional geodesic, are given (§ 197) by

$$\frac{1}{\gamma\sigma_\theta}-\frac{1}{\rho\tau_\theta}=0.$$

Also, the spatial direction-cosines of the binormal of the superficial geodesic are known. For the binormal is the line in the tangent plane of the surface drawn at right angles to the geodesic tangent; and therefore (§§ 95, 106), denoting a typical direction-cosine by λ_3, we have

$$V_0\lambda_3=\bar{u}_1\bar{y}_2-\bar{u}_2\bar{y}_1,$$

where

$$\bar{y}_1=y_1-c_1y_3,\quad \bar{y}_2=y_2-c_2y_3.$$

Consequently,

$$\lambda_3=-\frac{\theta_3}{\theta_n\Omega^{\frac{1}{2}}}\begin{vmatrix}u_1-c_1u_3, & u_2-c_2u_3\\ y_1-c_1y_3, & y_2-c_2y_3\end{vmatrix}$$

$$=-\frac{1}{\theta_n\Omega^{\frac{1}{2}}}\begin{vmatrix}u_1, & u_2, & u_3\\ y_1, & y_2, & y_3\\ \theta_1, & \theta_2, & \theta_3\end{vmatrix}.$$

But other deductions can be made from this expression. From the Frenet equations for the frame of the geodesic, we have

$$\frac{dY_0}{ds_0} = \frac{\lambda_3}{\sigma_0} - \frac{y'}{\rho_0};$$

and therefore

$$\frac{dY_0}{ds_0} = \frac{\Omega^{-\frac{1}{2}}}{\theta_n \sigma_0} \begin{vmatrix} u_1, & u_2, & u_3 \\ y_1, & y_2, & y_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix} - \frac{1}{\rho_0}(y_1 p' + y_2 q' + y_3 r'),$$

thus giving an expression for $\frac{dY_0}{ds_0}$ which is linear in the regional magnitudes along the superficial geodesic. Such an expression is entirely different in character from that given in § 203; and a comparison of the two forms leads to analytical inferences.

The foregoing expression for the spatial torsion of the superficial geodesic can be derived by means of the value of λ_3 and of the Frenet equation; for, from the latter, because

$$\sum \lambda_3 y' = 0,$$

the binormal and the tangent being at right angles, we have

$$\frac{1}{\sigma_0} = \sum \lambda_3 Y_0'$$

$$= \frac{1}{\theta_n \Omega^{\frac{1}{2}}} \begin{vmatrix} u_1, & \sum y_1 Y_0', & \theta_1 \\ u_2, & \sum y_2 Y_0', & \theta_2 \\ u_3, & \sum y_3 Y_0', & \theta_3 \end{vmatrix}.$$

Now we have

$$\sum y_1 Y_0 = \rho_0 \sum \left\{ y_1 \left(\frac{Y}{\rho} + \frac{1}{\gamma} \frac{dy}{dn} \right) \right\}$$

$$= \frac{\rho_0}{\gamma} \left(A \frac{dp}{dn} + H \frac{dq}{dn} + G \frac{dr}{dn} \right) = \frac{\rho_0}{\gamma} \frac{\theta_1}{\theta_n};$$

and therefore

$$\sum y_1 Y_0' = -\sum y_1' Y_0 + \frac{\rho_0}{\gamma \theta_n} \frac{d\theta_1}{ds_0} + \theta_1 \frac{d}{ds_0} \left(\frac{\rho_0}{\gamma \theta_n} \right).$$

But

$$\sum y_1' Y_0 = \sum Y_0 (y_{11} p' + y_{12} q' + y_{13} r')$$

$$= \frac{\rho_0}{\rho} \sum Y (y_{11} p' + y_{12} q' + y_{13} r') + \frac{\rho_0}{\gamma} \sum \frac{dy}{dn} (y_{11} p' + y_{12} q' + y_{13} r'),$$

and

$$\sum Y (y_{11} p' + y_{12} q' + y_{13} r') = v_1;$$

also

$$\sum \frac{dy}{dn} (y_{11} p' + y_{12} + y_{13} r')$$

$$= \frac{1}{\theta_n} [p' (\theta_1 \Gamma_{11} + \theta_2 \Delta_{11} + \theta_3 \Theta_{11}) + q' (\theta_1 \Gamma_{12} + \theta_2 \Delta_{12} + \theta_3 \Theta_{12}) + r' (\theta_1 \Gamma_{13} + \theta_2 \Delta_{13} + \theta_3 \Theta_{13})],$$

while

$$\frac{d\theta_1}{ds_0}=\theta_{11}p'+\theta_{12}q'+\theta_{13}r'.$$

Hence gathering the terms, we have

$$\sum y_1 Y_0'=-\frac{\rho_0}{\rho}v_1+\frac{\rho_0}{\gamma\theta_n}\vartheta_1+\theta_1\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma\theta_n}\right).$$

Similarly

$$\sum y_2 Y_0'=-\frac{\rho_0}{\rho}v_2+\frac{\rho_0}{\gamma\theta_n}\vartheta_2+\theta_2\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma\theta_n}\right).$$

$$\sum y_3 Y_0'=-\frac{\rho_0}{\rho}v_3+\frac{\rho_0}{\gamma\theta_n}\vartheta_3+\theta_3\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma\theta_n}\right).$$

Let these values be substituted in the foregoing determinantal expression for $1/\sigma_0$. The terms arising through the arc-derivatives of $\rho_0/\gamma\theta_n$ disappear; and we find

$$\frac{\theta_n\Omega^{\frac{1}{2}}}{\sigma_0}=-\frac{\rho_0}{\rho}\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}+\frac{\rho_0}{\gamma\theta_n}\begin{vmatrix} u_1, & u_2, & u_3 \\ \vartheta_1, & \vartheta_2, & \vartheta_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

leading as before, by using the results in §§ 201, 204, to the former equation

$$\frac{1}{\rho_0\sigma_0}=\frac{1}{\gamma\sigma_\theta}-\frac{1}{\rho\tau_\theta}.$$

With the former significance of the angle ψ as given (§ 203) by

$$Y_0=Y\cos\psi+\frac{dy}{dn}\sin\psi,\qquad \frac{\cos\psi}{\rho_0}=\frac{1}{\rho},\qquad \frac{\sin\psi}{\rho_0}=\frac{1}{\gamma},$$

this equation can be written

$$\frac{1}{\sigma_0}=\frac{\sin\psi}{\sigma_0}-\frac{\cos\psi}{\tau_0}.$$

Spatial tilt of a superficial geodesic.

205. We proceed similarly to determine the spatial tilt, and the direction of the trinormal, of a superficial geodesic.

Denoting the typical spatial direction-cosine of the trinormal by λ_4, and the magnitude of the spatial tilt by $1/\tau_0$, we have (§ 132)

$$\frac{\lambda_4}{\tau_0}-\frac{Y_0}{\sigma_0}=\frac{1}{V_0}(\bar{u}_1\zeta_2-\bar{u}_2\zeta_1),$$

$$\frac{\lambda_4}{\rho_0\tau_0}=\frac{1}{V_0}(\bar{v}_1\zeta_2-\bar{v}_2\zeta_1),$$

where

$$\zeta_1 = \zeta_{11}p' + \zeta_{12}q' = (\eta_1 - c_1\eta_3) - \frac{1}{\theta_n}(\vartheta_1 - c_1\vartheta_3)\frac{dy}{dn},$$

$$\zeta_2 = \zeta_{12}p' + \zeta_{22}q' = (\eta_2 - c_2\eta_3) - \frac{1}{\theta_n}(\vartheta_2 - c_2\vartheta_3)\frac{dy}{dn}.$$

The two formulæ for λ_4 are equivalent to one another. In the first of them, let these values of ζ_1 and ζ_2, as well as the values

$$\bar{u}_1 = u_1 - c_1 u_3, \quad \bar{u}_2 = u_2 - c_2 u_3,$$

be substituted; then

$$Y_0\left(\frac{\lambda_4}{\tau_0} - \frac{Y_0}{\sigma_0}\right) = \begin{vmatrix} u_1 - c_1u_3, & u_2 - c_2u_3 \\ \eta_1 - c_1\eta_3, & \eta_2 - c_2\eta_3 \end{vmatrix} - \frac{1}{\theta_n}\frac{dy}{dn}\begin{vmatrix} u_1 - c_1u_3, & u_2 - c_2u_3 \\ \vartheta_1 - c_1\vartheta_3, & \vartheta_2 - c_2\vartheta_3 \end{vmatrix}$$

$$= \frac{1}{\theta_3}\begin{vmatrix} u_1, & u_2, & u_3 \\ \eta_1, & \eta_2, & \eta_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix} - \frac{1}{\theta_3\theta_n}\frac{dy}{dn}\begin{vmatrix} u_1, & u_2, & u_3 \\ \vartheta_1, & \vartheta_2, & \vartheta_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

so that, with the value of $1/\sigma_\theta$ in § 204, we have

$$\frac{\lambda_4}{\tau_0} - \frac{Y_0}{\sigma_0} + \frac{1}{\sigma_\theta}\frac{dy}{dn} = \frac{1}{\Omega^{\frac{1}{2}}\theta_n}\begin{vmatrix} u_1, & u_2, & u_3 \\ \eta_1, & \eta_2, & \eta_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}.$$

Now (§ 188)

$$\eta_1 = Yv_1 + l_5u_5, \quad \eta_2 = Yv_2 + l_5v_5, \quad \eta_3 = Yv_3 + l_5w_5;$$

on the substitution of these values in the determinant, the coefficient of Y

$$= \frac{1}{\Omega^{\frac{1}{2}}\theta_n}\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix} = \frac{1}{\tau_\theta},$$

by § 201; and so

$$\frac{\lambda_4}{\tau_0} - \frac{Y_0}{\sigma_0} + \frac{1}{\sigma_\theta}\frac{dy}{dn} - \frac{Y}{\tau_\theta} = \frac{l_5}{\Omega^{\frac{1}{2}}\theta_n}\begin{vmatrix} u_1, & u_2, & u_3 \\ u_5, & v_5, & w_5 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}.$$

But (§ 189)

$$\frac{u_5}{\rho} = \bar{\mathfrak{c}}^{\frac{1}{2}}q' + \bar{\mathfrak{b}}^{\frac{1}{2}}r', \quad \frac{v_5}{\rho} = -\bar{\mathfrak{c}}^{\frac{1}{2}}p' + \bar{\mathfrak{a}}^{\frac{1}{2}}r', \quad \frac{w_5}{\rho} = -\bar{\mathfrak{b}}^{\frac{1}{2}}p' - \bar{\mathfrak{a}}^{\frac{1}{2}}q',$$

with the significance of the symbols $\bar{\mathfrak{a}}$, $\bar{\mathfrak{b}}$, $\bar{\mathfrak{c}}$, (and other connected symbols), as

given in § 188. When these values are inserted in the determinant, the coefficient of $\bar{\mathfrak{a}}^{\frac{1}{2}}\rho$

$$=(\theta_3 u_1-\theta_1 u_3)r'-(\theta_1 u_2-\theta_2 u_1)q'$$
$$=-\theta_1(u_3 r'+u_2 q')+u_1(\theta_3 r'+\theta_2 q')=-\theta_1,$$

because of the relations $\theta_1 p'+\theta_2 q'+\theta_3 r'=0$, $u_1 p'+u_2 q'+u_3 r'=1$; and similarly the coefficient of $\bar{\mathfrak{b}}^{\frac{1}{2}}\rho$

$$=(\theta_2 u_3-\theta_3 u_2)r'-(\theta_1 u_2-\theta_2 u_1)p'=\quad\theta_2,$$

and the coefficient of $\bar{\mathfrak{c}}^{\frac{1}{2}}\rho$

$$-(\theta_2 u_3-\theta_3 u_2)q'-(\theta_3 u_1-\theta_1 u_3)p'=-\theta_3.$$

We therefore have

$$\frac{\lambda_4}{\tau_0}-\frac{Y_0}{\sigma_0}+\frac{1}{\sigma_\theta}\frac{dy}{dn}-\frac{Y}{\tau_\theta}=-\frac{l_5\rho}{\Omega^{\frac{1}{2}}\theta_n}(\bar{\mathfrak{a}}^{\frac{1}{2}}\theta_1-\bar{\mathfrak{b}}^{\frac{1}{2}}\theta_2+\bar{\mathfrak{c}}^{\frac{1}{2}}\theta_3),$$

the typical equation for the direction-cosines of the trinormal of the superficial geodesic, expressed in terms of variables of the regional directions associated with the surface and with the tangent regional geodesic. Of the direction-cosines involved, Y and l_5 belong to the tangent regional geodesic, both being at right angles to the tangent flat of the region at the point, and their two directions determine a plane orthogonal to the regional and superficial geodesic; while Y_0 is linearly expressible in terms of Y and $\frac{dy}{dn}$, the typical spatial direction-cosine of the regional normal to the surface, by the relation

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn}.$$

In the first place, multiply the equation throughout by Y_0 and add the products. We have

$$\sum Y_0\lambda_4=0,$$

from the necessary orthogonality of the prime normal and the trinormal of the superficial geodesic: also

$$\sum YY_0=\cos\psi,\quad \sum\frac{dy}{dn}Y_0=\sin\psi,$$

with the former significance of ψ: and

$$\sum l_5 Y_0=\frac{\rho_0}{\rho}\sum l_5 Y+\frac{1}{\gamma}\sum l_5\frac{dy}{dn}=0,$$

the first term vanishing from the necessary orthogonality of the prime normal and the quartinormal of the regional geodesic, and the second term vanishing

because the quartinormal is at right angles to every direction in the tangent flat of the region. Hence

$$\frac{1}{\sigma_0}+\frac{1}{\sigma_\theta}\sin\psi-\frac{1}{\tau_\theta}\cos\psi=0,$$

in accordance with the preceding result (p. 41).

In the next place, we take the direction-equation in the form

$$\frac{\lambda_4}{\tau_0}-\frac{Y_0}{\sigma_0}=\frac{Y}{\tau_\theta}-\frac{1}{\sigma_\theta}\frac{dy}{dn}-\frac{l_5\rho}{\Omega^{\frac{1}{2}}\theta_n}(\bar{\mathfrak{a}}^{\frac{1}{2}}\theta_1-\bar{\mathfrak{b}}^{\frac{1}{2}}\theta_2+\bar{\mathfrak{c}}^{\frac{1}{2}}\theta_3),$$

where the directions typified by λ_4 and Y_0 are perpendicular, and the directions typified by $\frac{dy}{dn}$, Y, l_5, are perpendicular in pairs; we square the equation, and add the squares of the sides for all dimensions of the plenary space. Then

$$\frac{1}{\tau_0{}^2}+\frac{1}{\sigma_0{}^2}=\frac{1}{\tau_\theta{}^2}+\frac{1}{\sigma_\theta{}^2}+\frac{\rho^2}{\Omega\theta_n{}^2}(\bar{\mathfrak{a}}^{\frac{1}{2}}\theta_1-\bar{\mathfrak{b}}^{\frac{1}{2}}\theta_2+\bar{\mathfrak{c}}^{\frac{1}{2}}\theta_3)^2.$$

In evaluating the last square on the right-hand side, we use the relations (§ 188)

$$\bar{\mathfrak{f}}=-(\bar{\mathfrak{b}}\bar{\mathfrak{c}})^{\frac{1}{2}},\quad \bar{\mathfrak{g}}=-(\bar{\mathfrak{c}}\bar{\mathfrak{a}})^{\frac{1}{2}},\quad \bar{\mathfrak{h}}=-(\bar{\mathfrak{a}}\bar{\mathfrak{b}})^{\frac{1}{2}};$$

and therefore

$$\frac{1}{\tau_0{}^2}+\frac{1}{\sigma_0{}^2}=\frac{1}{\tau_\theta{}^2}+\frac{1}{\sigma_\theta{}^2}+\frac{\rho^2}{\Omega\theta_n{}^2}Y,$$

where

$$Y=\bar{\mathfrak{a}}\theta_1{}^2+2\bar{\mathfrak{h}}\theta_1\theta_2+2\bar{\mathfrak{g}}\theta_1\theta_3+\bar{\mathfrak{b}}\theta_2{}^2+2\bar{\mathfrak{f}}\theta_2\theta_3+\bar{\mathfrak{c}}\theta_3{}^2.$$

But

$$\frac{1}{\tau_\theta{}^2}+\frac{1}{\sigma_\theta{}^2}-\frac{1}{\sigma_0{}^2}=\frac{1}{\tau_\theta{}^2}+\frac{1}{\sigma_\theta{}^2}-\left(\frac{\sin\psi}{\sigma_\theta}-\frac{\cos\psi}{\tau_\theta}\right)^2$$

$$=\left(\frac{\sin\psi}{\tau_\theta}+\frac{\cos\psi}{\sigma_\theta}\right)^2;$$

and therefore

$$\frac{1}{\tau_0{}^2}=\left(\frac{\sin\psi}{\tau_\theta}+\frac{\cos\psi}{\sigma_\theta}\right)^2+\frac{\rho^2}{\Omega\theta_n{}^2}Y.$$

We note that, when the plenary space of the region is quadruple, the quantities $\bar{\mathfrak{a}}$, $\bar{\mathfrak{b}}$, $\bar{\mathfrak{c}}$, $\bar{\mathfrak{f}}$, $\bar{\mathfrak{g}}$, $\bar{\mathfrak{h}}$, all vanish: for example, because (§ 183) the quantities u_5, v_5, w_5, then vanish,

$$\bar{\mathfrak{a}}=\mathfrak{bc}-\mathfrak{f}^2=v_2{}^2v_3{}^2-(v_2v_3)^2=0$$

from the values in § 188. Thus, when the region exists in a quadruple space, we have

$$\frac{1}{\tau_0}=\frac{\sin\psi}{\tau_\theta}+\frac{\cos\psi}{\sigma_\theta},$$

$$\frac{1}{\sigma_0}=\frac{\sin\psi}{\sigma_\theta}-\frac{\cos\psi}{\tau_\theta},$$

in accordance with the known results for a parametric surface in such a region*.

* *G.F.D.*, vol. ii, chapter xxi.

Ex. 1. Obtain the foregoing equation for the tilt of the superficial geodesic by proceeding from the equation

$$\frac{\lambda_4}{\rho_0\tau_0}=\frac{1}{V_0}(\bar{v}_1\zeta_2-\bar{v}_2\zeta_1).$$

The equivalence of this equation to the equation

$$\frac{\lambda_4}{\tau_0}-\frac{Y_0}{\sigma_0}=\frac{1}{V_0}(\bar{u}_1\zeta_2-\bar{u}_2\zeta_1),$$

is established, by the help of the equation

$$\frac{Y_0}{\rho_0}=\zeta_1 p'+\zeta_2 q'\,;$$

for the elimination of ζ_1 and ζ_2 leads to the condition

$$\begin{vmatrix} \dfrac{\lambda_4}{\rho_0\tau_0}, & \dfrac{\lambda_4}{\tau_0}-\dfrac{Y_0}{\sigma_0}, & \dfrac{1}{V_0}\dfrac{Y_0}{\rho_0} \\ \bar{v}_1, & \bar{u}_1, & q' \\ \bar{v}_2, & \bar{u}_2, & -p' \end{vmatrix}=0$$

which is an identity under the equation

$$\frac{V_0}{\sigma_0}=\bar{v}_1\bar{u}_2-\bar{v}_2\bar{u}_1.$$

Ex. 2. Evaluate the covariantive magnitudes

$$\begin{vmatrix} v_1, & v_2, & v_3 \\ u_5, & v_5, & w_5 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}, \qquad \begin{vmatrix} \vartheta_1, & \vartheta_2, & \vartheta_3 \\ u_5, & v_5, & w_5 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

connected with the parametric θ-surface; and prove that

$$-\Omega^{\frac{1}{2}}\theta_n^{\,2}\left(\frac{1}{\rho\sigma_\theta}+\frac{1}{\gamma\tau_\theta}\right)=\begin{vmatrix} \vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r', & v_1, & \theta_1 \\ \vartheta_{12}p'+\vartheta_{22}q'+\vartheta_{23}r', & v_2, & \theta_2 \\ \vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r', & v_3, & \theta_3 \end{vmatrix}.$$

The determinant on the right-hand side, with its value, should be compared with the similar determinant in § 200, which has u_1, u_2, u_3, for its second column, and is equal to $-\Omega^{\frac{1}{2}}\theta_n^{\,2}/\sigma_\theta$.

Geometrical construction for the trinormal of a superficial geodesic.

206. Returning to the equation of the typical direction-cosine of the trinormal of the superficial geodesic, which may be written in the form

$$\frac{\lambda_1}{\tau_0}=\frac{1}{\tau_\theta}Y+\frac{1}{\sigma_0}Y_0-\frac{1}{\sigma_\theta}\frac{dy}{dn}-\left(\frac{Y\rho^2}{\Omega\theta_n^{\,2}}\right)^{\frac{1}{2}}l_5,$$

we substitute the value of Y_0 from § 195, that is,

$$Y_0 = Y\cos\psi + \frac{dy}{dn}\sin\psi.$$

On the right-hand side, the coefficient of Y

$$= \left(\frac{\cos\psi}{\sigma_\theta} + \frac{\sin\psi}{\tau_\theta}\right)\sin\psi,$$

and the coefficient of $\frac{dy}{dn}$

$$= -\left(\frac{\cos\psi}{\sigma_\theta} + \frac{\sin\psi}{\tau_\theta}\right)\cos\psi;$$

and therefore, if we write

$$\frac{1}{\mu} = \frac{\cos\psi}{\sigma_\theta} + \frac{\sin\psi}{\tau_\theta},$$

we have

$$\frac{\lambda_4}{\tau_0} = \frac{1}{\mu}\left(Y\sin\psi - \frac{dy}{dn}\cos\psi\right) - \left(\frac{Y\rho^2}{\Omega\theta_n^{\,2}}\right)^{\frac{1}{2}} l_5,$$

$$= -\frac{1}{\mu}\overline{Y}_0 - \left(\frac{Y\rho^2}{\Omega\theta_n^{\,2}}\right)^{\frac{1}{2}} l_5,$$

where $\overline{Y}_0$ is the magnitude in § 202, and is the typical direction-cosine of the direction OY_0 in Fig. 21 which lies in the plane, drawn through the prime normal of the tangent regional geodesic and the regional normal to the surface, and is at right angles to the prime normal to the superficial geodesic in that plane. The trinormal of the superficial geodesic therefore lies in the plane, through this direction typified by $\overline{Y}_0$ and through the quartinormal of the tangent regional geodesic; and if, in that plane, it makes an angle $\frac{1}{2}\pi - \omega$ with the quartinormal, we have

$$-\lambda_4 = \overline{Y}_0\cos\omega + l_5\sin\omega,$$

where

$$\frac{\cos\omega}{\tau_0} = \frac{\cos\psi}{\sigma_\theta} + \frac{\sin\psi}{\tau_\theta}, \quad \frac{\sin\omega}{\tau_0} = \left(\frac{Y\rho^2}{\Omega\theta_n^{\,2}}\right)^{\frac{1}{2}}.$$

As the covariantive expressions for $1/\sigma_\theta$, $1/\tau_\theta$, Y, are known, the magnitude and the direction of the radius of tilt of the superficial geodesic are thus obtained.

CHAPTER XVII

Surfaces Geodesic to a Region

Variations of the normal dilatation of a regional surface.

207. We have seen that the normal dilatation of a regional surface, which has been denoted by θ_n, is given by the equation

$$\Omega\theta_n^2 = \sum a\theta_1^2;$$

and that the direction-variables $\frac{dp}{dn}, \frac{dq}{dn}, \frac{dr}{dn}$, of the regional normal to the surface, that is, of the direction of the radius of regional flexure, are given by equations

$$\Omega\theta_n \frac{dp}{dn} = a\theta_1 + h\theta_2 + g\theta_3, \quad \Omega\theta_n \frac{dq}{dn} = h\theta_1 + b\theta_2 + f\theta_3, \quad \Omega\theta_n \frac{dr}{dn} = g\theta_1 + f\theta_2 + c\theta_3.$$

We need the variations of θ_n in various directions in the region, especially those which are normal to the surface or are tangential to the surface. We need also the variations of the direction-variables of the regional normal.

We begin by obtaining the parametric variations of θ_n. Differentiating the relation

$$\theta_n^2 = \sum \frac{a}{\Omega}\theta_1^2,$$

with respect to p, we have

$$\theta_n \frac{\partial\theta_n}{\partial p} = \frac{1}{\Omega}\{(a\theta_1 + h\theta_2 + g\theta_3)\theta_{11} + (h\theta_1 + b\theta_2 + f\theta_3)\theta_{12} + (g\theta_1 + f\theta_2 + c\theta_3)\theta_{13}\}$$
$$+ \tfrac{1}{2}\sum \theta_1^2 \frac{\partial}{\partial p}\left(\frac{a}{\Omega}\right).$$

By the results in § 160, the last line

$$\begin{aligned}
= -\frac{1}{\Omega}[&\theta_1^2(a\Gamma_{11} + h\Gamma_{12} + g\Gamma_{13}) + \theta_2^2(h\Delta_{11} + b\Delta_{12} + f\Delta_{13}) + \theta_3^2(g\Theta_{11} + f\Theta_{12} + c\Theta_{13}) \\
&+ \theta_2\theta_3\{(g\Delta_{11} + f\Delta_{12} + c\Delta_{13}) + (h\Theta_{11} + b\Theta_{12} + f\Theta_{13})\} \\
&+ \theta_3\theta_1\{(a\Theta_{11} + h\Theta_{12} + g\Theta_{13}) + (g\Gamma_{11} + f\Gamma_{12} + c\Gamma_{13})\} \\
&+ \theta_1\theta_2\{(h\Gamma_{11} + b\Gamma_{12} + f\Gamma_{13}) + (a\Delta_{11} + h\Delta_{12} + g\Delta_{13})\}] \\
= -\frac{1}{\Omega}\{&(a\theta_1 + h\theta_2 + g\theta_3)(\theta_1\Gamma_{11} + \theta_2\Delta_{11} + \theta_3\Theta_{11}) \\
&+ (h\theta_1 + b\theta_2 + f\theta_3)(\theta_1\Gamma_{12} + \theta_2\Delta_{12} + \theta_3\Theta_{12}) \\
&+ (g\theta_1 + f\theta_2 + c\theta_3)(\theta_1\Gamma_{13} + \theta_2\Delta_{13} + \theta_3\Theta_{13});
\end{aligned}$$

and therefore

$$\Omega\theta_n \frac{\partial\theta_n}{\partial p} = (a\theta_1 + h\theta_2 + g\theta_3)\vartheta_{11} + (h\theta_1 + b\theta_2 + f\theta_3)\vartheta_{12} + (g\theta_1 + f\theta_2 + c\theta_3)\vartheta_{13}.$$

Proceeding in the same way for the other two parametric derivatives of θ_n, we find

$$\Omega\theta_n \frac{\partial\theta_n}{\partial q} = (a\theta_1 + h\theta_2 + g\theta_3)\,\vartheta_{21} + (h\theta_1 + b\theta_2 + f\theta_3)\,\vartheta_{22} + (g\theta_1 + f\theta_2 + c\theta_3)\,\vartheta_{23},$$

$$\Omega\theta_n \frac{\partial\theta_n}{\partial r} = (a\theta_1 + h\theta_2 + g\theta_3)\,\vartheta_{31} + (h\theta_1 + b\theta_2 + f\theta_3)\,\vartheta_{32} + (g\theta_1 + f\theta_2 + c\theta_3)\,\vartheta_{33};$$

and these three results may also be written in the forms

$$\left.\begin{aligned} \frac{\partial\theta_n}{\partial p} &= \vartheta_{11}\frac{dp}{dn} + \vartheta_{12}\frac{dq}{dn} + \vartheta_{13}\frac{dr}{dn} \\ \frac{\partial\theta_n}{\partial q} &= \vartheta_{21}\frac{dp}{dn} + \vartheta_{22}\frac{dq}{dn} + \vartheta_{23}\frac{dr}{dn} \\ \frac{\partial\theta_n}{\partial r} &= \vartheta_{31}\frac{dp}{dn} + \vartheta_{32}\frac{dq}{dn} + \vartheta_{33}\frac{dr}{dn} \end{aligned}\right\}.$$

Hence using θ_{nn} to denote $\frac{d\theta_n}{dn}$, the second normal derivative of the parametric magnitude θ within the region, we have

$$\begin{aligned} \theta_{nn} &= \frac{\partial\theta_n}{\partial p}\frac{dp}{dn} + \frac{\partial\theta_n}{\partial q}\frac{dq}{dn} + \frac{\partial\theta_n}{\partial r}\frac{dr}{dn} \\ &= \left(\vartheta_{11},\ \vartheta_{22},\ \vartheta_{33},\ \vartheta_{23},\ \vartheta_{31},\ \vartheta_{12}\right\rangle\!\!\left\langle \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn}\right)^2, \end{aligned}$$

and the equivalent form

$$\begin{aligned} &\Omega^2\theta_n{}^2\theta_{nn} \\ &= (\vartheta_{11},\ \vartheta_{22},\ \vartheta_{33},\ \vartheta_{23},\ \vartheta_{31},\ \vartheta_{12}\rangle\!\langle a\theta_1 + h\theta_2 + g\theta_3,\ h\theta_1 + b\theta_2 + f\theta_3,\ g\theta_1 + f\theta_2 + c\theta_3)^2, \end{aligned}$$

as covariantive expressions for θ_{nn}.

Similarly, for the variation of θ_n for any direction p', q', r', in the region, whether the direction lies on or off the surface at O, we have

$$\begin{aligned} \frac{d\theta_n}{ds} &= \frac{\partial\theta_n}{\partial p}p' + \frac{\partial\theta_n}{\partial q}q' + \frac{\partial\theta_n}{\partial r}r' \\ &= \left(\vartheta_{11},\ \vartheta_{22},\ \vartheta_{33},\ \vartheta_{23},\ \vartheta_{31},\ \vartheta_{21}\right\rangle\!\!\left\langle \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn}\right\rangle\!\!\left\langle p',\ q',\ r'\right) \end{aligned}$$

as a covariantive expression; and an equivalent form is

$$\begin{aligned} \Omega\theta_n\frac{d\theta_n}{ds} &= (\vartheta_{ij}\rangle\!\langle p',\ q',\ r'\rangle\!\langle a\theta_1 + h\theta_2 + g\theta_3,\ h\theta_1 + f\theta_2 + f\theta_3,\ g\theta_1 + f\theta_2 + c\theta_3) \\ &= (\vartheta_{11}p' + \vartheta_{12}q' + \vartheta_{13}r')(a\theta_1 + h\theta_2 + g\theta_3) \\ &\quad + (\vartheta_{12}p' + \vartheta_{22}q' + \vartheta_{23}r')(h\theta_1 + b\theta_2 + f\theta_3) \\ &\quad\quad + (\vartheta_{13}p' + \vartheta_{23}q' + \vartheta_{33}r')(g\theta_1 + f\theta_2 + c\theta_3) = \sum a\theta_1\vartheta_1. \end{aligned}$$

One inference may be noted: the operators $\frac{d}{ds}$ and $\frac{d}{dn}$ are not interchangeable.

Thus $\frac{d\theta}{ds}$ on a surface always vanishes, and therefore $\frac{d}{dn}\left(\frac{d\theta}{ds}\right)$ always vanishes, whereas $\frac{d\theta_n}{ds}$ usually does not vanish. It will, however, be found (p. 62) that $\frac{d\theta_n}{ds}$ does vanish for a class of parametric surfaces which have second-order contact with the geodesic surface in the same regional orientation.

Next, we have

$$\theta_n \frac{dp}{dn} = \frac{1}{\Omega}(a\theta_1 + h\theta_2 + g\theta_3),$$

and therefore

$$\frac{\partial}{\partial p}\left(\theta_n \frac{dp}{dn}\right) = \frac{1}{\Omega}(a\theta_{11} + h\theta_{12} + g\theta_{13}) + \theta_1 \frac{\partial}{\partial p}\left(\frac{a}{\Omega}\right) + \theta_2 \frac{\partial}{\partial p}\left(\frac{h}{\Omega}\right) + \theta_3 \frac{\partial}{\partial p}\left(\frac{g}{\Omega}\right)$$

$$= \frac{1}{\Omega}(a\vartheta_{11} + h\vartheta_{12} + g\vartheta_{13}) - \theta_n\left(\Gamma_{11}\frac{dp}{dn} + \Gamma_{12}\frac{dq}{dn} + \Gamma_{13}\frac{dr}{dn}\right),$$

after reduction similar to that used for the parametric derivatives of θ_n; also

$$\frac{\partial}{\partial q}\left(\theta_n \frac{dp}{dn}\right) = \frac{1}{\Omega}(a\vartheta_{21} + h\vartheta_{22} + g\vartheta_{23}) - \theta_n\left(\Gamma_{21}\frac{dp}{dn} + \Gamma_{22}\frac{dq}{dn} + \Gamma_{23}\frac{dr}{dn}\right),$$

$$\frac{\partial}{\partial r}\left(\theta_n \frac{dp}{dn}\right) = \frac{1}{\Omega}(a\vartheta_{31} + h\vartheta_{32} + g\vartheta_{33}) - \theta_n\left(\Gamma_{31}\frac{dp}{dn} + \Gamma_{32}\frac{dq}{dn} + \Gamma_{33}\frac{dr}{dn}\right).$$

Consequently

$$\begin{aligned}\frac{d}{dn}\left(\theta_n \frac{dp}{dn}\right) = \frac{1}{\Omega}\Big\{ & a\left(\vartheta_{11}\frac{dp}{dn} + \vartheta_{12}\frac{dq}{dn} + \vartheta_{13}\frac{dr}{dn}\right) \\ & + h\left(\vartheta_{12}\frac{dp}{dn} + \vartheta_{22}\frac{dq}{dn} + \vartheta_{23}\frac{dr}{dn}\right) \\ & + g\left(\vartheta_{13}\frac{dp}{dn} + \vartheta_{23}\frac{dq}{dn} + \vartheta_{33}\frac{dr}{dn}\right)\Big\} \\ & - \theta_n\left(\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{23}, \Gamma_{31}, \Gamma_{12} \between \frac{dp}{dn}, \frac{dq}{dn}, \frac{dr}{dn}\right)^2,\end{aligned}$$

so that

$$\begin{aligned}\frac{d^2p}{dn^2} + & \left(\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{23}, \Gamma_{31}, \Gamma_{12} \between \frac{dp}{dn}, \frac{dq}{dn}, \frac{dr}{dn}\right)^2 + \frac{\theta_{nn}}{\theta_n}\frac{dp}{dn} \\ = \frac{1}{\Omega\theta_n}\Big[& a\left(\vartheta_{11}\frac{dp}{dn} + \vartheta_{12}\frac{dq}{dn} + \vartheta_{13}\frac{dr}{dn}\right) \\ & + h\left(\vartheta_{21}\frac{dp}{dn} + \vartheta_{22}\frac{dq}{dn} + \vartheta_{23}\frac{dr}{dn}\right) \\ & + g\left(\vartheta_{31}\frac{dp}{dn} + \vartheta_{32}\frac{dq}{dn} + \vartheta_{33}\frac{dr}{dn}\right)\Big] \\ = \frac{1}{\Omega\theta_n} & (a\bar{\vartheta}_1 + h\bar{\vartheta}_2 + g\bar{\vartheta}_3),\end{aligned}$$

with the significance

$$\bar{\vartheta}_i = \vartheta_{i1}\frac{dp}{dn} + \vartheta_{i2}\frac{dq}{dn} + \vartheta_{i3}\frac{dr}{dn},$$

for $i=1, 2, 3$. In the same way, we find

$$\frac{d^2q}{dn^2} + \left(\Delta_{11},\ \Delta_{22},\ \Delta_{33},\ \Delta_{23},\ \Delta_{31},\ \Delta_{12} \between \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn}\right)^2 + \frac{\theta_{nn}}{\theta_n}\frac{dq}{dn}$$

$$= \frac{1}{\Omega\theta_n}(h\bar{\vartheta}_1 + b\bar{\vartheta}_2 + f\bar{\vartheta}_3),$$

$$\frac{d^2r}{dn^2} + \left(\Theta_{11},\ \Theta_{22},\ \Theta_{33},\ \Theta_{23},\ \Theta_{31},\ \Theta_{12} \between \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn}\right)^2 + \frac{\theta_{nn}}{\theta_n}\frac{dr}{dn}$$

$$= \frac{1}{\Omega\theta_n}(g\bar{\vartheta}_1 + f\bar{\vartheta}_2 + c\bar{\vartheta}_3).$$

The covariantive expression for θ_{nn} is known; manifestly it can be taken in the form

$$\theta_{nn} = \bar{\vartheta}_1\frac{dp}{dn} + \bar{\vartheta}_2\frac{dq}{dn} + \bar{\vartheta}_3\frac{dr}{dn}.$$

Hence the values of the second normal-derivatives of p, q, r, are known.

Again, we have

$$\frac{d^2\theta}{dn^2} = \theta_1\frac{d^2p}{dn^2} + \theta_2\frac{d^2q}{dn^2} + \theta_3\frac{d^2r}{dn^2} + \sum\theta_{11}\left(\frac{dp}{dn}\right)^2$$

$$= \sum\vartheta_{11}\left(\frac{dp}{dn}\right)^2 + \theta_1\left\{\frac{d^2p}{dn^2} + \left(\Gamma_{11} \between \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn}\right)^2\right\}$$

$$+ \theta_2\left\{\frac{d^2q}{dn^2} + \left(\Delta_{11} \between \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn}\right)^2\right\}$$

$$+ \theta_3\left\{\frac{d^2r}{dn^2} + \left(\Theta_{11} \between \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn}\right)^2\right\};$$

when the preceding values of $\frac{d^2p}{dn^2}, \frac{d^2q}{dn^2}, \frac{d^2r}{dn^2}$, are substituted, the resulting equation merely repeats the known value of θ_{nn}.

Similarly if we take any direction p', q', r', in the region, whether the direction be on the parametric surface $\theta=0$ or not, and if, as before (§ 197),

$$\vartheta_1 = \vartheta_{11}p' + \vartheta_{12}q' + \vartheta_{13}r', \quad \vartheta_2 = \vartheta_{21}p' + \vartheta_{22}q' + \vartheta_{23}r', \quad \vartheta_3 = \vartheta_{31}p' + \vartheta_{32}q' + \vartheta_{33}r',$$

we have

$$\frac{d}{ds}\left(\theta_n\frac{dp}{dn}\right) = \frac{1}{\Omega}\left(a\vartheta_1 + h\vartheta_2 + g\vartheta_3\right) - \theta_n\left(\Gamma_{11} \between \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn} \between p',\ q',\ r'\right),$$

$$\frac{d}{ds}\left(\theta_n\frac{dq}{dn}\right) = \frac{1}{\Omega}\left(h\vartheta_1 + b\vartheta_2 + f\vartheta_3\right) - \theta_n\left(\Delta_{11} \between \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn} \between p',\ q',\ r'\right),$$

$$\frac{d}{ds}\left(\theta_n\frac{dr}{dn}\right) = \frac{1}{\Omega}\left(g\vartheta_1 + f\vartheta_2 + c\vartheta_3\right) - \theta_n\left(\Theta_{11} \between \frac{dp}{dn},\ \frac{dq}{dn},\ \frac{dr}{dn} \between p',\ q',\ r'\right).$$

The value of $\dfrac{d\theta_n}{ds}$ has already (p. 45) been obtained and it can be expressed in the form

$$\frac{d\theta_n}{ds}=\vartheta_1\frac{dp}{dn}+\vartheta_2\frac{dq}{dn}+\vartheta_3\frac{dr}{dn};$$

and thus the arc-derivatives of $\dfrac{dp}{dn}, \dfrac{dq}{dn}, \dfrac{dr}{dn}$, taken in any direction in the region, are known.

Arc-variation of the direction of a regional normal to a surface.

208. By means of the foregoing relations, we can obtain the value of $\dfrac{d}{ds}\left(\dfrac{dy}{dn}\right)$ which will be needed later. For

$$\begin{aligned}\frac{dy_1}{ds}&=y_{11}p'+y_{12}q'+y_{13}r'\\ &=\eta_{11}p'+\eta_{12}q'+\eta_{13}r'+y_1\alpha+y_2\xi+y_3\phi\\ &=\eta_1+y_1\alpha+y_2\xi+y_3\phi,\end{aligned}$$

with the notation of § 172; and similarly for the arc-derivatives of y_2 and y_3. Hence

$$\begin{aligned}\frac{d}{ds}\left(\frac{dy}{dn}\right)&=\frac{d}{ds}\left(y_1\frac{dp}{dn}+y_2\frac{dq}{dn}+y_3\frac{dr}{dn}\right)\\ &=\sum(\eta_1+y_1\alpha+y_2\xi+y_3\phi)\frac{dp}{dn}\\ &\quad+y_1\frac{d}{ds}\left(\frac{dp}{dn}\right)+y_2\frac{d}{ds}\left(\frac{dq}{dn}\right)+y_3\frac{d}{ds}\left(\frac{dr}{dn}\right).\end{aligned}$$

But by the use of results in § 207, we have three formulæ of the type

$$\theta_n\frac{d}{ds}\left(\frac{dp}{dn}\right)+\theta_n'\frac{dp}{dn}=\frac{1}{\Omega}(a\vartheta_1+h\vartheta_2+g\vartheta_3)-\theta_n\left(\alpha\frac{dp}{dn}+\beta\frac{dq}{dn}+\gamma\frac{dr}{dn}\right),$$

so that

$$\begin{aligned}\theta_n\sum y_1\frac{d}{ds}\left(\frac{dp}{dn}\right)&+\theta_n'\frac{dy}{dn}\\ &=\frac{1}{\Omega}(a\between\vartheta_1,\vartheta_2,\vartheta_3\between y_1,y_2,y_3)-\theta_n\sum(y_1\alpha+y_2\xi+y_3\phi)\frac{dp}{dn}.\end{aligned}$$

Consequently

$$\begin{aligned}\frac{d}{ds}\left(\frac{dy}{dn}\right)&+\frac{\theta_n'}{\theta_n}\frac{dy}{dn}\\ &=\eta_1\frac{dp}{dn}+\eta_2\frac{dq}{dn}+\eta_3\frac{dr}{dn}+\frac{1}{\Omega\theta_n}(a\between\vartheta_1,\vartheta_2,\vartheta_3\between y_1,y_2,y_3),\end{aligned}$$

the formula in question.

Ex. Verify that

$$\sum \left\{ y_i \frac{d}{ds}\left(\frac{dy}{dn}\right) \right\} = \theta_i \frac{d}{ds}\left(\frac{1}{\theta_n}\right) + \frac{1}{\theta_n} \vartheta_i,$$

for $i=1, 2, 3$. These results were obtained in an earlier investigation (§ 200).

Magnitudes involving third-order derivatives for a regional surface.

209. For all values $i, j, =1, 2, 3$ (and with the current conventions $p=x_1$, $q=x_2$, $r=x_3$), we have

$$\vartheta_{ij} = \theta_{ij} - \theta_1 \Gamma_{ij} - \theta_2 \Delta_{ij} - \theta_3 \Theta_{ij},$$

so that

$$\frac{\partial \vartheta_{ij}}{\partial x_k} = \theta_{ijk} - \theta_{1k}\Gamma_{ij} - \theta_{2k}\Delta_{ij} - \theta_{3k}\Theta_{ij}$$

$$- \theta_1 \frac{\partial \Gamma_{ij}}{\partial x_k} - \theta_2 \frac{\partial \Delta_{ij}}{\partial x_k} - \theta_3 \frac{\partial \Theta_{ij}}{\partial x_k}.$$

In the values of the derivatives of the quantities Γ_{ij}, Δ_{ij}, Θ_{ij}, as given by the results of § 163, it is convenient to write

$$\alpha_k(ij) = (1i, jk) + (1j, ik),$$
$$\beta_k(ij) = (2i, jk) + (2j, ik),$$
$$\gamma_k(ij) = (3i, jk) + (3j, ik),$$

all the quantities on the right-hand sides being Riemann four-index symbols (ab, cd). Also, we write

$$\bar{\theta}_{ijk} = \theta_{ijk} - \theta_1\{\Gamma_{ijk} + \Gamma_i(jk) + \Gamma_j(ki) + \Gamma_k(ij)\}$$
$$- \theta_2\{\Delta_{ijk} + \Delta_i(jk) + \Delta_j(ki) + \Delta_k(ij)\}$$
$$- \theta_3\{\Theta_{ijk} + \Theta_i(jk) + \Theta_j(ki) + \Theta_k(ij)\}.$$

Then after reduction we find

$$\frac{\partial \vartheta_{ij}}{\partial x_k} = \bar{\theta}_{ijk} - \vartheta_{1k}\Gamma_{ij} - \vartheta_{2k}\Delta_{ij} - \vartheta_{3k}\Theta_{ij}$$

$$+ \tfrac{1}{3}\theta_n \left\{ \frac{dp}{dn}\alpha_k(ij) + \frac{dq}{dn}\beta_k(ij) + \frac{dr}{dn}\gamma_k(ij) \right\},$$

holding for all values of $i, j, k, =1, 2, 3$, repetitions being admissible. In the symbols $\alpha_k(ij)$, $\beta_k(ij)$, $\gamma_k(ij)$, the integers i and j (when different) are interchangeable without affecting the value of the magnitudes α_k, β_k, γ_k; and the form of $\bar{\theta}_{ijk}$ shews that the integers i, j, k, (when different, wholly or in part) can be interchanged without affecting the value of the quantity denoted by $\bar{\theta}_{ijk}$.

Accordingly, we take the combination

$$\frac{\partial \vartheta_{ij}}{\partial x_k} + \frac{\partial \vartheta_{jk}}{\partial x_i} + \frac{\partial \vartheta_{ki}}{\partial x_j}.$$

In this magnitude, the aggregate of terms involving the four-index symbols is

$$\tfrac{1}{3}\theta_n\left(\frac{dp}{dn}P+\frac{dq}{dn}Q+\frac{dr}{dn}R\right),$$

where

$$\begin{aligned}P=&\,a_k(ij)+a_i(jk)+a_j(ki)\\ &=(1i, jk)+(1j, ik)\\ &\quad+(1j, ki)+(1k, ji)\\ &\qquad+(1k, ij)+(1i, kj)=0\,;\end{aligned}$$

and similarly $Q=0$, $R=0$. Consequently the aggregate of terms involving the four-index symbols in the specified combination of the derivatives of the quantities ϑ_{ij} is zero. We therefore have

$$\begin{aligned}&\frac{\partial\vartheta_{ij}}{\partial x_k}+\frac{\partial\vartheta_{jk}}{\partial x_i}+\frac{\partial\vartheta_{ki}}{\partial x_j}\\ &\qquad=3\bar{\theta}_{ijk}-\vartheta_i(jk)-\vartheta_j(ki)-\vartheta_k(ij),\end{aligned}$$

where

$$\vartheta_\lambda(\mu\nu)=\vartheta_{1\lambda}\Gamma_{\mu\nu}+\vartheta_{2\lambda}\Delta_{\mu\nu}+\vartheta_{3\lambda}\Theta_{\mu\nu},$$

for all the admissible combinations.

The value of the regional flexure of a superficial geodesic has been obtained in the form

$$-\frac{\theta_n}{\gamma}=\sum\vartheta_{11}p'^2=(\vartheta_{11}, \vartheta_{22}, \vartheta_{33}, \vartheta_{23}, \vartheta_{31}, \vartheta_{12}\between p', q', r')^2\,;$$

and therefore, differentiating along the superficial geodesic, we have

$$\begin{aligned}-\frac{d}{ds}\left(\frac{\theta_n}{\gamma}\right)&=2(\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r')p_0''+2(\vartheta_{12}p'+\vartheta_{22}q'+\vartheta_{23}r')q_0''\\ &\qquad+2(\vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r')r_0''+\left(\frac{d\vartheta_{11}}{ds}\between p', q', r'\right)^2\\ &=2[(\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r')p''+\vartheta(_{12}p'+\vartheta_{22}q'+\vartheta_{23}r')q''\\ &\qquad+(\vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r')r'']\\ &\qquad+\frac{2}{\gamma}\left(\vartheta_{11}, \vartheta_{22}, \vartheta_{33}, \vartheta_{23}, \vartheta_{31}, \vartheta_{12}\between\frac{dp}{dn}, \frac{dq}{dn}, \frac{dr}{dn}\between p', q', r'\right)\\ &\qquad+\left(\frac{d\vartheta_{11}}{ds}\between p', q', r'\right)^2.\end{aligned}$$

The expression in the third line of the right-hand side of the last expression

$$=\frac{2}{\gamma}\frac{d\theta_n}{ds},$$

after the result on p. 48; and therefore we have

$$-\theta_n \frac{d}{ds}\left(\frac{1}{\gamma}\right) - \frac{3}{\gamma}\frac{d\theta_n}{ds}$$
$$= \left(\frac{d\vartheta_{11}}{ds} \between p', q', r'\right)^2 + 2(\vartheta_1 p'' + \vartheta_2 q'' + \vartheta_3 r''),$$

with the notation of p. 15 for ϑ_1, ϑ_2, ϑ_3. Because

$$\frac{d\vartheta_{ij}}{ds} = p'\frac{\partial \vartheta_{ij}}{\partial p} + q'\frac{\partial \vartheta_{ij}}{\partial q} + r'\frac{\partial \vartheta_{ij}}{\partial r},$$

and because p'', q'', r'', are homogeneous quadratic functions of p', q', r', the right-hand side is a homogeneous cubic form in p', q', r'; we shall take

$$-\theta_n \frac{d}{ds}\left(\frac{1}{\gamma}\right) - \frac{3}{\gamma}\frac{d\theta_n}{ds} = \sum_i \sum_j \sum_k \vartheta_{ijk} x_1'^i x_2'^j x_3'^k,$$

where $x_1 = p$, $x_2 = q$, $x_3 = r$.

To obtain more explicit values of the coefficients ϑ_{ijk} in relation to the magnitudes $\bar{\theta}_{ijk}$, we take the respective coefficients of $x_1'^i x_2'^j x_3'^k$ on the two sides of the equivalent expressions. When they are equated, they give the relation

$$\begin{aligned} 6\vartheta_{ijk} = 2\left(\frac{\partial \vartheta_{ij}}{\partial x_k} + \frac{\partial \vartheta_{jk}}{\partial x_i} + \frac{\partial \vartheta_{ki}}{\partial x_j}\right) &- 4(\vartheta_{1j}\Gamma_{ki} + \vartheta_{1k}\Gamma_{ij} + \vartheta_{1i}\Gamma_{jk}) \\ &- 4(\vartheta_{2j}\Delta_{ki} + \vartheta_{2k}\Delta_{ij} + \vartheta_{2i}\Delta_{jk}) \\ &- 4(\vartheta_{3j}\Theta_{ki} + \vartheta_{3k}\Theta_{ij} + \vartheta_{3i}\Theta_{jk}) \\ = 6\bar{\theta}_{ijk} - 6\vartheta_i(jk) - 6\vartheta_j(ki) - 6\vartheta_k(ij),& \end{aligned}$$

so that

$$\bar{\theta}_{ijk} = \vartheta_{ijk} + \vartheta_i(jk) + \vartheta_j(ki) + \vartheta_k(ij).$$

We therefore have the value of $\dfrac{\partial \vartheta_{ij}}{\partial x_k}$, on the substitution of this value of $\bar{\theta}_{ijk}$, in the form

$$\frac{\partial \vartheta_{ij}}{\partial x_k} = \vartheta_{ijk} + \vartheta_i(jk) + \vartheta_j(ki) + \tfrac{1}{3}\theta_n \left\{\frac{dp}{dn}\alpha_k(ij) + \frac{dq}{dn}\beta_k(ij) + \frac{dr}{dn}\gamma_k(ij)\right\},$$

with the assigned definitions of the symbols $\vartheta_\lambda(\mu\nu)$, $\alpha_\lambda(\mu\nu)$, $\beta_\lambda(\mu\nu)$, $\gamma_\lambda(\mu\nu)$.

Next, we proceed to find the value of $\dfrac{d^3\theta}{dn^3}$ for the parametric surface in a region; the associated results will be required later in the consideration of geodesic surfaces.

When the relation

$$\frac{d^2\theta}{dn^2} = \sum \vartheta_{11}\left(\frac{dp}{dn}\right)^2$$

is differentiated along the regional normal to the surface, we have

$$\frac{d^3\theta}{dn^3} = 2\left(\bar{\vartheta}_1 \frac{d^2p}{dn^2} + \bar{\vartheta}_2 \frac{d^2q}{dn^2} + \bar{\vartheta}_3 \frac{d^2r}{dn^2}\right) + \sum \frac{d\vartheta_{11}}{dn}\left(\frac{dp}{dn}\right)^2.$$

From the values that have been obtained for the second normal-derivatives of p, q, r, it follows that

$$\bar{\vartheta}_1 \frac{d^2p}{dn^2} + \bar{\vartheta}_2 \frac{d^2q}{dn^2} + \bar{\vartheta}_3 \frac{d^2r}{dn^2}$$

$$= N - \left[\bar{\vartheta}_1 \sum \Gamma_{11} \left(\frac{dp}{dn}\right)^2 + \bar{\vartheta}_2 \sum \Delta_{11} \left(\frac{dp}{dn}\right)^2 + \bar{\vartheta}_3 \sum \Theta_{11} \left(\frac{dp}{dn}\right)^2\right]$$

where

$$N = \frac{1}{\Omega \theta_n} (a, b, c, f, g, h \between \bar{\vartheta}_1, \bar{\vartheta}_2, \bar{\vartheta}_3)^2 - \frac{1}{\theta_n} \theta_{nn}^2 ;$$

and therefore

$$\frac{d^3\theta}{dn^3} - 2N = \sum \frac{d\vartheta_{11}}{dn} \left(\frac{dp}{dn}\right)^2$$

$$- 2\left[\bar{\vartheta}_1 \sum \Gamma_{11} \left(\frac{dp}{dn}\right)^2 + \bar{\vartheta}_2 \sum \Delta_{11} \left(\frac{dp}{dn}\right)^2 + \bar{\vartheta}_3 \sum \Theta_{11} \left(\frac{dp}{dn}\right)^2\right].$$

The quantity on the right-hand side can be deduced from the quantity on the right-hand side of the foregoing expression for

$$-\theta_n \frac{d}{ds}\left(\frac{1}{\gamma}\right) - \frac{3}{\gamma} \frac{d\theta_n}{ds}$$

when, in the latter, we change p', q', r', into $\frac{dp}{dn}, \frac{dq}{dn}, \frac{dr}{dn}$, respectively; and therefore the value of the last quantity on the right-hand side is

$$\sum_i \sum_j \sum_k \vartheta_{ijk} \left(\frac{dp}{dn}\right)^i \left(\frac{dq}{dn}\right)^j \left(\frac{dr}{dn}\right)^k .$$

Accordingly, we have

$$\frac{d^3\theta}{dn^3} - 2N = \sum_i \sum_j \sum_k \vartheta_{ijk} \left(\frac{dp}{dn}\right)^i \left(\frac{dq}{dn}\right)^j \left(\frac{dr}{dn}\right)^k ,$$

with the foregoing value of N: and we associate this result with

$$-\frac{1}{\theta_n^2} \frac{d}{ds}\left(\frac{\theta_n^3}{\gamma}\right) = \sum_i \sum_j \sum_k \vartheta_{ijk} p'^i q'^j r'^k ,$$

while

$$\vartheta_{ijk} = \bar{\theta}_{ijk} - \vartheta_i (jk) - \vartheta_j (ki) - \vartheta_k (ij).$$

Surfaces geodesic to a region.

210. Any equation $\theta(p, q, r) = 0$, among the parameters of a region, determines a surface wholly comprised within the region; such a basis for a surface is manifestly of an analytic character in its initial stages. There is one species of surfaces in a region (and in any non-homaloidal configuration of more than two

dimensions) the basis of which, initially, is of a geometric character; the conception is due originally to Riemann (§ 74). At any point O of a region (or non-homaloidal configuration), he postulates two regional geodesics in different directions. The two directions determine a plane orientation at O; every other direction, passing through O and lying in that orientation, determines a regional geodesic through O; and the aggregate of all these regional geodesics through O is defined as a *geodesic surface* at O, the tangent plane of which at O is the plane orientation. (It is to be noted that the definition is concerned solely with the regional geodesics passing through the point O: an assumption that, if P and Q be points on two of the geodesics through O, the regional geodesic PQ lies in the surface, seems plausible but cannot be justified.) By means of this type of surface, Riemann postulates his measure of curvature of the region (or other non-homaloidal configuration) as given by his measure of curvature for that geodesic surface; and as has been seen (§ 117), the measure of curvature is what has been called the sphericity of the geodesic surface, and thus it is the sphericity of the region (or other configuration) estimated in the orientation of the surface.

We proceed to consider the analytical characteristics of geodesic surfaces at a point O within a region. Manifestly, all the superficial geodesics through O are regional geodesics: were any superficial geodesic through O less over a range than the regional geodesic, it would be an arc within the region less than the regional geodesic—inadmissible as a possibility. The determination of the character of surface geodesics, not passing through O, in relation to the region must be effected later. We shall also obtain relations, concerning grades of contact between a parametric surface and a geodesic surface in the same regional orientation at O.

Accordingly, we suppose initially that a relation $\theta(p, q, r)=0$ can represent a surface which, in the foregoing sense, is geodesic to the region at O; then every regional geodesic, which originates in a direction touching the surface at O, must lie wholly within the surface. Consider therefore a regional geodesic through O, the parametric point p, q, r, in any direction p', q', r', touching the surface; at any point T on this regional geodesic, at an arc-distance t from O measured along the regional geodesic, the regional parameters are

$$\begin{aligned} P&=p+tp'+\tfrac{1}{2}t^2p''+\tfrac{1}{6}t^3p'''+\dots,\\ Q&=q+tq'+\tfrac{1}{2}t^2q''+\tfrac{1}{6}t^3q'''+\dots,\\ R&=r+tr'+\tfrac{1}{2}t^2r''+\tfrac{1}{6}t^3r'''+\dots,\end{aligned}$$

where the second and higher derivatives of p, q, r, are their values at O, as determined for the region by equations of the type

$$\begin{aligned} -p''&=\sum\Gamma_{11}\,p'^2, & -q''&=\sum\Delta_{11}\,p'^2, & -r''&=\sum\Theta_{11}\,p'^2,\\ -p'''&=\sum\Gamma_{111}p'^3, & -q'''&=\sum\Delta_{111}p'^3, & -r'''&=\sum\Theta_{111}p'^3.\end{aligned}$$

If then the regional geodesic thus determined lies wholly within the surface $\theta=0$, the equation

$$\theta(P, Q, R)=0$$

must be satisfied for all values of t; and when the surface is postulated to be a geodesic surface, the various conditions must be satisfied for all directions p', q', r', touching the surface.

To satisfy the requirements, the first necessity is that when $\theta(P, Q, R)$ is expanded in powers of t, the coefficients of the successive powers of t shall vanish, each in turn. For present purposes, we shall consider such powers up to the third inclusive. Now, up to this third order inclusive, we have

$$\theta(P, Q, R)=\sum\theta_1(tp'+\tfrac{1}{2}t^2p''+\tfrac{1}{6}t^3p''')+\tfrac{1}{2}\sum\theta_{11}(t^2p'^2+t^3p'p'')+\tfrac{1}{6}\sum\theta_{300}t^3p'^3,$$

it being assumed that $\theta(p, q, r)=0$; and the coefficients of the powers of t must vanish successively.

(i) In order that the coefficient of the first power of t shall vanish, the condition

$$\theta_1p'+\theta_2q'+\theta_3r'=0$$

must be satisfied. Effectively, it is the expression of the property that the regional direction p', q', r', touches the surface.

(ii) In order that the coefficient of the second power of t shall vanish, the condition

$$\sum\theta_1p''+\sum\theta_{11}p'^2=0$$

must be satisfied. When the values of p'', q'', r'', are inserted, the condition becomes

$$\sum\vartheta_{11}p'^2=0.$$

Effectively, it is the expression of the property that the regional flexure of the superficial geodesic in the direction p', q', r', shall vanish.

Now it is possible that, for any arbitrarily assumed surface $\theta(p, q, r)=0$, the two equations

$$\theta_1p'+\theta_2q'+\theta_3r'=0, \quad \sum\vartheta_{11}p'^2=0,$$

shall be satisfied. As has been seen (p. 13), they are the equations which, at any point O of the surface, determine the two inflexional directions of that surface at O; consequently, the two conditions can be satisfied for a pair of directions on the arbitrarily assumed surface. But when the surface is required to be geodesic, the equations must be satisfied for all directions on the surface; the exceptional and particular instance passes from present consideration.

The analytical significance of this second condition will be developed later.

(iii) In order that the coefficient of the third power of t shall vanish, the condition

$$\sum\theta_1p'''+3\sum\theta_{11}p'p''+\sum\theta_{300}p'^3=0$$

must be satisfied. When the values of p'', q'', r'', and p''', q''', r''', are inserted, the total coefficient of

$$6\frac{p'^i q'^j r'^k}{i!\,j!\,k!}$$

on the left-hand side is

$$\begin{aligned}&\theta_{ijk}-(\theta_1\Gamma_{ijk}+\theta_2\Delta_{ijk}+\theta_3\Delta_{ijk})\\ -\{(\theta_{1i}\Gamma_{jk}+\theta_{1j}\Gamma_{ki}+\theta_{1k}\Gamma_{ij})&+(\theta_{2i}\Delta_{jk}+\theta_{2j}\Delta_{ki}+\theta_{2k}\Delta_{ij})+(\theta_{3i}\Theta_{jk}+\theta_{3j}\Theta_{ki}+\theta_{3k}\Theta_{ij})\}.\end{aligned}$$

In the last set of terms, we substitute

$$\theta_{lm}=\vartheta_{lm}+\theta_1\Gamma_{lm}+\theta_2\Delta_{lm}+\theta_3\Theta_{lm},$$

for all the combinations l, m; and then the coefficient can be arranged in the form

$$\begin{aligned}\theta_{ijk}&-\theta_1\{\Gamma_{ijk}+\Gamma_i(jk)+\Gamma_j(ki)+\Gamma_k(ij)\}\\ &-\theta_2\{\Delta_{ijk}+\Delta_i(jk)+\Delta_j(ki)+\Delta_k(ij)\}\\ &-\theta_3\{\Theta_{ijk}+\Theta_i(jk)+\Theta_j(ki)+\Theta_k(ij)\}-\{\vartheta_i(jk)+\vartheta_j(ki)+\vartheta_k(ij)\}:\end{aligned}$$

that is, the said total coefficient is ϑ_{ijk}. Hence the condition, which arises from the necessary vanishing of the terms in t^3, becomes

$$\sum\vartheta_{300}p'^3=0.$$

The analytical significance of this condition will be developed later.

It thus appears that, for the selected range up to the third power of t inclusive, there are three conditions

$$\sum\theta_1 p'=0,\quad \sum\vartheta_{11}p'^2=0,\quad \sum\vartheta_{300}p'^3=0,$$

which must be satisfied by every direction p', q', r', through the initial point O on the surface $\theta(p, q, r)=0$, if that surface is to be geodesic to the region at O. It must not, however, be assumed that the further conditions, which result from the evanescence of the coefficients of higher powers of t, also are satisfied.

Now the preceding discussion relates to the possibly geodesic quality of the postulated parametric surface $\theta=0$. The surface, which is geodesic to the region at O and has the same orientation at O as the parametric surface, can be represented in a different manner. It is given by the three earlier relations, which express P, Q, R, as functions of p', q', r', t; the values of p', q', r', are subject to the two relations

$$\sum Ap'^2=1,\quad \xi p'+\eta q'+\zeta r'=0,$$

where ξ, η, ζ, are coordinates of the orientation at O; and t is a variable for the geodesic surface. A single equation of the geodesic surface would be the result of eliminating t, p', q', r', among the five relations: the quantities p, q, r, then are merely the parametric coordinates of the individual point O in the region. If this eliminant actually had the form $\theta(P, Q, R)=0$, the postulated surface would

be geodesic in quality. When the eliminant is $G(P, Q, R)=0$, an equation functionally different from $\theta=0$, there still are approximations to agreement between the two functionally different equations.

When the values of P, Q, R, are substituted in the hypothetical equation $G=0$, it is satisfied identically for all values of t. When the same substitutions are effected in $\theta=0$, the foregoing conditions in their sequence indicate grades of geometrical relation between the parametric surface and the geodesic surface.

The vanishing of the term independent of t expresses the fact that both the surfaces pass through O.

The vanishing of the term, which involves the first power of t in the parametric equation, expresses the property that the two surfaces have a common tangent plane if the condition holds for all values of p', q', r'. This geometrical property may be called *contact of the first order* between the two surfaces. But if the condition does not hold for all values of p', q', r', the relations

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0, \quad \xi p' + \eta q' + \zeta r' = 0, \quad \sum A p'^2 = 1,$$

determine two sets of values of p', q', r', the two sets differing only in sign: that is, they determine the direction of the line through O which is the tangent to the curve of intersection of the surfaces. We shall assume that the first-order condition is satisfied for all values of p', q', r': then the two surfaces have the same orientation at O, with the variables ξ, η, ζ.

The vanishing of the term, which involves the second power of t in the parametric equation, expresses the property that every geodesic on G through O osculates the parametric surface if the new condition holds for all values of p', q', r'. This geometrical property may be called *contact of the second order* between the two surfaces. But if the condition does not hold for all directions p', q', r', in the orientation, the three equations

$$\sum A p'^2 = 1, \quad \xi p' + \eta q' + \zeta r' = 0, \quad \sum \vartheta_{11} p'^2 = 0,$$

determine four sets of values of p', q', r', in a couple of pairs, the pair in a set differing only in sign: that is, they give the inflexional tangents to the curve of intersection of the two surfaces at O, and the curve has a double point at O (or a cusp if the tangents coincide).

Similarly, the vanishing of the term, which involves the third power of t in the parametric equation, expresses the property that every geodesic on G through O, already presumed to osculate the parametric surface, has contact with that surface of the order next higher than osculation, if the condition is satisfied for all values of p', q', r', in the orientation. This geometrical property may be called *contact of the third order* between the two surfaces.

Corresponding inferences are associated with the vanishing of the terms which involve any further power of t. In effect, such a mode of procedure establishes the geodesic surface of a region in an assigned orientation at O as the uniquely

determinate surface of reference for all the regional parametric surfaces through O in that orientation; and it is the natural extension of the practice which has established the regional geodesic in any direction through O as the unique curve of reference for all regional parametric curves through O in that direction.

211. The analytical effect of the three conditions will now be considered in succession.

(i) The first-order condition is

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

to be satisfied for all directions in the surface. Let p_1', q_1', r_1', and p_2', q_2', r_2', denote two sets of direction-variables, so that

$$\theta_1 p_1' + \theta_2 q_1' + \theta_3 r_1' = 0, \quad \theta_1 p_2' + \theta_2 q_2' + \theta_3 r_2' = 0;$$

they are sufficient to define the orientation at O. Any other direction in that orientation can be represented by

$$p' = \alpha p_1' + \beta p_2', \quad q' = \alpha q_1' + \beta q_2', \quad r' = \alpha r_1' + \beta r_2',$$

where α and β are arbitrary parameters; and manifestly these variables satisfy the condition. Now, for this orientation, we have

$$\frac{\theta_1}{q_1'r_2' - r_1'q_2'} = \frac{\theta_2}{r_1'p_2' - p_1'r_2'} = \frac{\theta_3}{p_1'q_2' - q_1'p_2'} = \frac{\Omega^{\frac{1}{2}}}{\sin\epsilon}\,\theta_n,$$

where ϵ denotes the angle between the two directions in the orientation.

(ii) The condition, which secures contact of the second order between the parametric surface and the geodesic surface in the same orientation at O, is

$$\sum \vartheta_{11} p'^2 = 0;$$

and it is to be satisfied for all directions p', q', r', such that

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0.$$

Let the value of r', as given in terms of p' and q' by this last relation, be substituted in the second-order condition. The condition, thus modified, must now be an identity because p' and q' are independent of one another. The coefficients of p'^2, $p'q'$, q'^2, therefore vanish; and the three resulting relations can be expressed in the form

$$\left.\begin{aligned} 2\frac{\vartheta_{23}}{\theta_2\theta_3} &= \phantom{\frac{\vartheta_{11}}{\theta_1^2} + {}} \frac{\vartheta_{22}}{\theta_2^2} + \frac{\vartheta_{33}}{\theta_3^2} \\ 2\frac{\vartheta_{31}}{\theta_3\theta_1} &= \frac{\vartheta_{11}}{\theta_1^2} \phantom{{} + \frac{\vartheta_{22}}{\theta_2^2}} + \frac{\vartheta_{33}}{\theta_3^2} \\ 2\frac{\vartheta_{12}}{\theta_1\theta_2} &= \frac{\vartheta_{11}}{\theta_1^2} + \frac{\vartheta_{22}}{\theta_2^2} \end{aligned}\right\}.$$

These are relations which must be satisfied if there is to be contact of the second order at O. (If they are satisfied over the whole of the parametric surface, they are three simultaneous partial differential equations of the second order, serving to define all geodesic surfaces in the region.)

On the assumption that these relations at O are satisfied, two remarks may be made. In the first place, when the values of $\vartheta_{23}, \vartheta_{31}, \vartheta_{12}$, in terms of $\vartheta_{11}, \vartheta_{22}, \vartheta_{33}$, are used, we find

$$\sum \vartheta_{11} p'^2 = (\theta_1 p' + \theta_2 q' + \theta_3 r')\left(\frac{\vartheta_{11}}{\theta_1} p' + \frac{\vartheta_{22}}{\theta_2} q' + \frac{\vartheta_{33}}{\theta_3} r'\right),$$

identically; and we thus verify that, under the relations, the condition for contact of the second order between the surfaces is satisfied. In the second place, also by using these values of $\vartheta_{23}, \vartheta_{31}, \vartheta_{12}$, the determinant ∇, where

$$\nabla = \begin{vmatrix} \vartheta_{11}, & \vartheta_{12}, & \vartheta_{13} \\ \vartheta_{21}, & \vartheta_{22}, & \vartheta_{23} \\ \vartheta_{31}, & \vartheta_{32}, & \vartheta_{23} \end{vmatrix},$$

is found to vanish. But ∇ is the discriminant of the ternary quadratic form $\sum \vartheta_{11} p'^2$; and its evanescence is the analytic condition that the quadratic form should be the product of two factors linear in p', q', r'. The three conditions (which involve $\nabla = 0$) are necessary in order that one of these linear factors should be $\theta_1 p' + \theta_2 q' + \theta_3 r'$.

Ex. 1. Shew that, on a geodesic surface $\theta(p, q, r) = 0$ in the region,

$$\vartheta_{i1} p' + \vartheta_{i2} q' + \vartheta_{i3} r' = \frac{\theta_i}{\theta_n} \frac{d\theta_n}{ds},$$

for $i = 1, 2, 3$.

Ex. 2. A parametric surface $\theta = 0$ or $\theta =$ constant might, without any change of its geometric character, be represented analytically by an equation

$$\phi = f(\theta) = \text{constant},$$

where $f(\theta)$ denotes any function of its single argument θ. The condition

$$\phi_1 p' + \phi_2 q' + \phi_3 r' = 0$$

manifestly is satisfied, when the corresponding condition is satisfied for θ.

The relations, characteristic of contact of the second order, should remain unaltered in essence. Let $\bar{\phi}_{ij}$ denote the magnitude, which bears to ϕ the same relation as that borne to θ by ϑ_{ij}; then we find

$$\phi_i = \theta_i f', \quad \bar{\phi}_{ij} = \theta_i \theta_j f'' + \vartheta_{ij} f',$$

so that

$$\frac{\bar{\phi}_{ij}}{\phi_i \phi_j} = \frac{f''}{f'^2} + \frac{\vartheta_{ij}}{\theta_i \theta_j} \frac{1}{f'},$$

for all values of i, j. Consequently

$$\frac{\overline{\phi}_{ii}}{\phi_i^2}-2\frac{\overline{\phi}_{ij}}{\phi_i\phi_j}+\frac{\overline{\phi}_{jj}}{\phi_j^2}=\frac{1}{f'}\left(\frac{\vartheta_{ii}}{\theta_i^2}-2\frac{\vartheta_{ij}}{\theta_i\theta_j}+\frac{\vartheta_{jj}}{\theta_j^2}\right),$$

thus verifying the invariantive nature of the relations characteristic of second-order contact, so far as concerns the specified kind of change in the parametric equation.

Ex. 3. Construct the equations, characteristic of contact of the second order, between the regional surface

$$r=\psi(p, q)$$

and the geodesic surface in the same orientation at any point.

Ex. 4. A special class of parametric surfaces, having contact of the second order with the geodesic surface in the same orientation, is represented by the equations

$$\frac{\vartheta_{11}}{\theta_1^2}=\frac{\vartheta_{12}}{\theta_1\theta_2}=\frac{\vartheta_{13}}{\theta_1\theta_3}=\frac{\vartheta_{22}}{\theta_2^2}=\frac{\vartheta_{23}}{\theta_2\theta_3}=\frac{\vartheta_{33}}{\theta_3^2}.$$

Establish the following results for a surface of this class:

(i) The common value of the six fractions

$$=\frac{\theta_{nn}}{\theta_n^2}.$$

(ii) The normal dilatation is stationary in all directions on the surface at O.

(iii) The second variations of the regional parameters at O along the regional normal to the surface are given by

$$\frac{d^2p}{dn^2}+\sum\Gamma_{11}\left(\frac{dp}{dn}\right)^2=0,\quad \frac{d^2q}{dn^2}+\sum\Delta_{11}\left(\frac{dp}{dn}\right)^2=0,\quad \frac{d^2r}{dn^2}+\sum\Theta_{11}\left(\frac{dp}{dn}\right)^2=0.$$

(iv) The quantity N of § 209 vanishes, so that, at O,

$$\frac{d^3\theta}{dn^3}=\sum_i\sum_j\sum_k\vartheta_{ijk}\left(\frac{dp}{dn}\right)^i\left(\frac{dq}{dn}\right)^j\left(\frac{dr}{dn}\right)^k.$$

(iii) The further condition, which ensures contact of the third order between the parametric surface and the geodesic surface in the same orientation at O, is

$$\sum\vartheta_{300}p'^3=0\,;$$

and it is to be satisfied for all directions p', q', r', such that

$$\theta_1 p'+\theta_2 q'+\theta r'=0.$$

Let the value of r', as given in terms of p' and q' by this last relation, be substituted in the third-order condition. The condition, thus modified, must now be an identity because p' and q' are independent of one another. The coefficients of p'^3, p'^2q', $p'q'^2$, q'^3, therefore vanish; and the four resulting relations can be stated in the form

$$3\frac{\vartheta_{021}}{\theta_2{}^2\theta_3}-\frac{\vartheta_{030}}{\theta_2{}^3}=3\frac{\vartheta_{012}}{\theta_2\theta_3{}^2}-\frac{\vartheta_{103}}{\theta_3{}^3}=\frac{2\mathfrak{a}}{\theta_2\theta_3},$$

$$3\frac{\vartheta_{102}}{\theta_1\theta_3{}^2}-\frac{\vartheta_{003}}{\theta_3{}^3}=3\frac{\vartheta_{201}}{\theta_1{}^2\theta_2}-\frac{\vartheta_{300}}{\theta_1{}^3}=\frac{2\mathfrak{b}}{\theta_3\theta_1},$$

$$3\frac{\vartheta_{210}}{\theta_1{}^2\theta_2}-\frac{\vartheta_{300}}{\theta_1{}^3}=3\frac{\vartheta_{120}}{\theta_1\theta_2{}^2}-\frac{\vartheta_{030}}{\theta_2{}^3}=\frac{2\mathfrak{c}}{\theta_1\theta_2},$$

$$3\frac{\vartheta_{111}}{\theta_1\theta_2\theta_3}=\frac{\mathfrak{a}}{\theta_2\theta_3}+\frac{\mathfrak{b}}{\theta_3\theta_1}+\frac{\mathfrak{c}}{\theta_1\theta_2},$$

where the symbols $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, are to be regarded as defined in connection with the first three equalities.

These are relations which must be satisfied if there is to be contact of the third order between the parametric surface and the geodesic surface. On the assumption that they are satisfied, let the values of ϑ_{210}, ϑ_{201}, ϑ_{120}, ϑ_{102}, ϑ_{021}, ϑ_{012}, ϑ_{111}, in terms of ϑ_{300}, ϑ_{030}, ϑ_{003}, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, be substituted in the ternary cubic $\sum \vartheta_{300}p'^3$; then

$$\sum \vartheta_{300}p'^3$$
$$=(\theta_1p'+\theta_2q'+\theta_3r')\left(\frac{\vartheta_{300}}{\theta_1}p'^2+\frac{\vartheta_{030}}{\theta_2}q'^2+\frac{\vartheta_{003}}{\theta_3}r'^2+2\mathfrak{a}q'r'+2\mathfrak{b}r'p'+2\mathfrak{c}p'q'\right),$$

identically. Hence the four relations among the coefficients ϑ_{ijk} are sufficient, as well as necessary, to ensure that there is contact of the third order at O between the two surfaces.

Ex. 5. As in the preceding *Ex.* 2, where the second-order condition was shewn to be invariantive under any functional transformation, $\phi=f(\theta)=$ constant of the parametric equation, we may expect that the third-order condition will also be invariantive under such a transformation. The inference can be verified analytically, thus:

(*a*) If $\overline{\phi}_{ijk}$ denotes the same magnitude in connection with ϕ as ϑ_{ijk} denotes in connection with θ, then

$$\frac{\overline{\phi}_{ijk}}{\phi_i\phi_j\phi_k}=\frac{\vartheta_{ijk}}{\theta_i\theta_j\theta_k}\frac{1}{f'^2}+\frac{1}{\theta_j\theta_k}\left\{\theta(jk)+\frac{1}{\theta_k\theta_i}\theta(ki)+\frac{1}{\theta_i\theta_j}\theta(ij)\right\}\frac{f''}{f'^3}-\frac{f'''}{f'^3},$$

where, for all the values,

$$\theta(\lambda\mu)=\theta_1\Gamma_{\lambda\mu}+\theta_2\Delta_{\lambda\mu}+\theta_3\Theta_{\lambda\mu};$$

(*b*) When the critical equalities among the magnitudes $\overline{\phi}_{ijk}$ alone are formed, they are found to be satisfied in virtue of the critical equalities among the magnitudes ϑ_{ijk} alone.

Ex. 6. Prove that, when the relations for third-order contact between the two surfaces at O are satisfied, the equation

$$\begin{vmatrix} \vartheta_{300}p'+\vartheta_{210}q'+\vartheta_{201}r', & \vartheta_{210}p'+\vartheta_{120}q'+\vartheta_{111}r', & \vartheta_{201}p'+\vartheta_{111}q'+\vartheta_{102}r' \\ \vartheta_{210}p'+\vartheta_{120}q'+\vartheta_{111}r', & \vartheta_{120}p'+\vartheta_{030}q'+\vartheta_{021}r', & \vartheta_{111}p'+\vartheta_{021}q'+\vartheta_{012}r' \\ \vartheta_{201}p'+\vartheta_{111}q'+\vartheta_{102}r', & \vartheta_{111}p'+\vartheta_{021}q'+\vartheta_{012}r', & \vartheta_{102}p'+\vartheta_{012}q'+\vartheta_{003}r' \end{vmatrix}=0$$

holds for all directions in the common orientation of the two surfaces.

NOTE 1. The analytical results, connected with the second-order condition and the third-order condition, can be illustrated from the theory of plane curves, by regarding p', q', r', as the homogeneous coordinates of a point in a plane.

The second-order condition $\sum \vartheta_{11}p'^2=0$ then represents a conic in the plane. Under the three relations, the conic degenerates into two straight lines one of which is $\theta_1 p'+\theta_2 q'+\theta_3 r'=0$.

The third-order condition $\sum \vartheta_{300}p'^3=0$ then represents a cubic curve in the plane. Under the four relations, the cubic degenerates into a conic and the same straight line. It is a known property * that the Hessian of the cubic then vanishes as it contains the linear factor of the degenerate cubic.

NOTE 2. The foregoing relations, for contact of the second order and for contact of the third order, can be obtained simply by means of the umbral notation † for homogeneous quantities. The process will be used later (§ 344) in the similar investigation of the contact between a parametric surface and the tangent geodesic surface in a domain.

We take sets of umbral symbols $a, b, c, \ldots$, associated with $\sum \vartheta_{11}p'^2$, and sets of umbral symbols α, β, γ, associated with $\sum \vartheta_{300}p'^3$, so that

$$U=\sum \vartheta_{11}p'^2 = (a_1p'+a_2q'+a_3r')^2=a_{p'}{}^2,$$
$$W=\sum \vartheta_{300}p'^3=(\alpha_1p'+\alpha_2q'+\alpha_3r')^3=\alpha_{p'}{}^3;$$

and we write
$$L=\theta_1p'+\theta_2q'+\theta_3r',$$
the symbols θ_1, θ_2, θ_3, being non-umbral. Then

$$a_{p'}=G_1p'+G_2q'+\frac{L}{\theta_3}a_3,\quad \alpha_{p'}=\Gamma_1p'+\Gamma_2q'+\frac{L}{\theta_3}\alpha_3,$$

where
$$\left.\begin{aligned} G_1&=a_1-\frac{\theta_1}{\theta_3}a_3\\ G_2&=a_2-\frac{\theta_2}{\theta_3}a_3\end{aligned}\right\},\qquad \left.\begin{aligned}\Gamma_1&=\alpha_1-\frac{\theta_1}{\theta_3}\alpha_3\\ \Gamma_2&=\alpha_2-\frac{\theta_2}{\theta_3}\alpha_3\end{aligned}\right\};$$

and therefore
$$U=\left(G_1p'+G_2q'+a_3\frac{L}{\theta_3}\right)^2,\quad W=\left(\Gamma_1p'+\Gamma_2q'+\alpha_3\frac{L}{\theta_3}\right)^3,$$
in general.

For directions in the superficial orientation common to the parametric surface and the geodesic surface, we have $L=0$. Thus the condition for contact of the second order becomes
$$U=(G_1p'+G_2q')^2=0,$$
which must be evanescent for all values of p' and q'. The necessary relations are

$$0=G_1{}^2 \;=\left(a_1-\frac{\theta_1}{\theta_3}a_3\right)^2=\vartheta_{11}-2\vartheta_{13}\frac{\theta_1}{\theta_3}+\vartheta_{33}\frac{\theta_1{}^2}{\theta_3{}^2},$$
$$0=G_1G_2=\left(a_1-\frac{\theta_1}{\theta_3}a_3\right)\left(a_2-\frac{\theta_2}{\theta_3}a_3\right)=\vartheta_{12}-\vartheta_{13}\frac{\theta_2}{\theta_3}-\vartheta_{23}\frac{\theta_1}{\theta_3}+\vartheta_{33}\frac{\theta_1\theta_2}{\theta_3{}^2},$$
$$0=G_2{}^2 \;=\left(a_2-\frac{\theta_2}{\theta_3}a_3\right)^2=\vartheta_{22}-2\vartheta_{23}\frac{\theta_2}{\theta_3}+\vartheta_{33}\frac{\theta_2{}^2}{\theta_3{}^2};$$

* Salmon, *Higher Plane Curves* (3rd edn., 1879), §§ 240-242.

† J. H. Grace and A. Young, *Algebra of Invariants*.

and these can be changed to the form in the text. Further, when these relations are satisfied, we have

$$U=2(\vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r')\frac{L}{\theta_3}+\vartheta_{33}\frac{L^2}{\theta_3^{\,2}},$$

shewing that they are sufficient to secure contact of the second order between the surfaces for all directions given by $L=0$.

Similarly the condition for contact of the third order becomes

$$W=(\Gamma_1 p'+\Gamma_2 q')^3=0\,;$$

and the necessary relations, in umbral form, are

$$0=\Gamma_1^{\,3},\quad 0=\Gamma_1^{\,2}\Gamma_2,\quad 0=\Gamma_1\Gamma_2^{\,2},\quad 0=\Gamma_2^{\,3}.$$

The first of these becomes

$$\vartheta_{300}-3\vartheta_{201}\frac{\theta_1}{\theta_3}+3\vartheta_{102}\frac{\theta_1^{\,2}}{\theta_3^{\,2}}-\vartheta_{003}\frac{\theta_1^{\,3}}{\theta_3^{\,3}}=0,$$

and similarly for the other three; and the set can be changed into the form in the text. These relations are also seen to be sufficient; for, when they are satisfied, we have

$$\begin{aligned}W=3(\vartheta_{201},\ \vartheta_{021},\ \vartheta_{003},\ \vartheta_{012},\ \vartheta_{102},\ \vartheta_{111}\between p',\ q',\ r')^2\frac{L}{\theta_3}\\+3(\vartheta_{102}p'+\vartheta_{012}q'+\vartheta_{003}r')\frac{L^2}{\theta_3^{\,2}}+\vartheta_{003}\frac{L^3}{\theta_3^{\,2}}.\end{aligned}$$

Range of a region in the vicinity of any point.

212. For a more precise estimate of a surface, in its quality of being geodesic to the region, we shall consider the properties of the region in a range near a point and not solely the properties at the point. Accordingly, the geometrical parts of a regional triangle will be investigated.

In the course of the analysis, certain subsidiary combinations of magnitudes enter: and they are stated at once.

I. The parametric derivatives of Γ, Δ, Θ, have been obtained (§ 163); and so, when we take derivatives along a regional geodesic in the direction p', q', r', we have, with the notation of § 173,

$$\begin{aligned}\frac{d\Gamma_{11}}{ds}&=\Gamma_{300}p'+\Gamma_{210}q'+\Gamma_{201}r'+2(\Gamma_{11}a+\Gamma_{12}\xi+\Gamma_{13}\phi)\\&\quad+\frac{2}{3\Omega}[\qquad q'(hk_{33}-gk_{23})\quad+r'(-hk_{23}+gk_{22})],\end{aligned}$$

$$\begin{aligned}\frac{d\Gamma_{22}}{ds}&=\Gamma_{120}p'+\Gamma_{030}q'+\Gamma_{021}r'+2(\Gamma_{12}\beta+\Gamma_{22}\eta+\Gamma_{23}\chi)\\&\quad+\frac{2}{3\Omega}[p'(ak_{33}-gk_{13})\qquad+r'(-ak_{13}+gk_{11})],\end{aligned}$$

$$\begin{aligned}\frac{d\Gamma_{33}}{ds}&=\Gamma_{102}p'+\Gamma_{012}q'+\Gamma_{003}r'+2(\Gamma_{13}\gamma+\Gamma_{23}\zeta+\Gamma_{33}\psi)\\&\quad+\frac{2}{3\Omega}[p'(ak_{22}-hk_{12})+q'(-ak_{12}+hk_{11})\qquad],\end{aligned}$$

$$\frac{d\Gamma_{23}}{ds}=\Gamma_{111}p'+\Gamma_{021}q'+\Gamma_{012}r'+(\Gamma_{12}\gamma+\Gamma_{22}\zeta+\Gamma_{23}\psi)+(\Gamma_{13}\beta+\Gamma_{23}\eta+\Gamma_{33}\chi)$$
$$+\frac{1}{3\Omega}[p'(-2ak_{23}+hk_{13}+gk_{12})+q'(ak_{13}-gk_{11})+r'(ak_{12}-hk_{11})],$$

$$\frac{d\Gamma_{31}}{ds}=\Gamma_{201}p'+\Gamma_{111}q'+\Gamma_{102}r'+(\Gamma_{13}\alpha+\Gamma_{23}\xi+\Gamma_{33}\phi)+(\Gamma_{11}\gamma+\Gamma_{12}\zeta+\Gamma_{13}\psi)$$
$$+\frac{1}{3\Omega}[p'(hk_{23}-gk_{22})+q'(ak_{23}-2hk_{13}+gk_{12})+r'(-ak_{22}+hk_{12})],$$

$$\frac{d\Gamma_{12}}{ds}=\Gamma_{210}p'+\Gamma_{120}q'+\Gamma_{111}r'+(\Gamma_{11}\beta+\Gamma_{12}\eta+\Gamma_{13}\chi)+(\Gamma_{12}\alpha+\Gamma_{22}\xi+\Gamma_{23}\phi)$$
$$+\frac{1}{3\Omega}[p'(-hk_{33}+gk_{23})+q'(-ak_{33}+gk_{13})+r'(ak_{23}+hk_{13}-2gk_{12})].$$

Sets of quantities occur, admitting of abbreviated analytical expression. In connection with the quantities of the type α, β, γ, defined in § 173, we write

$$\left.\begin{aligned}\alpha_i p_j'+\beta_i q_j'+\gamma_i r_j' &= \sum\sum \Gamma_{ij}p_i'p_j'=\bar{\gamma}_{ij}\\ \xi_i p_j'+\eta_i q_j'+\zeta_i r_j' &= \sum\sum \Delta_{ij}p_i'p_j'=\bar{\delta}_{ij}\\ \phi_i p_j'+\chi_i q_j'+\psi_i r_j' &= \sum\sum \Theta_{ij}p_i'p_j'=\bar{\theta}_{ij}\end{aligned}\right\}.$$

(These magnitudes $\bar{\gamma}_{ij}$, $\bar{\delta}_{ij}$, $\bar{\theta}_{ij}$, are of persistent recurrence in the theory of geodesic parallels*. Also, we note that $\bar{\gamma}_{ii}=-p_i''$, $\bar{\delta}_{ii}=-q_i''$, $\bar{\theta}_{ii}=-r_i''$). And, in connection with the magnitudes u_1, u_2, u_3, we write

$$\left.\begin{aligned}Ap_l'+Hq_l'+Gr_l' &= u_1^{(l)}\\ Hp_l'+Bq_l'+Fr_l' &= u_2^{(l)}\\ Gp_l'+Fq_l'+Cr_l' &= u_3^{(l)}\end{aligned}\right\}.$$

Conformably with the notation (§ 159) for surface-variables, let

$$\xi_{ij}=q_i'r_j'-r_i'q_j',\quad \eta_{ij}=r_i'p_j'-p_i'r_j',\quad \zeta_{ij}=p_i'q_j'-q_i'p_j';$$

and, further, take symbols $K_1(l, mn)$, $K_2(l, mn)$, $K_3(l, mn)$, with the defined significance

$$K_1(l, mn)=\begin{vmatrix} a, & p_l', & k_{11}\xi_{mn}+k_{12}\eta_{mn}+k_{13}\zeta_{mn}\\ h, & q_l', & k_{12}\xi_{mn}+k_{22}\eta_{mn}+k_{23}\zeta_{mn}\\ g, & r_l', & k_{13}\xi_{mn}+k_{23}\eta_{mn}+k_{33}\zeta_{mn}\end{vmatrix},$$

while $K_2(l, mn)$ is obtained from $K_1(l, mn)$ by substituting h, b, f, for the constituents of the first column, and $K_3(l, mn)$ is similarly obtained from $K_1(l, mn)$ by substituting g, f, c, for the constituents of the first column. Also, for brevity, we write

$$\sum(\Gamma_{ijk}\between p_l', q_l', r_l'\between p_m', q_m', r_m'\between p_n', q_n', r_n')=(\Gamma_{300}p_l'p_m'p_n'),$$
$$\sum(\Delta_{ijk}\between p_l', q_l', r_l'\between p_m', q_m', r_m'\between p_n', q_n', r_n')=(\Delta_{300}p_l'p_m'p_n'),$$
$$\sum(\Theta_{ijk}\between p_l', q_l', r_l'\between p_m', q_m', r_m'\between p_n', q_n', r_n')=(\Theta_{300}p_l'p_m'p_n').$$

* They are the magnitudes $g_\mu^{(ty)}$ of § 60 when the general amplitude is a region.

Then, taking three different directions p_l', q_l', r_l'; p_m', q_m', r_m'; p_n', q_n', r_n'; which can be made to coincide in any manner that is convenient, we find

$$\left(\frac{d\Gamma_{11}}{ds_l}, \frac{d\Gamma_{22}}{ds_l}, \frac{d\Gamma_{33}}{ds_l}, \frac{d\Gamma_{23}}{ds_l}, \frac{d\Gamma_{31}}{ds_l}, \frac{d\Gamma_{12}}{ds_l} \between p_m', q_m', r_m' \between p_n', q_n', r_n'\right)$$
$$= (\alpha_m \bar{\gamma}_{ln} + \beta_m \bar{\delta}_{ln} + \gamma_m \bar{\theta}_{ln}) + (\alpha_n \bar{\gamma}_{lm} + \beta_n \bar{\delta}_{lm} + \gamma_n \bar{\theta}_{lm})$$
$$+ (\Gamma_{300} p_l' p_m' p_n') + \frac{1}{3\Omega} K_1(m, ln) + \frac{1}{3\Omega} K_1(n, lm).$$

Similarly

$$\left(\frac{d\Delta_{11}}{ds_l}, \frac{d\Delta_{22}}{ds_l}, \frac{d\Delta_{33}}{ds_l}, \frac{d\Delta_{23}}{ds_l}, \frac{d\Delta_{31}}{ds_l}, \frac{d\Delta_{12}}{ds_l} \between p_m', q_m', r_m' \between p_n', q_n', r_n'\right)$$
$$= (\xi_m \bar{\gamma}_{ln} + \eta_m \bar{\delta}_{ln} + \zeta_m \bar{\theta}_{ln}) + (\xi_n \bar{\gamma}_{lm} + \eta_n \bar{\delta}_{lm} + \zeta_n \bar{\theta}_{lm})$$
$$+ (\Delta_{300} p_l' p_m' p_n') + \frac{1}{3\Omega} K_2(m, ln) + \frac{1}{3\Omega} K_2(n, lm);$$

and

$$\left(\frac{d\Theta_{11}}{ds_l}, \frac{d\Theta_{22}}{ds_l}, \frac{d\Theta_{33}}{ds_l}, \frac{d\Theta_{23}}{ds_l}, \frac{d\Theta_{31}}{ds_l}, \frac{d\Theta_{12}}{ds_l} \between p_m', q_m', r_m' \between p_n', q_n', r_n'\right)$$
$$= (\phi_m \bar{\gamma}_{ln} + \chi_m \bar{\delta}_{ln} + \psi_m \bar{\theta}_{ln}) + (\phi_n \bar{\gamma}_{lm} + \chi_n \bar{\delta}_{lm} + \psi_n \bar{\theta}_{lm})$$
$$+ (\Theta_{300} p_l' p_m' p_n') + \frac{1}{3\Omega} K_3(m, ln) + \frac{1}{3\Omega} K_3(n, lm).$$

The three left-hand sides will, as usual, be denoted by the respective symbols

$$\sum \frac{d\Gamma_{11}}{ds_l} p_m' p_n', \quad \sum \frac{d\Delta_{11}}{ds_l} p_m' p_n', \quad \sum \frac{d\Theta_{11}}{ds_l} p_m' p_n'.$$

II. The values of the first arc-derivatives of A, B, C, F, G, H, have been stated in § 172. Combinations of these derivatives, in a set of forms similar to those affecting the derivatives of Γ, Δ, Θ, occur. In particular, we infer

$$p_m' \frac{dA}{ds_l} + q_m' \frac{dH}{ds_l} + r_m' \frac{dG}{ds_l} = A\bar{\gamma}_{lm} + H\bar{\delta}_{lm} + G\bar{\theta}_{lm} + \alpha_l u_1^{(m)} + \xi_l u_2^{(m)} + \phi_l u_3^{(m)},$$

$$p_m' \frac{dH}{ds_l} + q_m' \frac{dB}{ds_l} + r_m' \frac{dF}{ds_l} = H\bar{\gamma}_{lm} + B\bar{\delta}_{lm} + F\bar{\theta}_{lm} + \beta_l u_1^{(m)} + \eta_l u_2^{(m)} + \chi_l u_3^{(m)},$$

$$p_m' \frac{dG}{ds_l} + q_m' \frac{dF}{ds_l} + r_m' \frac{dC}{ds_l} = G\bar{\gamma}_{lm} + F\bar{\delta}_{lm} + C\bar{\theta}_{lm} + \gamma_l u_1^{(m)} + \zeta_l u_2^{(m)} + \psi_l u_3^{(m)};$$

and

$$\sum \frac{dA}{ds_l} p_m' p_n' = u_1^{(n)} \bar{\gamma}_{lm} + u_2^{(n)} \bar{\delta}_{lm} + u_3^{(n)} \bar{\theta}_{lm} + u_1^{(m)} \bar{\gamma}_{lm} + u_2^{(m)} \bar{\delta}_{ln} + u_3^{(m)} \bar{\theta}_{lm}.$$

III. Corresponding combinations of the second arc-derivatives of A, B, C, F, G, H, arise in the discussion of geodesic triangles. For their complete

expression, we require one additional set of abbreviated symbols: and we take

$$\overline{\Psi}_1=\tfrac{1}{3}\frac{\partial}{\partial p'}\{\textstyle\sum\Psi_{ijk}p'^iq'^jr'^k\},\quad \overline{\Psi}_2=\tfrac{1}{3}\frac{\partial}{\partial q'}\{\textstyle\sum\Psi_{ijk}p'^iq'^jr'^k\},$$

$$\overline{\Psi}_3=\tfrac{1}{3}\frac{\partial}{\partial r'}\{\textstyle\sum\Psi_{ijk}p'^iq'^jr'^k\},$$

for $\Psi=\Gamma, \Delta, \Theta$, in turn.

We proceed, by first forming the second parametric derivatives of these primary magnitudes: they can be deduced from the first parametric derivatives as given in § 160. Then we take

$$\frac{d^2A}{ds^2}=A_1p''+A_2q''+A_3r''+\textstyle\sum A_{11}p'^2,$$

the summation being over the direction-variables; and the values of p'', q'', r'', A_i, are inserted. Similarly for the others. The final values are as follows:

$$\begin{aligned}\frac{d^2A}{ds^2}&=-\tfrac{2}{3}(k_{33}q'^2-2k_{23}q'r'+k_{22}r'^2)+2(A\overline{\Gamma}_1+H\overline{\Delta}_1+G\overline{\Theta}_1)\\&\quad+2(A, B, C, F, G, H\between\alpha, \xi, \phi)^2\\&\quad+4\alpha(A\alpha+H\xi+G\phi)+4\xi(A\beta+H\eta+G\chi)+4\phi(A\gamma+H\zeta+G\psi),\end{aligned}$$

$$\begin{aligned}\frac{d^2B}{ds^2}&=-\tfrac{2}{3}(k_{11}r'^2-2k_{31}p'r'+k_{33}p'^2)+2(H\overline{\Gamma}_2+B\overline{\Delta}_2+F\overline{\Theta}_2)\\&\quad+2(A, B, C, F, G, H\between\beta, \eta, \chi)^2\\&\quad+4\beta(H\alpha+B\xi+F\phi)+4\eta(H\beta+B\eta+F\chi)+4\chi(H\gamma+B\zeta+F\psi),\end{aligned}$$

$$\begin{aligned}\frac{d^2C}{ds^2}&=-\tfrac{2}{3}(k_{22}p'^2-2k_{12}p'q'+k_{11}q'^2)+2(G\overline{\Gamma}_3+F\overline{\Delta}_3+C\overline{\Theta}_3)\\&\quad+2(A, B, C, F, G, H\between\gamma, \zeta, \psi)^2\\&\quad+4\gamma(G\alpha+F\xi+C\phi)+4\zeta(G\beta+F\eta+C\chi)+4\psi(G\gamma+F\zeta+C\psi),\end{aligned}$$

$$\begin{aligned}\frac{d^2F}{ds^2}&=\tfrac{2}{3}(k_{11}q'r'-k_{12}p'r'-k_{13}p'q'+k_{23}p'^2)+(H\overline{\Gamma}_3+B\overline{\Delta}_3+F\overline{\Theta}_3)+(G\overline{\Gamma}_2+F\overline{\Delta}_2+C\overline{\Theta}_2)\\&\quad+2(A, B, C, F, G, H\between\beta, \eta, \chi\between\gamma, \zeta, \psi)\\&\quad+2\beta(G\alpha+F\xi+C\phi)+2\eta(G\beta+F\eta+C\chi)+2\chi(G\gamma+F\zeta+C\psi)\\&\quad+2\gamma(H\alpha+B\xi+F\phi)+2\zeta(H\beta+B\eta+F\chi)+2\psi(H\gamma+B\zeta+F\psi),\end{aligned}$$

$$\begin{aligned}\frac{d^2G}{ds^2}&=\tfrac{2}{3}(k_{22}r'p'-k_{23}p'q'-k_{12}q'r'+k_{13}q'^2)+(G\overline{\Gamma}_1+F\overline{\Delta}_1+C\overline{\Theta}_1)+(A\overline{\Gamma}_3+H\overline{\Delta}_3+G\overline{\Theta}_3)\\&\quad+2(A, B, C, F, G, H\between\gamma, \zeta, \psi\between\alpha, \xi, \phi)\\&\quad+2\gamma(A\alpha+H\xi+G\phi)+2\zeta(A\beta+H\eta+G\chi)+2\psi(A\gamma+H\zeta+G\psi)\\&\quad+2\alpha(G\alpha+F\xi+C\phi)+2\xi(G\beta+F\eta+C\chi)+2\phi(G\gamma+F\zeta+C\psi),\end{aligned}$$

$$\begin{aligned}\frac{d^2H}{ds^2}&=\tfrac{2}{3}(k_{33}p'q'-k_{13}q'r'-k_{23}p'r'+k_{12}r'^2)+(A\overline{\Gamma}_2+H\overline{\Delta}_2+G\overline{\Theta}_2)+(H\overline{\Gamma}_1+B\overline{\Delta}_1+F\overline{\Theta}_1)\\&\quad+2(A, B, C, F, G, H\between\alpha, \xi, \phi\between\beta, \eta, \chi)\\&\quad+2\alpha(H\alpha+B\xi+F\phi)+2\xi(H\beta+B\eta+F\chi)+2\phi(H\gamma+B\zeta+F\psi)\\&\quad+2\beta(A\alpha+H\xi+G\phi)+2\eta(A\beta+H\eta+G\chi)+2\chi(A\gamma+H\zeta+G\psi).\end{aligned}$$

(The values can be checked by interchange of parameters, simultaneously with all necessarily associated interchanges of magnitudes dependent on the parameters.)

In the ensuing investigations, we shall require quantities of the type

$$\sum \frac{d^2A}{ds_l^2} p_m' p_n', \quad \sum \frac{d^2A}{ds_l^2} p_m'^2,$$

the second being derivable from the first, by making the two directions p_m', q_m', r_m', and p_n', q_n', r_n', coincide. It will therefore suffice to state the value of the first: we find

$$\begin{aligned}\sum \frac{d^2A}{ds_l^2} p_m' p_n' = &-\tfrac{2}{3}(k_{11}, k_{22}, k_{33}, k_{23}, k_{31}, k_{12} \between \xi_{lm}, \eta_{lm}, \zeta_{lm} \between \xi_{ln}, \eta_{ln}, \zeta_{ln}) \\ &+ u_1^{(m)}(\Gamma_{300} p_l'^2 p_n') + u_2^{(m)}(\Delta_{300} p_l'^2 p_n') + u_3^{(m)}(\Theta_{300} p_l'^2 p_n') \\ &+ u_1^{(n)}(\Gamma_{300} p_l'^2 p_m') + u_2^{(n)}(\Delta_{300} p_l'^2 p_m') + u_3^{(n)}(\Theta_{300} p_l'^2 p_m') \\ &+ 2(A, B, C, F, G, H \between \bar{\gamma}_{lm}, \bar{\delta}_{lm}, \bar{\theta}_{lm} \between \bar{\gamma}_{ln}, \bar{\delta}_{ln}, \bar{\theta}_{ln}) \\ &+ 2u_1^{(m)}(\alpha_l \bar{\gamma}_{ln} + \beta_l \bar{\delta}_{ln} + \gamma_l \bar{\theta}_{ln}) + 2u_1^{(n)}(\alpha_l \bar{\gamma}_{lm} + \beta_l \bar{\delta}_{lm} + \gamma_l \bar{\theta}_{lm}) \\ &+ 2u_2^{(m)}(\xi_l \bar{\gamma}_{ln} + \eta_l \bar{\delta}_{ln} + \zeta_l \bar{\theta}_{ln}) + 2u_2^{(n)}(\xi_l \bar{\gamma}_{lm} + \eta_l \bar{\delta}_{lm} + \zeta_l \bar{\theta}_{lm}) \\ &+ 2u_3^{(m)}(\phi_l \bar{\gamma}_{ln} + \chi_l \bar{\delta}_{ln} + \psi_l \bar{\theta}_{ln}) + 2u_3^{(n)}(\phi_l \bar{\gamma}_{lm} + \chi_l \bar{\delta}_{lm} + \psi_l \bar{\theta}_{lm}).\end{aligned}$$

In particular, if m or n should be the same as l, one set of the surface-variables ξ, η, ζ, vanishes; and then the term containing the symbols k_{ij} disappears.

213. Occasions arise when it is necessary to consider derivatives of the primary magnitudes taken concurrently in different regional directions: thus we shall need (§ 233) the magnitudes of the type $\dfrac{d^2A}{ds_1 ds_2}$. The actual directions ds_1 and ds_2 at O are independent of one another. The successive arc-derivatives of the parameters p, q, r, taken along OA alone, are expressible by means of the intrinsic equations of the geodesic OA, in terms of the initial values p_1', q_1', r_1', at O; and likewise for the corresponding derivatives, taken along OB alone, in terms of initial values p_2', q_2', r_2', at O. When we take

$$\frac{d^2A}{ds_1 ds_2} = \frac{d}{ds_1}\{2(A\alpha_2 + H\xi_2 + G\phi_2)\},$$

where (§ 173)

$$\alpha_2 = \Gamma_{11} p_2' + \Gamma_{12} q_2' + \Gamma_{13} r_2',$$

with similar values for ξ_2 and ϕ_2, the developed expression involves magnitudes $\dfrac{dp_2'}{ds_1}$, $\dfrac{dq_2'}{ds_1}$, $\dfrac{dr_2'}{ds_1}$; consequently it is incumbent to have, either explicitly or implicitly, a law establishing a relation between the length of the variable arc OU and a direction through the variable point U along OA corresponding to the direction OB through O. The simplest relation, for immediate needs, is that of geodesic parallelism (§§ 221, 222, 233, *post*). Even within this relation,

there are various types of parallelism, each leading to its own set of complete values of direction-variables for a parallel. But, in all the assigned types, the two leading terms in the direction-variables are the same, being given (up to the first order) by

$$P_2' = p_2' - t\bar{\gamma}_{12} + \dots, \quad Q_2' = q_2' - t\bar{\delta}_{12} + \dots, \quad R_2' = r_2' - t\bar{\theta}_{12} + \dots,$$

always subject to the permanent condition

$$\sum A_U P_2'^2 = 1$$

at the originating point U of the new geodesic; the further terms, involving squares and higher powers of t, have coefficients which differ according to the type of parallelism formulated. In such circumstances, the quantity $\dfrac{dp_2'}{ds_1}$

$$= \operatorname*{Lim}_{t \to 0} \left\{ \frac{1}{t} (P_2' - p_2') \right\} = -\bar{\gamma}_{12},$$

whatever type of parallelism be adopted, while $\dfrac{d^2 p_2'}{ds_1^2}$ and higher derivatives will be affected by the selected type of parallelism.

In the same way, and under the same assumptions, we have, for the first variation of p_1' along OB, the value

$$\frac{dp_1'}{ds_2} = -\bar{\gamma}_{12}.$$

Accordingly we infer that, when a law of parallelism is assigned for regional geodesics drawn at points U along OA and for regional geodesics drawn at points V along OB,

$$\frac{dp_2'}{ds_1} = \frac{dp_1'}{ds_2} = -\bar{\gamma}_{12} = -\sum \Gamma_{11} p_1' p_2',$$

$$\frac{dq_2'}{ds_1} = \frac{dq_1'}{ds_2} = -\bar{\delta}_{12} = -\sum \Delta_{11} p_1' p_2',$$

$$\frac{dr_2'}{ds_1} = \frac{dr_1'}{ds_2} = -\bar{\theta}_{12} = -\sum \Theta_{11} p_1' p_2'.$$

It follows that, if W denote any function solely of the position in the region,

$$\frac{d^2 W}{ds_1 ds_2} = \frac{d^2 W}{ds_2 ds_1}.$$

When application is made to the primary magnitudes, analysis similar to that in § 212 leads to the expressions

$$\begin{aligned}
\frac{d^2 A}{ds_1 ds_2} = & -\tfrac{2}{3}\{k_{33} q_1' q_2' - k_{23}(q_1' r_2' + q_2' r_1') + k_{22} r_1' r_2'\} \\
& + 2\{A(\Gamma_{300} p_1' p_2') + H(\Delta_{300} p_1' p_2') + G(\Theta_{300} p_1' p_2')\} \\
& + 2(A, B, C, F, G, H \between \alpha_1, \xi_1, \phi_1 \between \alpha_2, \xi_2, \phi_2) \\
& + 2\alpha_1(A\alpha_2 + H\xi_2 + G\phi_2) + 2\xi_1(A\beta_2 + H\eta_2 + G\chi_2) + 2\phi_1(A\gamma_2 + H\zeta_2 + G\psi_2) \\
& + 2\alpha_2(A\alpha_1 + H\xi_1 + G\phi_1) + 2\xi_2(A\beta_1 + H\eta_1 + G\chi_1) + 2\phi_2(A\gamma_1 + H\zeta_1 + G\psi_1),
\end{aligned}$$

with like expressions for the same derivatives of B and of C, to be obtained also by circular permutations of the symbols p, q, r; and

$$\begin{aligned}
\frac{d^2F}{ds_1 ds_2} = {} & \tfrac{1}{3}\{2k_{23}p_1'p_2' - k_{12}(p_1'r_2' + p_2'r_1') - k_{13}(p_1'q_2' + p_2'q_1') + k_{11}(q_1'r_2' + q_2'r_1')\} \\
& + H(\Gamma_{201}p_1'p_2') + B(\Delta_{201}p_1'p_2') + F(\Theta_{201}p_1'p_2') \\
& + G(\Gamma_{210}p_1'p_2') + F(\Delta_{210}p_1'p_2') + C(\Theta_{210}p_1'p_2') \\
& + (A, B, C, F, G, H \between \beta_1, \eta_1, \chi_1 \between \gamma_2, \zeta_2, \psi_2) \\
& + (A, B, C, F, G, H \between \beta_2, \eta_2, \chi_2 \between \gamma_1, \zeta_1, \psi_1) \\
& + \beta_1(G\alpha_2 + F\xi_2 + C\phi_2) + \eta_1(G\beta_2 + F\eta_2 + C\chi_2) + \chi_1(G\gamma_2 + F\zeta_2 + C\psi_2) \\
& + \beta_2(G\alpha_1 + F\xi_1 + C\phi_1) + \eta_2(G\beta_1 + F\eta_1 + C\chi_1) + \chi_2(G\gamma_1 + F\zeta_1 + C\psi_1) \\
& + \gamma_1(H\alpha_2 + B\xi_2 + F\phi_2) + \zeta_1(H\beta_2 + B\eta_2 + F\chi_2) + \psi_1(H\gamma_2 + B\zeta_2 + F\psi_2) \\
& + \gamma_2(H\alpha_1 + B\xi_1 + F\phi_1) + \zeta_2(H\beta_1 + B\eta_1 + F\chi_1) + \psi_2(H\gamma_1 + B\zeta_1 + F\psi_1),
\end{aligned}$$

with like expressions for the same derivatives of G and of H.

Moreover, if we take four directions represented by small arcs ds_k, ds_l, ds_m, ds_n, and use these results, we find the comprehensive relation

$$\begin{aligned}
\sum \frac{d^2A}{ds_k ds_l} p_m' p_n' = {} & -\tfrac{1}{3}(k_{ij} \between \xi_{km}, \eta_{km}, \zeta_{km} \between \xi_{ln}, \eta_{ln}, \zeta_{ln}) \\
& - \tfrac{1}{3}(k_{ij} \between \xi_{lm}, \eta_{lm}, \zeta_{lm} \between \xi_{kn}, \eta_{kn}, \zeta_{kn}) \\
& + u_1^{(m)}(\Gamma_{300}p_k'p_l'p_n') + u_2^{(m)}(\Delta_{300}p_k'p_l'p_n') + u_3^{(m)}(\Theta_{300}p_k'p_l'p_n') \\
& + u_1^{(n)}(\Gamma_{300}p_k'p_l'p_m') + u_2^{(n)}(\Delta_{300}p_k'p_l'p_m') + u_3^{(n)}(\Theta_{300}p_k'p_l'p_m') \\
& + (A \between \bar{\gamma}_{lm}, \delta_{lm}, \bar{\theta}_{lm} \between \bar{\gamma}_{kn}, \delta_{kn}, \bar{\theta}_{kn}) \\
& + (A \between \bar{\gamma}_{km}, \delta_{km}, \bar{\theta}_{km} \between \bar{\gamma}_{ln}, \delta_{ln}, \bar{\theta}_{ln}) \\
& + u_1^{(m)}\{(\alpha_l\bar{\gamma}_{kn} + \beta_l\delta_{kn} + \gamma_l\bar{\theta}_{kn}) + (\alpha_k\bar{\gamma}_{ln} + \beta_k\delta_{ln} + \gamma_k\bar{\theta}_{ln})\} \\
& + u_2^{(m)}\{(\xi_l\bar{\gamma}_{kn} + \eta_l\delta_{kn} + \zeta_l\bar{\theta}_{kn}) + (\xi_k\bar{\gamma}_{ln} + \eta_k\delta_{ln} + \zeta_k\bar{\theta}_{ln})\} \\
& + u_3^{(m)}\{(\phi_l\bar{\gamma}_{kn} + \chi_l\delta_{kn} + \psi_l\bar{\theta}_{kn}) + (\phi_k\bar{\gamma}_{ln} + \chi_k\delta_{ln} + \psi_k\bar{\theta}_{ln})\} \\
& + u_1^{(n)}\{(\alpha_l\bar{\gamma}_{km} + \beta_l\delta_{km} + \gamma_l\bar{\theta}_{km}) + (\alpha_k\bar{\gamma}_{lm} + \beta_k\delta_{lm} + \gamma_k\bar{\theta}_{lm})\} \\
& + u_2^{(n)}\{(\xi_l\bar{\gamma}_{km} + \eta_l\delta_{km} + \zeta_l\bar{\theta}_{km}) + (\xi_k\bar{\gamma}_{lm} + \eta_k\delta_{lm} + \zeta_k\bar{\theta}_{lm})\} \\
& + u_3^{(n)}\{(\phi_l\bar{\gamma}_{km} + \chi_l\delta_{km} + \psi_l\bar{\theta}_{km}) + (\phi_k\bar{\gamma}_{lm} + \psi_k\delta_{lm} + \psi_k\bar{\theta}_{lm})\}.
\end{aligned}$$

In particular, there are four sets of surface-variables, occurring in the terms connected with the four-index symbols k_{ij}; some of these vanish if k or l should be the same as one of the two numbers m and n, and then the corresponding terms in these symbols disappear.

Small geodesic triangle in a region.

214. Now consider the region in the vicinity of a point O. Through O draw two regional geodesics: OU, in a direction p_1', q_1', r_1', and OV, in a direction p_2', q_2', r_2'; and take small arcs $OU = x$ and $OV = y$ along those geodesics. Let

the points U and V be joined by a regional geodesic: so that a regional triangle is constituted. The orientation, defined by the two regional geodesics OU and OV at O, determines a geodesic surface of the region at O: in addition to finding the parts of the regional triangle UOV, we shall have to settle whether the regional geodesic UV lies wholly in that geodesic surface at O: or, alternatively, whether that regional geodesic UV is also the superficial geodesic joining the two points U and V on the surface.

Let the accurate length of the geodesic arc UV be denoted by t: it will be necessary to make approximations to the value of t, the quantities x and y being regarded as comparable small quantities of the same order of magnitude. Also let the accurate direction-variables of the regional geodesic UV at U in the direction UV be denoted by p', q', r': it will likewise be necessary to make approximations to the values of p', q', r'.

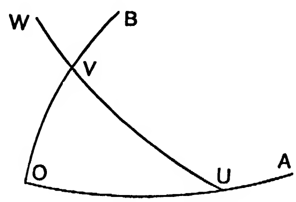

FIG. 22.

The point V can be reached from O either by a geodesic regional path OV or by a broken path, composed of the regional geodesic OU and the regional geodesic UV; and the values of the regional parameters specifying V must be the same by these two paths. As regards the parameter corresponding to p at O, the value at U is

$$p_U = p + xp_1' + \tfrac{1}{2}x^2p_1'' + \tfrac{1}{6}x^3p_1''' + \dots,$$

for the length of the geodesic arc OU is x; and the value at V, distant t from U along a regional geodesic in a direction p', q', r', at U, is

$$p_V = p_U + t(p')_U + \tfrac{1}{2}t^2(p'')_U + \tfrac{1}{6}t^3(p''')_U + \dots,$$

where the subscripts indicate the points at which the values of the quantities are to be taken: thus the value of the parameter at V, as attained by the path OUV,

$$\begin{aligned} &= p + xp_1' + \tfrac{1}{2}x^2p_1'' + \tfrac{1}{6}x^3p_1''' + \dots \\ &\qquad + t(p')_U + \tfrac{1}{2}t^2(p'')_U + \tfrac{1}{6}t^3(p''')_U + \dots. \end{aligned}$$

By the geodesic path OV, the value of the same parameter at V

$$= p + yp_2' + \tfrac{1}{2}y^2p_2'' + \tfrac{1}{6}y^3p_2''' + \dots.$$

Hence we have

$$\begin{aligned} &t(p')_U + \tfrac{1}{2}t^2(p'')_U + \tfrac{1}{6}t^3(p''')_U + \dots \\ &\qquad = (yp_2' - xp_1') + \tfrac{1}{2}(y^2p_2'' - x^2p_1'') + \tfrac{1}{6}(y^3p_2''' - x^3p_1''') + \dots. \end{aligned}$$

There are two other equations, arising similarly in association with the two other regional parameters: they are

$$\begin{aligned} &t(q')_U + \tfrac{1}{2}t^2(q'')_U + \tfrac{1}{6}t^3(q''')_U + \dots \\ &\qquad = (yq_2' - xq_1') + \tfrac{1}{2}(y^2q_2'' - x^2q_1'') + \tfrac{1}{6}(y^3q_2''' - x^3q_1''') + \dots, \\ &t(r')_U + \tfrac{1}{2}t^2(r'')_U + \tfrac{1}{6}t^3(r''')_U + \dots \\ &\qquad = (yr_2' - xr_1') + \tfrac{1}{2}(y^2r_2'' - x^2r_1'') + \tfrac{1}{6}(y^3r_2''' - x^3r_1''') + \dots. \end{aligned}$$

Also there is the permanent arc-relation at U which, for the regional geodesic UV, there takes the form

$$\sum A_U (p'^2)_U = 1.$$

For immediate purposes, it is found sufficient to retain small quantities of the third order.

In the first place, whatever their approximate values in terms of other quantities may prove to be, the values of the direction-variables at U along UV are accurate when actually denoted by p', q', r': thus

$$(p')_U = p', \quad (q')_U = q', \quad (r')_U = r',$$

accurately. Similarly

$$(p'')_U = -\sum (\Gamma_{11})_U p'^2,$$

where the values of $(\Gamma_{11})_U$ and other coefficients on the right-hand side are the values of Γ_{ij} at U: and such value is accurate. Also

$$(q'')_U = -\sum (\Delta_{11})_U p'^2, \quad (r'')_U = -\sum (\Theta_{11})_U p'^2;$$

$$(p''')_U = -\sum (\Gamma_{300})_U p'^3, \quad (q''')_U = -\sum (\Delta_{300})_U p'^3, \quad (r''')_U = -\sum (\Theta_{300})_U p'^3,$$

all of them accurate. We have to make, from these equations, approximations in successive orders of the small quantities.

As regards approximations involving only the first order, we neglect terms involving x^2, x^3, y^2, y^3, t^2, t^3; and there are three equations of the type

$$tp' = yp_2' - xp_1',$$

which, in this form, has significance only for the first order. For a first approximation, we can take

$$t = z + T, \quad p' = p_0' + P_1, \quad q' = q_0' + Q_1, \quad r' = r_0' + R_1,$$

where T is of an order higher than the first, and where P_1, Q_1, R_1, vanish with the small quantities x and y. Thus, up to the first order*, we have

$$tp' = zp_0',$$

and so for the other two equations; and therefore

$$zp_0' = yp_2' - xp_1', \quad zq_0' = yq_2' - xq_1', \quad zr_0' = yr_2' - xr_1'.$$

Also, the finite terms, in the permanent arc-relation at U for the geodesic UV, provide the equation

$$\sum Ap_0'^2 = 1.$$

We denote the (accurate) value of the angle UOV by ϵ, so that

$$\cos\epsilon = \sum Ap_1'p_2', \quad \sin^2\epsilon = \sum a(q_1'r_2' - r_1'q_2')^2.$$

* This form of phrase will frequently be used, to imply inclusion of the specified order.

We also denote by U and V the (accurate) values of the internal angles of the regional triangle at U and V respectively, so that

$$\cos(\pi - U) = \sum A_U p'(p_1')_U, \quad \cos V = \sum A_V (p_2')_V P'_V,$$

where $(p_1')_U$ is a direction-variable of the geodesic OUA at U in the direction UA and $(p_2')_V$ is a direction-variable of the geodesic OVB at V in the direction VB, while P'_V is used to denote the same direction-variable of the geodesic UVW at V in the direction VW. Let U_0 and V_0 represent the main values of the angles U and V; that is, the values of these angles in the limit when x and y vanish (so that they become inclinations at O): then

$$-\cos U_0 = \sum A p_1' p_0', \quad \cos V_0 = \sum A p_2' p_0'.$$

Then from the foregoing relations constituting the first approximation, we have the set of results

$$z^2 = x^2 + y^2 - 2xy\cos\epsilon,$$

$$\frac{\sin U_0}{y} = \frac{\sin V_0}{x} = \frac{\sin\epsilon}{z}, \quad \epsilon + U_0 + V_0 = \pi,$$

$$z\cos U_0 = x - y\cos\epsilon, \quad z\cos V_0 = y - x\cos\epsilon, \quad z = x\cos U_0 + y\cos V_0.$$

These relations are the characteristic relations satisfied by the sides x, y, z, and the angles V_0, U_0, ϵ, of a plane rectilinear triangle.

As x, y, ϵ, are given initially, as well as the direction-variables of OU and OV, we can regard z, U_0, V_0, and the direction-variables p_0', q_0', r_0', as known quantities.

Second approximation.

215. For approximations of the second order of small quantities, we neglect terms in x^3, y^3, t^3, and higher powers; and, in its first form, the p-equation now becomes

$$tp' + \tfrac{1}{2}t^2(p'')_U = (yp_2' - xp_1') + \tfrac{1}{2}(y^2p_2'' - x^2p_1'');$$

the terms, which ultimately are of the first order, will disappear on account of the first-order approximation already made. Now

$$-(p'')_U = \sum (\Gamma_{11})_U p'^2,$$

accurately, where on the right-hand side the values of the coefficients Γ_{ij} are to be taken at U. But, in the approximating equation, this magnitude is multiplied by t^2, a quantity of the second order (and higher orders); and therefore, in the expression for $t^2(p'')_U$, the adequate approximation will be obtained by taking the values of the coefficients Γ_{ij} at O, and

$$p' = p_0', \quad q' = q_0', \quad r' = r_0', \quad t = z.$$

Hence, in the second order, we have

$$\begin{aligned}-t^2(p'')_U &= z^2 \sum \Gamma_{11} p_0'^2 \\ &= \sum \Gamma_{11} (y p_2' - x p_1')^2 \\ &= -x^2 p_1'' - y^2 p_2'' - 2xy \sum \Gamma_{11} p_1' p_2' \\ &= -x^2 p_1'' - y^2 p_2'' - 2xy\bar{\gamma}_{12},\end{aligned}$$

with the significance for $\bar{\gamma}$ assigned in § 212.

Again,

$$tp' = (z + T)(p_0' + P_1) = zp_0' + zP_1 + p_0'T,$$

up to the second order, for TP_1 is certainly of the third order.

Thus the approximating p-equation, in this order, is

$$\begin{aligned}zp_0' + zP_1 + p_0'T + \tfrac{1}{2}(x^2 p_1'' + y^2 p_2'' + 2xy\bar{\gamma}_{12}) \\ = (yp_2' - xp_1') + \tfrac{1}{2}(y^2 p_2'' - x^2 p_1'').\end{aligned}$$

Here, the terms of the first order cancel, as is to be expected: and

$$\begin{aligned}xy\gamma_{12} + x^2 p_1'' &= xy \sum \Gamma_{11} p_1' p_2' - x^2 \sum \Gamma_{11} p_1'^2 \\ &= x\{\alpha_1(yp_2' - xp_1') + \beta_1(yq_2' - xq_1') + \gamma_1(yr_2' - xr_1')\} \\ &= xz(\alpha_1 p_0' + \beta_1 q_0' + \gamma_1 r_0') = xz\bar{\gamma}_{01};\end{aligned}$$

consequently, the approximate p-equation is

$$p_0'T + z(P_1 + x\bar{\gamma}_{01}) = 0.$$

Similarly, the approximate (second-order) equations for q and r are

$$\begin{aligned}q_0'T + z(Q_1 + x\bar{\delta}_{01}) &= 0, \\ r_0'T + z(R_1 + x\bar{\theta}_{01}) &= 0.\end{aligned}$$

Next, account of the approximation has to be taken in the permanent arc-relation

$$\sum A_U p'^2 = 1.$$

In approximation, there is a set of terms represented by

$$\sum A p_0'^2 = 1,$$

free from small quantities; it provides a condition already satisfied. Therefore, in the next stage for this relation, we keep small quantities of the first order. Now, up to this order,

$$A_U = A + x\frac{dA}{ds_1},$$

and similarly for the other primary magnitudes at U; also

$$\begin{aligned}p'^2 &= p_0'^2 + 2p_0'P_1, \\ p'q' &= p_0'q_0' + p_0'Q_1 + q_0'P_1,\end{aligned}$$

and similarly for the other combinations; therefore the arc-relation, up to this order, becomes

$$\sum Ap_0'^2 + x\sum \frac{dA}{ds_1} p_0'^2 + 2\{u_1^{(0)}P_1 + u_2^{(0)}Q_1 + u_3^{(0)}R_1\} = 1,$$

with the significance for the symbols $u^{(0)}$ assigned on p. 66. Also, by taking $l=1$, $m=0$, $n=0$, in the result given in § 212, II,

$$\sum \frac{dA}{ds_1} p_0'^2 = 2\{u_1^{(0)}\bar{\gamma}_{01} + u_2^{(0)}\bar{\delta}_{01} + u_3^{(0)}\bar{\theta}_{01}\};$$

thus the approximating form of the arc-relation, to this order, becomes

$$u_1^{(0)}\{P_1 + x\bar{\gamma}_{01}\} + u_2^{(0)}\{Q_1 + x\bar{\delta}_{01}\} + u_3^{(0)}\{R_1 + x\bar{\theta}_{01}\} = 0.$$

Let the approximate equations for p, q, r, be multiplied by $u_1^{(0)}$, $u_2^{(0)}$, $u_3^{(0)}$, respectively and the results be added; then, as

$$u_1^{(0)}p_0' + u_2^{(0)}q_0' + u_3^{(0)}r_0' = \sum Ap_0'^2 = 1,$$

we have

$$T=0,$$

that is, up to the (retained) second order of small quantities. Hence T is of at least the third order of small quantities.

Moreover, using this result in the approximate equations for p, q, r, we have

$$z(P_1 + x\bar{\gamma}_{01}) = 0, \quad z(Q_1 + x\bar{\delta}_{01}) = 0, \quad x(R_1 + x\bar{\theta}_{01}) = 0;$$

and therefore

$$P_1 + x\bar{\gamma}_{01} = 0, \quad Q_1 + x\bar{\delta}_{01} = 0, \quad R_1 + x\bar{\theta}_{01} = 0,$$

that is, up to the first order of small quantities, because the precedent approximation is up to the second order. Accordingly, for further approximation, we can take

$$P_1 = -x\bar{\gamma}_{01} + P,$$

and similarly for Q_1, R_1; that is,

$$\left.\begin{aligned} p' &= p_0' - x\bar{\gamma}_{01} + P \\ q' &= q_0' - x\bar{\delta}_{01} + Q \\ r' &= r_0' - x\bar{\theta}_{01} + R \end{aligned}\right\},$$

where P, Q, R, are of the second order (and higher orders) of small quantities; and we have seen, above, that T is of the third order (and higher orders) in the small quantities.

These equations (as follows from § 213) shew that, at U, the regional geodesic UV satisfies the first-order test for parallelism to the regional direction p_0', q_0', r_0', through O.

Third-order approximation.

216. For approximations of the third order, we neglect all powers and combinations of x, y, t, of higher than the third degree in all; thus, in the initial form, the approximate p-equation is

$$tp'+\tfrac{1}{2}t^2(p'')_U+\tfrac{1}{6}t^3(p''')_U=(yp_2'-xp_1')+\tfrac{1}{2}(y^2p_2''-x^2p_1'')+\tfrac{1}{6}(y^3p_2'''-x^3p_1''').$$

The terms, ultimately of the first order, must balance because of the first approximation; and the terms, ultimately of the second order, must balance because of the second approximation. The residue provides the contribution of the p-equation towards the complete approximation for the third order.

The terms on the left-hand side must be taken in succession. The first term gives

$$tp'=(z+T)p';$$

as T is of the third order at least, we have

$$\begin{aligned} tp' &= zp'+Tp_0' \\ &= z(p_0'-x\bar{\gamma}_{01}+P)+Tp_0' \\ &= yp_2'-xp_1'-xz\bar{\gamma}_{01}+zP+Tp_0', \end{aligned}$$

accurately up to the third order.

The second term $\frac{1}{2}t^2(p'')_U$, contains the factor t^2, that is, $(z+T)^2$: or, as T is of the third order, we can take $t^2=z^2$ for a third-order approximation. As the factor now contains explicitly only the second power of the small quantity z, quantities up to the first order must be retained in $(p'')_U$ for the present approximation. Now, always

$$-(p'')_U=\sum(\Gamma_{11})_U p'^2,$$

accurately; and therefore, as

$$(\Gamma_{ij})_U=\Gamma_{ij}+x\frac{d\Gamma_{ij}}{ds_1},$$

$$p'=p_0'-x\bar{\gamma}_{01},\quad q'=q_0'-x\bar{\delta}_{01},\quad r'=r_0'-x\bar{\theta}_{01},$$

up to the first order, the required approximation for $-(p'')_U$ is

$$\begin{aligned} -(p'')_U &= \sum\left(\Gamma_{11}+x\frac{d\Gamma_{11}}{ds_1}\right)(p_0'^2-2xp_0'\bar{\gamma}_{01}) \\ &= \sum\Gamma_{11}p_0'^2+x\sum\frac{d\Gamma_{11}}{ds_1}p_0'^2-2x(\alpha_0\bar{\gamma}_{01}+\beta_0\bar{\delta}_{01}+\gamma_0\bar{\theta}_{01}). \end{aligned}$$

Consequently, up to the third order,

$$-t^2(p'')_U=z^2\sum\Gamma_{11}p_0'^2+xz^2\sum\frac{d\Gamma_{11}}{ds_1}p_0'^2-2xz^2(\alpha_0\bar{\gamma}_{01}+\beta_0\bar{\delta}_{01}+\gamma_0\bar{\theta}_{01}).$$

But

$$\begin{aligned}
z^2(\sum \Gamma_{11} p_0'^2) &= \sum \Gamma_{11}(z p_0')^2 \\
&= \sum \Gamma_{11}(y p_2' - x p_1')^2 \\
&= y^2(\sum \Gamma_{11} p_2'^2) - 2xy\bar{\gamma}_{12} + x^2(\sum \Gamma_{11} p_1'^2) \\
&= -y^2 p_2'' - x^2 p_1'' - 2xy\bar{\gamma}_{12} \\
&= -y^2 p_2'' + x^2 p_1'' - 2x(y\bar{\gamma}_{12} + x p_1'')\,;
\end{aligned}$$

also, as before (p. 75),

$$y\bar{\gamma}_{12} + x p_1'' = z\bar{\gamma}_{01},$$

so that

$$z^2(\sum \Gamma_{11} p_0'^2) = -y^2 p_2'' + x^2 p_1'' - 2xz\bar{\gamma}_{01}.$$

It follows that, when the results are combined for the third-order approximation, we have

$$\tfrac{1}{2}t^2(p'')_U = \tfrac{1}{2}(y^2 p_2'' - x^2 p_1'') + xz\bar{\gamma}_{01} - \tfrac{1}{2}xz^2 W,$$

where

$$W = \left(\sum \frac{d\Gamma_{11}}{ds_1} p_0'^2\right) - 2(\alpha_0\bar{\gamma}_{01} + \beta_0\bar{\delta}_{01} + \gamma_0\bar{\theta}_{01}).$$

In § 212, an expression has been obtained for $\sum \dfrac{d\Gamma_{11}}{ds_l} p_m' p_n'$; hence, taking $l=1$, $m=0$, $n=0$, in that expression, we have the value of the first term in W in the form

$$2(\alpha_0\bar{\gamma}_{01} + \beta_0\bar{\delta}_{01} + \gamma_0\bar{\theta}_{01}) + (\Gamma_{300} p_0'^2 p_1') + \frac{2}{3\Omega} K_1(0, 10).$$

Accordingly

$$W = (\Gamma_{300} p_0'^2 p_1') + \frac{2}{3\Omega} K_1(0, 10)\,;$$

and therefore

$$\tfrac{1}{2}t^2(p'')_U = \tfrac{1}{2}(y^2 p_2'' - x^2 p_1'') + xz\bar{\gamma}_{01} - \tfrac{1}{2}xz^2\{(\Gamma_{300} p_0'^2 p_1') + \frac{2}{3\Omega} K_1(0, 10)\},$$

which gives the value to be substituted in the p-equation for the third-order approximation.

The third term on the left-hand side of the approximate p-equation, being $\tfrac{1}{6}t^3(p''')_U$, contains a factor t^3 which, to the order under consideration, can be taken as z^3, already of the third order. Hence, for the present approximation, only that part of $(p''')_U$ which is independent of small quantities need be retained; or, as

$$-(p''')_U = \sum (\Gamma_{300})_U p'^3$$

accurately, our approximation is made by taking

$$-(p''')_U = \sum \Gamma_{300} p_0'^3.$$

Thus, for the third term in the left-hand side, we have

$$\tfrac{1}{6}t^3(p''')_U = -\tfrac{1}{6}z^3\sum \Gamma_{300}p_0'^3$$
$$= -\tfrac{1}{6}\sum \Gamma_{300}(yp_2' - xp_1')^3.$$

But

$$(yp_2' - xp_1')^3 = y^3p_2'^3 - x^3p_1'^3 - 3(yp_2' - xp_1')yp_2'xp_1'$$
$$= y^3p_2'^3 - x^3p_1'^3 - 3xyzp_0'p_1'p_2' :$$

and therefore

$$\tfrac{1}{6}t^3(p''')_U = \tfrac{1}{6}(y^3p_2''' - x^3p_1''') + \tfrac{1}{2}xyz(\Gamma_{300}p_0'p_1'p_2'),$$

as the value to be substituted in the p-equation.

(i) Now let the values of the three t-terms be substituted in the p-equation; it becomes

$$\begin{aligned}(yp_2' - xp_1') - xz\bar{\gamma}_{01} + zP + Tp_0'&\\ + \tfrac{1}{2}(y^2p_2'' - x^2p_1'') + xz\bar{\gamma}_{01} - \tfrac{1}{2}xz^2&\left\{(\Gamma_{300}p_0'^2p_1') + \frac{2}{3\Omega}K_1(0, 10)\right\}\\ + \tfrac{1}{6}(y^3p_2''' - x^3p_1''') + \tfrac{1}{2}xyz&(\Gamma_{300}p_0'p_1'p_2')\\ = (yp_2' - xp_1') + \tfrac{1}{2}(y^2p_2'' - x^2p_1'') &+ \tfrac{1}{6}(y^3p_2''' - x^3p_1''').\end{aligned}$$

Consequently,

$$zP + Tp_0' = \tfrac{1}{2}xz^2\left\{(\Gamma_{300}p_0'^2p_1') + \frac{2}{3\Omega}K_1(0, 10)\right\} - \tfrac{1}{2}xyz(\Gamma_{300}p_0'p_1'p_2').$$

Again, in the aggregate of terms involving the magnitudes Γ_{ijk},

$$zp_0' - yp_2' = -xp_1',$$

so that

$$z(\Gamma_{300}p_0'^2p_1') - y(\Gamma_{300}p_0'p_1'p_2') = -x(\Gamma_{300}p_0'p_1'^2);$$

and therefore the third-order approximating form of the p-equation is

$$Tp_0' + z\{P + \tfrac{1}{2}x^2(\Gamma_{300}p_0'p_1'^2)\} = \frac{1}{3\Omega}xz^2K_1(0, 10).$$

Similar treatment of the q-equation and of the r-equation, for the third-order approximation, leads to the results

$$Tq_0' + z\{Q + \tfrac{1}{2}x^2(\Delta_{300}p_0'p_1'^2)\} = \frac{1}{3\Omega}xz^2K_2(0, 10),$$

$$Tr_0' + z\{R + \tfrac{1}{2}x^2(\Theta_{300}p_0'p_1'^2)\} = \frac{1}{3\Omega}xz^2K_3(0, 10).$$

(ii) Also, the permanent arc-relation at U for the regional geodesic UV provides its own contribution towards the general third-order approximation. In the relation

$$\sum A_U p'^2 = 1,$$

it is necessary to retain terms up to the second order of small quantities; consequently, for this purpose, there are values of the type

$$A_U = A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2}$$

for each of the primary magnitudes, and there are the three values

$$p' = p_0' - x\bar{\gamma}_{01} + P, \quad q' = q_0' - x\bar{\delta}_{01} + Q, \quad r' = r_0' - x\bar{\theta}_{01} + R.$$

When substitution is effected, and terms up to the second order of small quantities are retained, there are three sets to be taken into account.

The terms in the arc-relation, which are free from small quantities, lead to the condition

$$\sum A p_0'^2 = 1;$$

it is satisfied.

The terms in the arc-relation, which involve the first powers of small quantities, lead to the condition

$$\sum \frac{dA}{ds_1} p_0'^2 - 2\{u_1^{(0)}\bar{\gamma}_{01} + u_2^{(0)}\bar{\delta}_{01} + u_3^{(0)}\bar{\theta}_{01}\} = 0.$$

By the result in § 212, II, on inserting values $l=1$, $m=0$, $n=0$, we have

$$\sum \frac{dA}{ds_1} p_0'^2 = 2\{u_1^{(0)}\bar{\gamma}_{01} + u_2^{(0)}\bar{\delta}_{01} + u_3^{(0)}\bar{\theta}_{01}\},$$

so that the required condition is satisfied, as is to be expected after the earlier use of the arc-relation in framing the general second-order approximation.

The terms in the arc-relation, which involve the second powers of small quantities, lead to the condition

$$\sum A(2p_0'P + x^2\bar{\gamma}_{01}^2) - 2x^2\sum \frac{dA}{ds_1} p_0'\bar{\gamma}_{01} + \tfrac{1}{2}x^2\sum \frac{d^2A}{ds_1^2} p_0'^2 = 0.$$

The first summation-term on the left-hand side

$$= 2\{u_1^{(0)}P + u_2^{(0)}Q + u_3^{(0)}R\} + x^2\sum A\bar{\gamma}_{01}^2.$$

For the second summation-term, the sum in which can be written

$$\bar{\gamma}_{01}\left(p_0'\frac{dA}{ds_1} + q_0'\frac{dH}{ds_1} + r_0'\frac{dG}{ds_1}\right)$$
$$+\bar{\delta}_{01}\left(p_0'\frac{dH}{ds_1} + q_0'\frac{dB}{ds_1} + r_0'\frac{dF}{ds_1}\right)$$
$$+\bar{\theta}_{01}\left(p_0'\frac{dG}{ds_1} + q_0'\frac{dF}{ds_1} + r_0'\frac{dC}{ds_1}\right),$$

we use the relations in § 212, II, inserting the values $l=1$, $m=0$; thus

$$p_0'\frac{dA}{ds_1} + q_0'\frac{dH}{ds_1} + r_0'\frac{dG}{ds_1} = A\bar{\gamma}_{01} + H\bar{\delta}_{01} + G\bar{\theta}_{01} + a_1u_1^{(0)} + \xi_1u_2^{(0)} + \phi_1u_3^{(0)},$$

with like expressions for the other two, so that the sum can be taken in the form

$$\sum A\bar{\gamma}_{01}^2 + (\alpha_1\bar{\gamma}_{01} + \beta_1\bar{\delta}_{01} + \gamma_1\bar{\theta}_{01})\,u_1^{(0)} + (\xi_1\bar{\gamma}_{01} + \eta_1\bar{\delta}_{01} + \zeta_1\bar{\theta}_{01})\,u_2^{(0)} \\ + (\phi_1\bar{\gamma}_{01} + \chi_1\bar{\delta}_{01} + \psi_1\bar{\theta}_{01})\,u_3^{(0)};$$

and therefore the second summation-term on the left-hand side (with its sign) becomes

$$-2x^2\sum A\bar{\gamma}_{01}^2 \\ -2x^2\{(\alpha_1\bar{\gamma}_{01} + \beta_1\bar{\delta}_{01} + \gamma_1\bar{\theta}_{01})\,u_1^{(0)} + (\xi_1\bar{\gamma}_{01} + \eta_1\bar{\delta}_{01} + \zeta_1\bar{\theta}_{01})\,u_2^{(0)} \\ + (\phi_1\bar{\gamma}_{01} + \chi_1\bar{\delta}_{01} + \psi_1\bar{\theta}_{01})u_3^{(0)}\}.$$

The expression of the third summation-term on the left-hand side can be evaluated by taking $l=1$, $m=0$, $n=0$, in the result of § 212 for $\sum \dfrac{d^2A}{ds_1^{\,2}}\,p_m'p_n'$; and thus the third summation-term in the condition becomes

$$-\tfrac{1}{3}x^2\sum k_{11}\xi_{10}^2 \\ + x^2\{u_1^{(0)}(\Gamma_{300}p_0'p_1'^2) + u_2^{(0)}(\Delta_{300}p_0'p_1'^2) + u_3^{(0)}(\Theta_{300}p_0'p_1'^2)\} + x^2\sum A\bar{\gamma}_{01}^2 \\ + 2x^2\{(\alpha_1\bar{\gamma}_{01} + \beta_1\bar{\delta}_{01} + \gamma_1\bar{\theta}_{01})u_1^{(0)} + (\xi_1\bar{\gamma}_{01} + \eta_1\bar{\delta}_{01} + \zeta_1\bar{\theta}_{01})\,u_2^{(0)} \\ + (\phi_1\bar{\gamma}_{01} + \chi_1\bar{\delta}_{01} + \psi_1\bar{\theta}_{01})u_3^{(0)}\}.$$

When these respective values are inserted, the surviving form of the condition arising out of the permanent arc-relation at U for the geodesic UV becomes

$$u_1^{(0)}\{2P + x^2(\Gamma_{300}p_0'p_1'^2)\} + u_2^{(0)}\{2Q + x^2(\Delta_{300}p_0'p_1'^2)\} + u_3^{(0)}\{2R + x^2(\Theta_{300}p_0'p_1'^2)\} \\ = \tfrac{1}{3}x^2\sum k_{11}\xi_{10}^2.$$

(iii) Thus there are four linear non-homogeneous equations involving the magnitudes T, P, Q, R. When the values of the three quantities of the type $2P + x^2(\Gamma_{300}p_0'p_1'^2)$, as given by the former conditions, are multiplied by $u_1^{(0)}$, $u_2^{(0)}$, $u_3^{(0)}$, respectively, when the results are added, and this last equation, as well as the relation

$$u_1^{(0)}p_0' + u_2^{(0)}q_0' + u_3^{(0)}r_0' = \sum Ap_0'^2 = 1,$$

is used, we find

$$T + \tfrac{1}{6}x^2z\sum k_{11}\xi_{10}^2 = \frac{1}{3\Omega}\,xz^2\{u_1^{(0)}K_1(0, 10) + u_2^{(0)}K_2(0, 10) + u_3^{(0)}K_3(0, 10)\}.$$

The values of the quantities K_1, K_2, K_3, on the right-hand side are given from p. 66, by the substitution $l=0$, $m=1$, $n=0$. The total coefficient of a quantity $k_{11}\xi_{10} + k_{12}\eta_{10} + k_{13}\zeta_{10}$

$$= (hr_0' - gq_0')\,u_1^{(0)} + (br_0' - fq_0')\,u_2^{(0)} + (fr_0' - cq_0')\,u_3^{(0)},$$

which, on the substitution of the values of $u_1^{(0)}$, $u_2^{(0)}$, $u_3^{(0)}$, is easily seen to be

zero ; and similarly for the total coefficients of the quantities $k_{12}\xi_{10}+k_{22}\eta_{10}+k_{23}\zeta_{10}$, $k_{13}\xi_{10}+k_{23}\eta_{10}+k_{33}\zeta_{10}$. Thus the right-hand side is zero ; and therefore

$$T=-\tfrac{1}{6}x^2z\sum k_{11}\xi_{10}^2.$$

Also, we have

$$z\xi_{10}=z(q_1'r_0'-r_1'q_0')=y(q_1'p_2'-p_1'q_2')=y\xi_{12},$$

and, similarly,

$$z\eta_{10}=y\eta_{12},\quad z\zeta_{10}=y\zeta_{12};$$

consequently, we have

$$T=-\tfrac{1}{6}\frac{x^2y^2}{z}\sum k_{11}\xi_{12}^2.$$

The Riemann measure of curvature of a region, called (§§ 65, 117) the sphericity and denoted by K, when estimated in the orientation determined by the directions OU and OV, is given by

$$K=\frac{\sum k_{11}\xi_{12}^2}{\sum a\xi_{12}^2}=\frac{1}{\sin^2\epsilon}\sum k_{11}\xi_{12}^2;$$

and therefore we have

$$T=-\tfrac{1}{6}K\frac{x^2y^2}{z}\sin^2\epsilon.$$

Thus, up to the third order of small quantities, the length of the third side of the regional triangle is

$$z-\tfrac{1}{6}K\frac{x^2y^2}{z}\sin^2\epsilon,$$

where $z^2=x^2+y^2-2xy\cos\epsilon$.

The identification, for the region, of the general significance (§ 65) of the magnitude K for an amplitude and its specific significance (§ 112) for a surface will be made later (§ 219). Meanwhile, it is to be noted the values of P, Q, R, now can be inferred, in the forms

$$\left.\begin{aligned}P&=-\tfrac{1}{2}x^2(\Gamma_{300}p_0'p_1'^2)+\tfrac{1}{6}p_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon+\frac{xz}{3\Omega}K_1(0,10)\\Q&=-\tfrac{1}{2}x^2(\Delta_{300}p_0'p_1'^2)+\tfrac{1}{6}q_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon+\frac{xz}{3\Omega}K_2(0,10)\\R&=-\tfrac{1}{2}x^2(\Theta_{300}p_0'p_1'^2)+\tfrac{1}{6}r_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon+\frac{xz}{3\Omega}K_3(0,10)\end{aligned}\right\},$$

the direction-variables p', q', r', at U of the regional geodesic UV in the direction UV being given by

$$p'=p_0'-x\bar{\gamma}_{01}+P,\quad q'=q_0'-x\bar{\delta}_{01}+Q,\quad r'=r_0'-x\bar{\theta}_{01}+R,$$

where

$$zp_0'=yp_2'-xp_1',\quad zq_0'=yq_2'-xq_1',\quad zr_0'=yr_2'-xr_1'.$$

These results give the direction-variables of the regional geodesic UV at U in the direction UV. Interchange of the directions of OU and OV at O, with simultaneous interchange of x and y, leads to the direction-variables $[p']$, $[q']$, $[r']$, of the same geodesic at V in the direction VU, in the form

$$[p'] = -p_0' + y\bar{\gamma}_{02} + [P], \quad [q'] = -q_0' + y\bar{\delta}_{02} + [Q], \quad [r'] = -r_0' + y\bar{\theta}_{02} + [R],$$

where

$$\left.\begin{aligned}
[P] &= \tfrac{1}{2}y^2(\Gamma_{300}p_0'p_2'^2) - \tfrac{1}{6}p_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon + \frac{yz}{3\Omega}K_1(0, 20)\\
[Q] &= \tfrac{1}{2}y^2(\Delta_{300}p_0'p_2'^2) - \tfrac{1}{6}q_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon + \frac{yz}{3\Omega}K_2(0, 20)\\
[R] &= \tfrac{1}{2}y^2(\Theta_{300}p_0'p_2'^2) - \tfrac{1}{6}r_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon + \frac{yz}{3\Omega}K_3(0, 20)
\end{aligned}\right\},$$

with the same significance for z, p_0', q_0', r_0', as before.

Angles of the small geodesic triangle: angular excess.

217. To find the corresponding approximations for the angles U and V of the regional triangle OUV, the values U_0 and V_0 being the approximations without regard to small quantities, we note that the direction-variables of OU at U in the direction UA are

$$p_1' + xp_1'' + \tfrac{1}{2}x^2p_1''', \quad q_1' + xq_1'' + \tfrac{1}{2}x^2q_1''', \quad r_1' + xr_1'' + \tfrac{1}{2}x^2r_1''',$$

to the order of approximation that has been considered, while those of UV at U in the direction UV are p', q', r'; hence, as U is the internal angle of the triangle, we have

$$\begin{aligned}
-\cos U &= \sum A_U(p_1' + xp_1'' + \tfrac{1}{2}x^2p_1''')p'\\
&= \sum\left(A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2}\right)(p_1' + xp_1'' + \tfrac{1}{2}x^2p_1''')(p_0' - x\bar{\gamma}_{01} + P)\\
&= \sum Ap_1'p_0'\\
&\qquad + x\sum\left(\frac{dA}{ds_1}p_1'p_0' + Ap_1''p_0' - Ap_1'\bar{\gamma}_{01}\right)\\
&\qquad + \sum\left[\tfrac{1}{2}x^2\frac{d^2A}{ds_1^2}p_1'p_0' + \tfrac{1}{2}x^2Ap_0'p_1''' + Ap_1'P\right.\\
&\qquad\qquad \left. + x^2\frac{dA}{ds_1}p_0'p_1'' - x^2Ap_1''\bar{\gamma}_{01} - x^2\frac{dA}{ds_1}p_1'\bar{\gamma}_{01}\right],
\end{aligned}$$

accurately up to the second order of small quantities.

The terms in the first line are the value of $-\cos U_0$.

For the terms of the first order of small quantities, being the aggregate in the second line, we have (by § 212, II)

$$p_1'\frac{dA}{ds_1}+q_1'\frac{dH}{ds_1}+r_1'\frac{dG}{ds_1}=-(Ap_1''+Hq_1''+Gr_1'')+a_1u_1^{(1)}+\xi_1u_2^{(1)}+\phi_1u_3^{(1)},$$

with similar expressions; and therefore

$$\begin{aligned}\sum\left(\frac{dA}{ds_1}p_1'p_0'+Ap_1''p_0'\right)&=u_1^{(1)}(a_1p_0'+\beta_1q_0'+\gamma_1r_0')\\&\quad+u_2^{(1)}(\xi_1p_0'+\eta_1q_0'+\zeta_1r_0')\\&\quad+u_3^{(1)}(\phi_1p_0'+\chi_1q_0'+\psi_1r_0')\\&=u_1^{(1)}\bar{\gamma}_{01}+u_2^{(1)}\bar{\delta}_{01}+u_3^{(1)}\bar{\theta}_{01}\\&=\sum Ap_1'\bar{\gamma}_{01}.\end{aligned}$$

Consequently the aggregate of terms of the first order vanishes.

For the terms of the second order of small quantities, we have, on putting $l=1$, $m=1$, $n=0$, in the result obtained in § 212, III,

$$\begin{aligned}\sum\frac{d^2A}{ds_1^2}p_1'p_0'&=u_1^{(1)}(\Gamma_{300}p_1'^2p_0')+u_2^{(1)}(\Delta_{300}p_1'^2p_0')+u_3^{(1)}(\Theta_{300}p_1'^2p_0')\\&\quad+u_1^{(0)}(\Gamma_{300}p_1'^3)+u_2^{(0)}(\Delta_{300}p_1'^3)+u_3^{(0)}(\Theta_{300}p_1'^3)\\&\quad-2\sum Ap_1''\bar{\gamma}_{01}\\&\quad+2u_1^{(1)}(a_1\bar{\gamma}_{01}+\beta_1\bar{\delta}_{01}+\gamma_1\bar{\theta}_{01})+2u_2^{(1)}(\xi\bar{\gamma}_{01}+\eta_1\bar{\delta}_{01}+\zeta_1\bar{\theta}_{01})\\&\quad+2u_3^{(1)}(\phi_1\bar{\gamma}_{01}+\chi_1\bar{\delta}_{01}+\psi_1\bar{\theta}_{01})\\&\quad-2u_1^{(0)}(a_1p_1''+\beta_1q_1''+\gamma_1r_1'')-2u_2^{(0)}(\xi p_1''+\eta_1q_1''+\zeta_1r_1'')\\&\quad-2u_3^{(0)}(\phi_1p_1''+\chi_1q_1''+\psi_1r_1''),\end{aligned}$$

the aggregate of terms involving the symbols k_{ij} disappearing because of the vanishing of one of the sets of surface-variables. Also

$$\begin{aligned}\sum Ap_0'p_1'''&=u_1^{(0)}p_1'''+u_2^{(0)}q_1'''+u_3^{(0)}r_1'''\\&=-u_1^{(0)}(\Gamma_{300}p_1'^3)-u_2^{(0)}(\Delta_{300}p_1'^3)-u_3^{(0)}(\Theta_{300}p_1'^3).\end{aligned}$$

Next, by the results in § 212, II, we have

$$\begin{aligned}\sum\frac{dA}{ds_1}p_0'p_1''&=p_1''\{A\bar{\gamma}_{01}+H\bar{\delta}_{01}+G\bar{\theta}_{01}+a_1u_1^{(0)}+\xi_1u_2^{(0)}+\phi_1u_3^{(0)}\}\\&\quad+q_1''\{H\bar{\gamma}_{01}+B\bar{\delta}_{01}+F\bar{\theta}_{01}+\beta_1u_1^{(0)}+\eta_1u_2^{(0)}+\chi_1u_3^{(0)}\}\\&\quad+r_1''\{G\bar{\gamma}_{01}+F\bar{\delta}_{01}+C\bar{\theta}_{01}+\gamma_1u_1^{(0)}+\zeta_1u_2^{(0)}+\psi_1u_3^{(0)}\}\\&=\sum Ap_1''\bar{\gamma}_{01}\\&\quad+u_1^{(0)}(a_1p_1''+\beta_1q_1''+\gamma_1r_1'')+u_2^{(0)}(\xi_1p_1''+\eta_1q_1''+\zeta_1r_1'')\\&\quad+u_3^{(0)}(\phi_1p_1''+\chi_1q_1''+\psi_1r_1'').\end{aligned}$$

Also, by the same results,

$$\begin{aligned}\sum \frac{dA}{ds_1} p_1'\bar{\gamma}_{01} = \quad & \bar{\gamma}_{01}\{a_1 u_1^{(1)} + \xi_1 u_2^{(1)} + \phi_1 u_3^{(1)} - Ap_1'' - Hq_1'' - Gr_1''\} \\ & + \bar{\delta}_{01}\{\beta_1 u_1^{(1)} + \eta_1 u_2^{(1)} + \chi_1 u_3^{(1)} - Hp_1'' - Bq_1'' - Fr_1''\} \\ & + \bar{\theta}_{01}\{\gamma_1 u_1^{(1)} + \zeta_1 u_2^{(1)} + \psi_1 u_3^{(1)} - Gp_1'' - Fq_1'' - Cr_1''\} \\ = & - \sum Ap_1''\bar{\gamma}_{01} \\ & + u_1^{(1)}(a_1\bar{\gamma}_{01} + \beta_1\bar{\delta}_{01} + \gamma_1\bar{\theta}_{01}) + u_2^{(1)}(\xi_1\bar{\gamma}_{01} + \eta_1\bar{\delta}_{01} + \zeta_1\bar{\theta}_{01}) \\ & + u_3^{(1)}(\phi_1\bar{\gamma}_{01} + \chi_1\bar{\delta}_{01} + \psi_1\bar{\theta}_{01}).\end{aligned}$$

When these values of the various summations are substituted in the equation for the second approximation, it becomes

$$\begin{aligned}-\cos U + \cos U_0 = & \sum Ap_1'P \\ & + \tfrac{1}{2}x^2\{u_1^{(1)}(\Gamma_{300}p_1'^2 p_0') + u_2^{(1)}(\Delta_{300}p_1'^2 p_0') + u_3^{(1)}(\Theta_{300}p_1'^2 p_0')\}.\end{aligned}$$

When the values of P, Q, R, are inserted, then

$$\begin{aligned}-\cos U + \cos U_0 = & \tfrac{1}{6}(\sum Ap_1'p_0')\frac{x^2y^2}{z^2} K \sin^2 \epsilon \\ & + \frac{xz}{3\Omega}\{u_1^{(1)}K_1(0, 10) + u_2^{(1)}K_2(0, 10) + u_3^{(1)}K_3(0, 10)\}.\end{aligned}$$

Thus, in small quantities, $\cos U - \cos U_0$ (and therefore $U - U_0$) is of the second order: for the present approximation, we take

$$-\cos U + \cos U_0 = (U - U_0) \sin U_0.$$

Further, we have

$$\sum Ap_1'p_0' = -\cos U_0, \quad y \sin \epsilon = z \sin U_0, \quad x \sin \epsilon = z \sin V_0,$$

so that the first term on the right-hand side

$$= -\tfrac{1}{6}xyK \cos U_0 \sin U_0 \sin V_0.$$

Again, when the values of K_1, K_2, K_3, in the coefficient of $\frac{xz}{3\Omega}$, are taken as on p. 66, the coefficient of $k_{11}\xi_{10} + k_{12}\eta_{10} + k_{13}\zeta_{10}$

$$\begin{aligned}& = u_1^{(1)}(hr_0' - gq_0') + u_2^{(1)}(br_0' - fq_0') + u_3^{(1)}(fr_0' - cq_0') \\ & = \Omega(q_1'r_0' - r_1'q_0') = \Omega\zeta_{10},\end{aligned}$$

and so for the other like combinations. Hence the aggregate of terms in the second line of the right-hand side of the expression for $-\cos U + \cos U_0$

$$\begin{aligned}& = \tfrac{1}{3}xz \sum k_{11}\xi_{10}^2 \\ & = \tfrac{1}{3}xz \frac{y^2}{z^2} \sum k_{11}\xi_{12}^2 \\ & = \tfrac{1}{3}\frac{xy^2}{z} K \sin^2 \epsilon = \tfrac{1}{3}xyK \sin \epsilon \sin U_0.\end{aligned}$$

Accordingly, our equation (after division by $\sin U_0$) becomes

$$U - U_0 = \tfrac{1}{3}Kxy \sin \epsilon - \tfrac{1}{6}Kxy \cos U_0 \sin V_0,$$

thus determining the angle U up to the second order.

We immediately infer the angle V, likewise to the second order, in the form

$$V - V_0 = \tfrac{1}{3}Kxy \sin \epsilon - \tfrac{1}{6}Kxy \cos V_0 \sin U_0.$$

In these expressions, the symbol K denotes the Riemann measure of curvature of the region at O, estimated in the superficial orientation which is determined by the two directions OU and OV at O; and in the small curvilinear triangle OUV, the three sides are regional geodesics between the angular points. Now

$$\epsilon + U_0 + V_0 = \pi;$$

and therefore

$$\begin{aligned}(\epsilon + U + V) - \pi &= U - U_0 + V - V_0 \\ &= \tfrac{2}{3}Kxy \sin \epsilon - \tfrac{1}{6}Kxy \sin (U_0 + V_0) \\ &= \tfrac{1}{2}Kxy \sin \epsilon.\end{aligned}$$

The magnitude on the left-hand side is the excess of the sum of the three angles of the curvilinear triangle over the sum of the three angles of a rectilinear triangle; we shall call it the *angular excess* of the regional geodesic triangle, as in § 112 for a superficial geodesic triangle. The magnitude $\tfrac{1}{2}xy \sin \epsilon$ is the value, to the present approximation, of the area of the small regional geodesic triangle; and therefore the equation can be stated in the form

$$\text{area of regional geodesic triangle} = \frac{1}{K}\,(\text{angular excess}),$$

the geodesic sides of the triangle being small. Consequently, as in § 112, the geometrical magnitude denoted by K is called the *sphericity* of the region in the regional orientation defined by the directions OU and OV of the regional geodesics.

Principal values of the sphericity for a region.

218. From the preceding investigations it follows that, at any orientation within a region specified by orientation-variables ξ, η, ζ, the sphericity of the region, estimated in that orientation and denoted by K, is given by the relation

$$K = \frac{\sum k_{11}\xi^2}{\sum a\xi^2},$$

with the significance of the regional symbols k_{ij} as defined (§ 162)

$$\left.\begin{aligned} k_{11} &= \sum (\eta_{22}\eta_{33} - \eta_{23}{}^2), & k_{23} &= \sum (\eta_{12}\eta_{13} - \eta_{11}\eta_{23}) \\ k_{22} &= \sum (\eta_{33}\eta_{11} - \eta_{31}{}^2), & k_{31} &= \sum (\eta_{23}\eta_{21} - \eta_{22}\eta_{31}) \\ k_{33} &= \sum (\eta_{11}\eta_{22} - \eta_{12}{}^2), & k_{12} &= \sum (\eta_{31}\eta_{32} - \eta_{33}\eta_{12}) \end{aligned}\right\}.$$

It is convenient (but not necessary) to have the variables ξ, η, ζ, so chosen that the relation $\sum a\xi^2=1$ is satisfied—a relation corresponding to the permanent equation $\sum Ap'^2=1$ affecting line-variables. If it should have happened that ξ, η, ζ, are determined by two regional directions p_1', q_1', r_1', and p_2', q_2', r_2', in the orientation inclined at an angle ϵ, and are

$$\xi=q_1'r_2'-r_1'q_2',\quad \eta=r_1'p_2'-p_1'r_2',\quad \zeta=p_1'q_2'-q_1'p_2',$$

the corresponding relation is $\sum a\xi^2=\sin^2\epsilon$.

The quantity K, being a function of the orientation-variables, will have maximum and minimum values among all those which arise from all the sets of values; and these maximum and minimum values will be obtained by making the algebraical quantity K, where

$$K=\sum k_{11}\xi^2,$$

a maximum or a minimum for all possible values of the variables ξ, η, ζ, subject to the relation

$$\sum a\xi^2=1.$$

The customary kind of analysis leads to the critical equations

$$\left.\begin{aligned} k_{11}\xi+k_{12}\eta+k_{13}\zeta&=K(a\xi+h\eta+g\zeta)\\ k_{21}\xi+k_{22}\eta+k_{23}\zeta&=K(h\xi+b\eta+f\zeta)\\ k_{31}\xi+k_{32}\eta+k_{33}\zeta&=K(g\xi+f\eta+c\zeta)\end{aligned}\right\},$$

for the determination of maximum or minimum values of K (called the *principal* values) and for the determination of the corresponding orientations (called the *principal* orientations) for the region.

(i) The principal values of the sphericity of a region are the roots of the cubic equation

$$\begin{vmatrix} aK-k_{11}, & hK-k_{12}, & gK-k_{13}\\ hK-k_{21}, & bK-k_{22}, & fK-k_{23}\\ gK-k_{31}, & fK-k_{32}, & cK-k_{33}\end{vmatrix}=0.$$

Let S be used to denote the determinant

$$\begin{vmatrix} k_{11}, & k_{12}, & k_{13}\\ k_{12}, & k_{22}, & k_{23}\\ k_{13}, & k_{23}, & k_{33}\end{vmatrix};$$

also, let $\bar{k}_{ij}$ denote the minor of k_{ij} in S, so that (*e.g.*)

$$\bar{k}_{11}=k_{22}k_{33}-k_{23}^2,\quad \bar{k}_{23}=k_{12}k_{13}-k_{11}k_{23};$$

and write

$$S_1=Ak_{11}+2Hk_{12}+Bk_{22}+2Gk_{13}+2Fk_{23}+Ck_{33},$$
$$S_2=a\bar{k}_{11}+2h\bar{k}_{12}+b\bar{k}_{22}+2g\bar{k}_{13}+2f\bar{k}_{23}+c\bar{k}_{33}.$$

Then the cubic equation assumes the form

$$\Omega^2 K^3 - \Omega S_1 K^2 + S_2 K - S = 0.$$

There are therefore three *measures of sphericity* for a region, being the magnitudes composed of the simplest symmetric functions of the principal values of K: they are

$$\frac{S_1}{\Omega}, \quad \frac{S_2}{\Omega^2}, \quad \frac{S}{\Omega^2}.$$

The measures of sphericity are superficial measures for the region: and, in their source, they are not related to any principal measure or measures of circular curvature of regional geodesics which are of the nature of curvilinear measures.

(ii) Corresponding to each of the three principal values of the sphericity K, there is a set of orientation-variables; so that there are three principal orientations in a region. We denote by K_1, K_2, K_3, the roots of the sphericity cubic; and the variables for the respective orientations, providing these values, by ξ_i, η_i, ζ_i, for $i=1, 2, 3$. The three orientations are at right angles to one another in pairs, a result established as follows.

Let the three critical equations, for K_1, ξ_1, η_1, ζ_1, be multiplied by ξ_2, η_2, ζ_2, respectively, and the results be added: then

$$\sum k_{11}\xi_1\xi_2 = K_1 \sum a\xi_1\xi_2.$$

Similarly let the three critical equations, for K_2, ξ_2, η_2, ζ_2, be multiplied by ξ_1, η_1, ζ_1, respectively, and the results be added: then

$$\sum k_{11}\xi_1\xi_2 - K_2 \sum a\xi_1\xi_2.$$

On the underlying assumption that the roots of the sphericity cubic are unequal, so that the principal orientations are determinate, K_2 is not equal to K_1; hence

$$\sum k_{11}\xi_1\xi_2 = 0, \quad \sum a\xi_1\xi_2 = 0.$$

The last relation shews that the two orientations with the sets of variables ξ_1, η_1, ζ_1, and ξ_2, η_2, ζ_2, are perpendicular to one another (not orthogonal, in the sense of § 7). Similarly for the other two pairs of principal orientations.

Moreover, these superficial orientations in pairs intersect in three linear directions. Let P_1', Q_1', R_1', be the intersection of the orientations ξ_2, η_2, ζ_2, and ξ_3, η_3, ζ_3; and similarly for P_2', Q_2', R_2', and P_3', Q_3', R_3'. Then we have

$$\frac{P_i'}{\eta_j\zeta_k - \zeta_j\eta_k} = \frac{Q_i'}{\zeta_j\xi_k - \xi_j\zeta_k} = \frac{R_i'}{\xi_j\eta_k - \eta_j\xi_k} = \Omega^{\frac{1}{2}},$$

$$\frac{\xi_i}{Q_j'R_k' - R_j'Q_k'} = \frac{\eta_i}{R_j'P_k' - P_j'R_k'} = \frac{\zeta_i}{P_j'Q_k' - Q_j'P_k'} = 1,$$

for $i, j, k, =1, 2, 3$, taken in cyclical rotation. But these directions P_i', Q_i', R_i', do not provide curves of curvature in the region, unless the plenary homaloidal space of the region is quadruple *.

Ex. Certain umbral forms can be associated with the preceding results and are placed on record.

We have seen that, in the tangent flat of the region, any direction touching the region can be made a leading line; and two other convenient leading lines can be associated with it, being the directions of the binormal and the trinormal of a geodesic drawn in the direction of the first.

Consider the three preceding directions and the principal orientations. Denoting any such direction by P', Q', R', and the orthogonal principal orientation by ξ, η, ζ, we have

$$\begin{aligned} AP' + HQ' + GR' &= \xi\Omega^{\frac{1}{2}}, & a\xi + h\eta + g\zeta &= P'\Omega^{\frac{1}{2}}, \\ HP' + BQ' + FR' &= \eta\Omega^{\frac{1}{2}}, & h\xi + b\eta + f\zeta &= Q'\Omega^{\frac{1}{2}}, \\ GP' + FQ' + CR' &= \zeta\Omega^{\frac{1}{2}}, & g\xi + f\eta + c\zeta &= R'\Omega^{\frac{1}{2}}. \end{aligned}$$

Thus the critical equations for the principal sphericities and the principal orientations of the region become

$$\begin{aligned} (k_{11}A + k_{12}H + k_{13}G)P' + (k_{11}H + k_{12}B + k_{13}F)Q' + (k_{11}G + k_{12}F + k_{13}C)R' &= \Omega KP', \\ (k_{12}A + k_{22}H + k_{23}G)P' + (k_{12}H + k_{22}B + k_{23}F)Q' + (k_{12}G + k_{22}F + k_{23}C)R' &= \Omega KQ', \\ (k_{13}A + k_{23}H + k_{33}G)P' + (k_{13}H + k_{23}B + k_{33}F)Q' + (k_{13}G + k_{23}F + k_{33}C)R' &= \Omega KR'. \end{aligned}$$

For the umbral notation, let

$$K = \sum k_{11}\xi^2 = (\kappa_1\xi + \kappa_2\eta + \kappa_3\zeta)^2,$$
$$1 = \sum AP'^2 = (a_1P' + a_2Q' + a_3R')^2,$$

so that

$$k_{ij} = \kappa_i\kappa_j, \text{ for all values } i, j = 1, 2, 3;$$
$$A = a_1^2, \quad H = a_1a_2, \quad B = a_2^2, \quad G = a_1a_3, \quad F = a_2a_3, \quad C = a_3^2.$$

* Whatever be the dimensionality of the plenary space, the quantity K is a measure of superficial curvature of the region. In the special instance when the plenary space is quadruple, K is only one of the two measures of superficial curvature of the region, which then are analogous to the two measures of curvature of a Gaussian surface in triple homaloidal space. Also in this special instance of a quadruple plenary space, the principal values of the sphericity K of a region are analytically connected with the principal circular curvatures of regional geodesics: and the principal orientations of the region are, in fact, the pair-combinations of the directions of the three curves of curvature of a region at any point. (See *G.F.D.*, vol. ii, §§ 320, 321.)

The associations of K with the product of the principal curvatures of a surface in triple space (a primary surface), and with one of the measures of superficial curvature of a primary region, are made possible by the limitations in the dimensionality of the respective plenary spaces.

Let $\alpha, \beta, \gamma, \ldots$, be any umbral symbols cogredient with κ; and let $b, c, \ldots$, be any umbral symbols cogredient with a. Then

$$k_{11}A + k_{12}H + k_{13}G = a_1\kappa_1 a_\kappa,$$
$$k_{11}H + k_{12}B + k_{13}F = a_2\kappa_1 a_\kappa,$$
$$k_{11}G + k_{12}F + k_{13}C = a_3\kappa_1 a_\kappa;$$

and similarly for the other combinations of the coefficients k_{ij} with the primary magnitudes of the region. The critical equations now become

$$a_1\alpha_1 a_\alpha P' + a_2\alpha_1 a_\alpha Q' + a_3\alpha_1 a_\alpha R' = \Omega K P',$$
$$b_1\beta_2 b_\beta P' + b_2\beta_2 b_\beta Q' + b_3\beta_2 b_\beta R' = \Omega K Q',$$
$$c_1\gamma_3 c_\gamma P' + c_2\gamma_3 c_\gamma Q' + c_3\gamma_3 c_\gamma R' = \Omega K R';$$

and therefore the sphericity-cubic is

$$\begin{vmatrix} a_1\alpha_1 a_\alpha - \Omega K, & a_2\alpha_1 a_\alpha, & a_3\alpha_1 a_\alpha \\ b_1\beta_2 b_\beta, & b_2\beta_2 b_\beta - \Omega K, & b_3\beta_3 b_\beta \\ c_1\gamma_3 c_\gamma, & c_2\gamma_3 c_\gamma, & c_3\gamma_3 c_\gamma - \Omega K \end{vmatrix} = 0$$

which can be expressed in the form

$$\tfrac{1}{6}(\alpha\beta\gamma)^2 - \tfrac{1}{2}K(a\alpha\beta)^2 + \tfrac{1}{2}K^2(ab\alpha)^2 - \Omega^2 K^3 = 0.$$

The result may also be obtained by noting that the sphericity-cubic is the condition that the discriminant of the ternary quadratic

$$\sum k_{11}\xi^2 - K\sum a\xi^2$$

in surface-variables shall vanish. If, in umbral notation, we write

$$\sum k_{11}\xi^2 - K\sum a\xi^2 = (\theta_1\xi + \theta_2\eta + \theta_3\zeta)^2 = \theta_\xi^{\,2},$$

and if θ, ϕ, ψ, be cogredient symbols, the vanishing discriminant of $\theta_\xi^{\,2}$ is the equation

$$\tfrac{1}{6}(\theta\phi\psi)^2 = 0,$$

which can be changed into the foregoing form.

It should be noted that the introduction of the analytical measure of the Riemann sphericity requires the association of the quadratic form $\sum k_{11}\xi^2$ with the rest of the system of invariantive concomitants of the region.

Sphericity of a parametric surface in a region.

219. Now consider any parametric surface $\theta(p, q, r) = 0$ in a region, not restricted to be a geodesic surface. If p_1', q_1', r_1', and p_2', q_2', r_2', be any two directions touching that surface, giving rise to orientation-variables ξ, η, ζ, in the region, then (§ 193)

$$\frac{\theta_1}{\xi} = \frac{\theta_2}{\eta} = \frac{\theta_3}{\zeta} = \Omega^{\frac{1}{2}}\theta_n.$$

Let K denote the sphericity of the region in the orientation ξ, η, ζ, being the sphericity of a geodesic surface in that orientation; then

$$K = \frac{\sum k_{11}\xi^2}{\sum a\xi^2} = \frac{\sum k_{11}\theta_1^{\,2}}{\sum a\theta_1^{\,2}} = \frac{1}{\Omega\theta_n^{\,2}}\sum k_{11}\theta_1^{\,2}.$$

The sphericity of the parametric surface $\theta=0$, which is not a geodesic surface except under a group of limitations as investigated in §§ 210, 211, is distinct from K. Let its sphericity be denoted by K_θ: we proceed to find a relation between K and K_θ.

It has been proved (§ 195) that the circular curvature and the direction of the prime normal of a superficial geodesic, the circular curvature and the direction of the prime normal of a regional geodesic touching the superficial geodesic, together with the regional flexure and the prime radius of that regional flexure of the superficial geodesic, are connected by relations, typified by the equation

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn}.$$

Let p', q', r', be the direction-variables in the region of the two geodesics; as it touches the surface, there is the relation

$$\theta_1 p'+\theta_2 q'+\theta_3 r'=0.$$

Any surface can be represented by two parameters; accordingly, we shall take p and q to be the two parameters for the surface $\theta=0$; and the inclusion of that surface within the region is secured by taking the value of r, and derivatives of the value of r, given by the parametric equation. The element of arc along the surface is

$$ds^2=\sum Adp^2,$$

where

$$\theta_1\, dp+\theta_2\, dq+\theta_3\, dr=0,$$

so that

$$ds^2=E_\theta\, dp^2+2F_\theta\, dp\, dq+G_\theta\, dq^2,$$

where

$$E_\theta=A-2G\frac{\theta_1}{\theta_3}+C\frac{\theta_1^2}{\theta_3^2},$$

$$F_\theta=H-G\frac{\theta_2}{\theta_3}-F\frac{\theta_1}{\theta_3}+C\frac{\theta_1\theta_2}{\theta_3^2},$$

$$G_\theta=B-2F\frac{\theta_2}{\theta_3}+C\frac{\theta_2^2}{\theta_3^2}.$$

By direct substitution, we have

$$\begin{aligned}V_\theta^2&=E_\theta G_\theta-F_\theta^2\\&=c+2f\frac{\theta_2}{\theta_3}+2g\frac{\theta_1}{\theta_3}+b\frac{\theta_2^2}{\theta_3^2}+2h\frac{\theta_1\theta_2}{\theta_3^2}+a\frac{\theta_1^2}{\theta_3^2}\\&=\Omega\frac{\theta_n^2}{\theta_3^2}.\end{aligned}$$

Again, we have

$$\begin{aligned}\frac{Y}{\rho}&=\sum\eta_{11}p'^2\\&=\bar\eta_{11}p'^2+2\bar\eta_{12}p'q'+\bar\eta_{22}q'^2,\end{aligned}$$

where

$$\bar{\eta}_{11}=\eta_{11}-2\eta_{13}\frac{\theta_1}{\theta_3}+\eta_{33}\frac{\theta_1^2}{\theta_3^2},\quad \bar{\eta}_{22}=\eta_{22}-2\eta_{23}\frac{\theta_2}{\theta_3}+\eta_{33}\frac{\theta_2^2}{\theta_3^2},$$

$$\bar{\eta}_{12}=\eta_{12}-\eta_{13}\frac{\theta_2}{\theta_3}-\eta_{23}\frac{\theta_1}{\theta_3}+\eta_{33}\frac{\theta_1\theta_2}{\theta_3^2};$$

and, similarly,

$$\frac{\theta_n}{\gamma}=\sum\vartheta_{11}p'^2$$
$$=\bar{\vartheta}_{11}p'^2+2\bar{\vartheta}_{12}p'q'+\bar{\vartheta}_{22}q'^2,$$

with the like formations for $\bar{\vartheta}_{11}$, $\bar{\vartheta}_{12}$, $\bar{\vartheta}_{22}$, given by

$$\bar{\vartheta}_{11}-\vartheta_{11}-2\vartheta_{13}\frac{\theta_1}{\theta_2}+\vartheta_{33}\frac{\theta_1^2}{\theta_3^2},\quad \bar{\vartheta}_{22}=\vartheta_{22}-2\vartheta_{23}\frac{\theta_2}{\theta_3}+\vartheta_{33}\frac{\theta_2^2}{\theta_3^2},$$

$$\bar{\vartheta}_{12}=\vartheta_{12}-\vartheta_{13}\frac{\theta_2}{\theta_3}-\vartheta_{23}\frac{\theta_1}{\theta_3}+\vartheta_{33}\frac{\theta_1\theta_2}{\theta_3^2}.$$

The surface has been referred to the parameters p, q. The circular curvature of a superficial geodesic in the direction p', q', is $1/\rho_0$, and the typical direction-cosine of the prime normal of that geodesic is Y_0; we therefore have a relation (§ 93) of the form

$$\frac{Y_0}{\rho_0}=\eta_{11}^{(\theta)}p'^2+2\eta_{12}^{(\theta)}p'q'+\eta_{22}^{(\theta)}q'^2.$$

Thus the initial relation connecting the two circular curvatures and the regional flexure becomes

$$\eta_{11}^{(\theta)}p'^2+2\eta_{12}^{(\theta)}p'q'+\eta_{22}^{(\theta)}q'^2=\bar{\eta}_{11}p'^2+2\bar{\eta}_{12}p'q'+\bar{\eta}_{22}q'^2$$
$$-\frac{1}{\theta_n}\frac{dy}{dn}(\bar{\vartheta}_{11}p'^2+2\bar{\vartheta}_{12}p'q'+\bar{\vartheta}_{22}q'^2),$$

a homogeneous quadratic relation between the direction-variables p', q', of the surface; and it holds for all directions on the surface. Hence there are the relations *

$$\left.\begin{aligned}\eta_{11}^{(\theta)}&=\bar{\eta}_{11}-\frac{\bar{\vartheta}_{11}}{\theta_n}\frac{dy}{dn}\\ \eta_{12}^{(\theta)}&=\bar{\eta}_{12}-\frac{\bar{\vartheta}_{12}}{\theta_n}\frac{dy}{dn}\\ \eta_{22}^{(\theta)}&=\bar{\eta}_{22}-\frac{\bar{\vartheta}_{22}}{\theta_n}\frac{dy}{dn}\end{aligned}\right\}.$$

* These relations can be obtained directly from the equations of the type

$$\eta_{11}^{(\theta)}=\frac{\partial^2 y}{\partial p^2}-\frac{\partial y}{\partial p}\Gamma_{11}^{(\theta)}-\frac{\partial y}{\partial q}\Delta_{11}^{(\theta)}$$

of the surface by means of the necessary differential transformations, as these are affected by the existence of the surface-equation $\theta(p, q, r)=0$.

The sphericity of the surface, K_θ, is given by (§ 112)

$$K_\theta = \frac{\sum \eta_{11}{}^{(\theta)}\eta_{22}{}^{(\theta)} - \sum\{\eta_{12}{}^{(\theta)}\}^2}{E_\theta G_\theta - F_\theta{}^2}.$$

To evaluate the numerator, we note that, as

$$\sum \eta_{ij} y_k = 0,$$

for all values $i, j, k, = 1, 2, 3$, we have

$$\sum \eta_{ij} \frac{dy}{dn} = \sum \eta_{ij} \left(y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn} \right) = 0,$$

for all values of i, j; and therefore

$$\sum \bar{\eta}_{11} \frac{dy}{dn} = 0, \quad \sum \bar{\eta}_{12} \frac{dy}{dn} = 0, \quad \sum \bar{\eta}_{22} \frac{dy}{dn} = 0.$$

Also, we have

$$\sum \left(\frac{dy}{dn}\right)^2 = 1.$$

Hence

$$\begin{aligned}\sum \eta_{11}{}^{(\theta)}\eta_{22}{}^{(\theta)} &- \sum\{\eta_{12}{}^{(\theta)}\}^2 \\ &= \sum \bar{\eta}_{11}\bar{\eta}_{22} - \sum \bar{\eta}_{12}{}^2 + \frac{1}{\theta_n{}^2}\{\sum \bar{\vartheta}_{11}\bar{\vartheta}_{22} - \sum \bar{\vartheta}_{12}{}^2\}.\end{aligned}$$

After direct substitution of the values of $\bar{\eta}_{11}, \bar{\eta}_{12}, \bar{\eta}_{22}$, and reduction, we find

$$\begin{aligned}\sum \bar{\eta}_{11}\bar{\eta}_{22} &- \sum \bar{\eta}_{12}{}^2 \\ &= \frac{1}{\theta_3{}^2}(k_{11}\theta_1{}^2 + 2k_{12}\theta_1\theta_2 + k_{22}\theta_2{}^2 + 2k_{13}\theta_1\theta_3 + 2k_{23}\theta_2\theta_3 + k_{33}\theta_3{}^2) \\ &= \Omega K \frac{\theta_n{}^2}{\theta_3{}^2},\end{aligned}$$

where K is the sphericity of the region in the orientation of the surface.

Similarly, after direct substitution of the values of $\bar{\vartheta}_{11}, \bar{\vartheta}_{12}, \bar{\vartheta}_{22}$, and reduction, we find

$$\begin{aligned}\sum \bar{\vartheta}_{11}\bar{\vartheta}_{22} &- \sum \bar{\vartheta}_{12}{}^2 \\ &= \frac{1}{\theta_3{}^2}\{\sum (\vartheta_{22}\vartheta_{33} - \vartheta_{23}{}^2)\theta_1{}^2\} = \frac{\theta_n{}^4}{\theta_3{}^2}\frac{\Omega}{\gamma_1\gamma_2}\end{aligned}$$

by § 196, where γ_1 and γ_2 are the two principal radii of regional flexure of the surface.

Consequently, the numerator in the expression for K_θ

$$= \frac{\Omega\theta_n{}^2}{\theta_3{}^2}\left(K + \frac{1}{\gamma_1\gamma_2}\right).$$

The denominator in the expression for K_θ

$$=V_\theta{}^2=\Omega\frac{\theta_n{}^2}{\theta_3{}^2}.$$

We therefore have

$$K_\theta=K+\frac{1}{\gamma_1\gamma_2},$$

as the relation between the sphericity of a surface in the region and the sphericity of the region in the orientation of the surface. The difference between the two sphericities is equal to the bilinear measure of regional flexure of the surface.

This result, valid for a region in a plenary homaloidal space of any dimensionality, is known * in the particular instance when the plenary homaloidal space is quadruple.

Further, if it should happen that the surface $\theta=0$ is geodesic to the region at the central point O considered, the regional flexure of all geodesics on the surface (being also geodesics of the region) is zero, so that both $1/\gamma_1$ and $1/\gamma_2$ vanish: thus

$$K_\theta=K,$$

as is to be expected when the surface is geodesic.

Curvatures of a geodesic on a parametric surface.

220. In passing, it may be noted that the preceding results of the form

$$\eta_{11}{}^{(\theta)}=\bar{\eta}_{11}-\frac{\bar{\vartheta}_{11}}{\theta_n}\frac{dy}{dn}$$

can be used to obtain the spatial torsion of the superficial geodesic and, incidentally, to verify the geometrical relation (§ 197)

$$\frac{1}{\rho\tau_\theta}-\frac{1}{\gamma\sigma_\theta}=0$$

characteristic of the curves of spatial circular curvature on the surface.

The typical direction-cosine Y_0 of the prime normal of a superficial geodesic in the region is given by the equation

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn}.$$

Let the secondary magnitudes of the surface be denoted by L_θ, M_θ, N_θ, being defined by the values

$$L_\theta=\sum\{Y_0\eta_{11}{}^{(\theta)}\},\quad M_\theta=\sum\{Y_0\eta_{12}{}^{(\theta)}\},\quad N_\theta=\sum\{Y_0\eta_{22}{}^{(\theta)}\}.$$

* *G.F.D.* vol. ii, § 368.

Then we have

$$\frac{L_\theta}{\rho_0}=\sum\left\{\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn}\right\}\left\{\bar{\eta}_{11}-\frac{\bar{\vartheta}_{11}}{\theta_n}\frac{dy}{dn}\right\}.$$

Now for all values of i, j, k, we have, in the region,

$$\sum y_k\eta_{ij}=0;$$

and therefore

$$\sum y_k\bar{\eta}_{ij}=0,$$

for $ij=11, 12, 22$. Consequently

$$\sum\frac{dy}{dn}\bar{\eta}_{11}=0.$$

Also we have

$$\sum Y\frac{dy}{dn}=0,\quad \sum\left(\frac{dy}{dn}\right)^2=1;$$

and therefore

$$\begin{aligned}\frac{L_\theta}{\rho_0}&=\frac{1}{\rho}\sum Y\bar{\eta}_{11}-\frac{1}{\gamma\theta_n}\bar{\vartheta}_{11}\\&=\frac{1}{\rho}\left\{\bar{A}-2\bar{G}\frac{\theta_1}{\theta_3}+\bar{C}\frac{\theta_1^2}{\theta_3^2}\right\}-\frac{1}{\gamma\theta_n}\left(\vartheta_{11}-2\vartheta_{13}\frac{\theta_1}{\theta_3}+\vartheta_{33}\frac{\theta_1^2}{\theta_3^2}\right).\end{aligned}$$

Similarly

$$\frac{M_\theta}{\rho_0}=\frac{1}{\rho}\left\{\bar{H}-\bar{G}\frac{\theta_2}{\theta_3}-\bar{F}\frac{\theta_1}{\theta_3}+\bar{C}\frac{\theta_1\theta_2}{\theta_3^2}\right\}-\frac{1}{\gamma\theta_n}\left(\vartheta_{12}-\vartheta_{13}\frac{\theta_2}{\theta_3}-\vartheta_{23}\frac{\theta_1}{\theta_3}+\vartheta_{33}\frac{\theta_1\theta_2}{\theta_3^2}\right),$$

$$\frac{N_\theta}{\rho_0}=\frac{1}{\rho}\left\{\bar{B}-2\bar{F}\frac{\theta_2}{\theta_3}+\bar{C}\frac{\theta_2^2}{\theta_3^2}\right\}-\frac{1}{\gamma\theta_n}\left(\vartheta_{22}-2\vartheta_{23}\frac{\theta_2}{\theta_3}+\vartheta_{33}\frac{\theta_2^2}{\theta_3^2}\right).$$

Accordingly

$$\frac{1}{\rho_0}(L_\theta p'+M_\theta q')=\frac{1}{\rho}P_1-\frac{1}{\gamma\theta_n}Q_1,$$

where

$$\begin{aligned}P_1&=\bar{A}p'+\bar{H}q'-\frac{\bar{G}}{\theta_3}(\theta_1p'+\theta_2q')-\frac{\theta_1}{\theta_3}\left\{\bar{G}p'+\bar{F}q'-\frac{\bar{C}}{\theta_3}(\theta_1p'+\theta_2q')\right\}\\&=\bar{A}p'+\bar{H}q'+\bar{G}r'-\frac{\theta_1}{\theta_3}(\bar{G}p'+\bar{F}q'+\bar{C}r')\\&=v_1-\frac{\theta_1}{\theta_3}v_3;\end{aligned}$$

and, in the same way,

$$Q_1=\vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r'-\frac{\theta_1}{\theta_3}(\vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r').$$

Similarly, we have

$$\frac{1}{\rho_0}(M_\theta p'+N_\theta q')=\frac{1}{\rho}P_2-\frac{1}{\gamma\theta_n}Q_2,$$

where

$$P_2 = v_2 - \frac{\theta_2}{\theta_3} v_3,$$

$$Q_2 = \vartheta_{12} p' + \vartheta_{22} q' + \vartheta_{23} r' - \frac{\theta_2}{\theta_3}(\vartheta_{13} p' + \vartheta_{23} q' + \vartheta_{33} r').$$

Likewise we find

$$E_\theta p' + F_\theta q' = u_1 - \frac{\theta_1}{\theta_3} u_3, \quad F_\theta p' + G_\theta q' = u_2 - \frac{\theta_2}{\theta_3} u_3,$$

and we had (p. 91)

$$V_\theta = \Omega^{\frac{1}{2}} \frac{\theta_n}{\theta_3}.$$

Now (§ 106) the torsion $1/\sigma_0$ of a geodesic on a surface with E_θ, F_θ, G_θ, as its primary magnitudes, and L_θ, M_θ, N_θ, as its secondary magnitudes, is given by the equation

$$\frac{V_\theta}{\rho_0 \sigma_0} = \begin{vmatrix} \frac{1}{\rho_0}(L_\theta p' + M_\theta q'), & \frac{1}{\rho_0}(M_\theta p' + N_\theta q') \\ E_\theta p' + F_\theta q' \;, & F_\theta p' + G_\theta q' \end{vmatrix},$$

$1/\rho_0$ denoting the circular curvature of the geodesic. Hence, for the regional surface in question, we have

$$\frac{\Omega^{\frac{1}{2}}}{\rho_0 \sigma_0} \frac{\theta_n}{\theta_3} = \frac{1}{\rho} T - \frac{1}{\gamma \theta_n} S,$$

where

$$T = \begin{vmatrix} v_1 - \frac{\theta_1}{\theta_3} v_3, & v_2 - \frac{\theta_2}{\theta_3} v_3 \\ u_1 - \frac{\theta_1}{\theta_3} u_3, & u_2 - \frac{\theta_2}{\theta_3} u_3 \end{vmatrix}$$

$$= \frac{1}{\theta_3} \begin{vmatrix} v_1, & v_2, & v_3 \\ u_1, & u_2, & u_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}$$

$$= \frac{\Omega^{\frac{1}{2}}}{\tau_\theta} \frac{\theta_n}{\theta_3},$$

where $1/\tau_\theta$ is the regional tilt of the superficial geodesic (§ 201); and

$$S = \begin{vmatrix} \vartheta_{11} p' + \vartheta_{12} q' + \vartheta_{13} r' - \frac{\theta_1}{\theta_3}(\vartheta_{13} p' + \vartheta_{23} q' + \vartheta_{33} r'), & u_1 - \frac{\theta_1}{\theta_3} u_3 \\ \vartheta_{12} p' + \vartheta_{22} q' + \vartheta_{23} r' - \frac{\theta_2}{\theta_3}(\vartheta_{13} p' + \vartheta_{23} q' + \vartheta_{33} r'), & u_2 - \frac{\theta_2}{\theta_3} u_3 \end{vmatrix}$$

$$= \frac{1}{\theta_3} \begin{vmatrix} \vartheta_{11}p' + \vartheta_{12}q' + \vartheta_{13}r', & u_1, & \theta_1 \\ \vartheta_{21}p' + \vartheta_{22}q' + \vartheta_{23}r', & u_2, & \theta_2 \\ \vartheta_{31}p' + \vartheta_{32}q' + \vartheta_{33}r', & u_3, & \theta_3 \end{vmatrix}$$

$$= -\frac{\Omega^{\frac{1}{2}}}{\sigma_\theta} \cdot \frac{\theta_n^2}{\theta_3},$$

where $1/\sigma_\theta$ is the regional torsion of the superficial geodesic (§ 200). We therefore have

$$\frac{1}{\rho_0 \sigma_0} = \frac{1}{\gamma \sigma_\theta} - \frac{1}{\rho \tau_\theta},$$

the geometrical relation (§ 204) connecting the spatial circular curvature and the spatial torsion of the superficial geodesic, with the regional torsion and regional tilt of that geodesic, with the regional flexure of that geodesic, and with the circular curvature of the tangent regional geodesic *.

The curves of spatial circular curvature on the surface are characterised by the property $1/\sigma_0 = 0$; and thus, for the regional surface, they are given (as in § 197) by the relation

$$\frac{1}{\rho \tau_\theta} - \frac{1}{\gamma \sigma_\theta} = 0.$$

* When the region is a flat, we have a surface in triple homaloidal space; then $1/\gamma = 1/\rho_0$, while $1/\rho = 0$, $1/\sigma_\theta = 1/\sigma_0$, and $1/\tau_\theta = 0$.

CHAPTER XVIII

Geodesic Parallels in a Region

Parallel geodesics, after Levi-Civita.

221. The determination of parallelism for geodesics has been effected for surfaces (§ 119). There, along any curve OUA, when any geodesic OVB on the surface is drawn through O, a superficial geodesic UW through U is said to be parallel when the angle WUA is equal to the angle VOU; and it appeared sufficient to take a superficial geodesic for the basic curve OUA.

This characteristic property emerges as a result from the original definition. The matter is less simple when the containing amplitude is more extensive than a surface: thus, even for a region, at a point U on the basic curve OUA, there is a quadric cone with its vertex at U every generator of which makes the same angle with UA equal to VOU; and more selective precision is necessary. As already stated, the notion of parallel geodesics was first propounded by Levi-Civita; and it was related to considerations connected with the plenary homaloidal space of the curved amplitude *. For simplicity, we shall take the curved amplitude to be a region in multiple space.

Through O, let a curve C be drawn in the region; and let X specify a point moving along the curve. With this moving point, let a moving direction XZ be associated under some assigned law of continuous change. When X coincides with O, let the regional direction-variables of the initial position OD be P', Q', R'; the typical spatial direction-cosine, c, of OD is given by

$$c = y_1 P' + y_2 Q' + y_3 R'.$$

We select an aggregate of lines through O in the plenary space, as a fixed set for reference; they are not related to the curve C, nor are they affected by it. We postulate the law that, as X moves, the associated direction XZ shall be required to make, with the several lines of the fixed aggregate through O, the same several angles as are made with those lines by the initial position OD of the direction. Let l denote the typical spatial direction-cosine of any line in the aggregate through O: the postulated law requires the conditions

$$\sum cl = \text{constant},$$

where this constant differs from one fixed line to another but is unaffected by the

* The following investigation is based upon Levi-Civita's original memoir in the *Rendiconti del Circolo Matematico di Palermo*, t. xlii (1917), §§ 1-4.

movement of X. Hence, denoting differentiation along the curve by ds_1, we have the equations

$$\sum \frac{dc}{ds_1} l = 0,$$

which must be satisfied along the curve.

Were the requirement exacted for all possible directions l through O in plenary space, the equations could be satisfied only if

$$\frac{dc}{ds_1} = 0$$

for every direction-cosine: that is, a merely constant direction in the region. Such a result is not possible in general. Accordingly, we take the selected lines to be the aggregate of directions which, passing through O, lie in the tangent flat of the region at O; and we therefore make l proportional to

$$y_1\lambda + y_2\mu + y_3\nu,$$

with no limitations on the quantities λ, μ, ν. For our purpose, the conditions must be satisfied for all possible variations λ, μ, ν. Thus the complete set of equations, to be satisfied along the curve, becomes

$$\sum \frac{dc}{ds_1} (y_1\lambda + y_2\mu + y_3\nu) = 0,$$

for all possible variations λ, μ, ν; and therefore, for the complete fulfilment of the imposed law, the conditions are

$$\sum y_1 \frac{dc}{ds_1} = 0, \quad \sum y_2 \frac{dc}{ds_1} = 0, \quad \sum y_3 \frac{dc}{ds_1} = 0.$$

Now

$$\frac{dc}{ds_1} = y_1 \frac{dP'}{ds_1} + y_2 \frac{dQ'}{ds_1} + y_3 \frac{dR'}{ds_1}$$
$$+ P'(y_{11}p_1' + y_{12}q_1' + y_{13}r_1') + Q'(y_{12}p_1' + y_{22}q_1' + y_{23}r_1') + R'(y_{13}p_1' + y_{23}q' + y_{33}r').$$

When this value is substituted in the first of the three conditions, the coefficient of P'

$$= A\alpha_1 + H\xi_1 + G\phi_1,$$

the coefficient of Q'

$$= A\beta_1 + H\eta_1 + G\chi_1,$$

and the coefficient of R'

$$= A\gamma_1 + H\zeta_1 + G\psi_1,$$

with the notation of § 172; and thus the first condition is

$$A\left(\frac{dP'}{ds_1} + U_\gamma\right) + H\left(\frac{dQ'}{ds_1} + U_8\right) + G\left(\frac{dR'}{ds_1} + U_0\right) = 0,$$

where

$$U_\gamma = P'\alpha_1 + Q'\beta_1 + R'\gamma_1 = (\Gamma \between p_1', q_1', r_1' \between P', Q', R'),$$
$$U_\delta = P'\xi_1 + Q'\eta_1 + R'\zeta_1 = (\Delta \between p_1', q_1', r_1' \between P', Q', R'),$$
$$U_\theta = P'\phi_1 + Q'\chi_1 + R'\psi_1 = (\Theta \between p_1', q_1', r_1' \between P', Q', R').$$

Similarly, the second condition and the third condition become

$$H\left(\frac{dP'}{ds_1} + U_\gamma\right) + B\left(\frac{dQ'}{ds_1} + U_\delta\right) + F\left(\frac{dR'}{ds_1} + U_\theta\right) = 0,$$
$$G\left(\frac{dP'}{ds_1} + U_\gamma\right) + F\left(\frac{dQ'}{ds_1} + U_\delta\right) + C\left(\frac{dR'}{ds_1} + U_\theta\right) = 0.$$

When these three equations are resolved, the expression of the law is

$$\left.\begin{aligned} &\frac{dP'}{ds_1} + (\Gamma \between p_1', q_1', r_1' \between P', Q', R') = 0 \\ &\frac{dQ'}{ds_1} + (\Delta \between p_1, \ q_1, \ r_1 \between P', Q', R') = 0 \\ &\frac{dR'}{ds_1} + (\Theta \between p_1', q_1', r_1' \between P', Q', R') = 0 \end{aligned}\right\}.$$

It is easy to verify that the equation

$$\frac{d}{ds}(\sum AP'^2) = 0$$

is satisfied: the position XY (consecutive to the position OD) of the carried direction lies in the tangent flat of the region at the point X consecutive to the point O.

Now imagine a deformation of the region, within its plenary homaloidal space, to be effected without stretching and without rupture of continuity, such that the tangent flats of the region at successive points along the basic curve C are brought into coincidence with one another or, what is the same thing, into coincidence with the tangent flat at O. The character of the specified deformation leaves the lengths of all arcs and all their inclinations unaltered. Hence the deformed positions of the carried direction, in this one comprehensive flat, all make the same several angles with the aggregate of lines as does the initial line. Within this comprehensive flat, all these deformed positions are therefore actually parallel to the initial line. Accordingly, in these circumstances, the position of the carried direction in the undeformed region is defined to be *parallel* to the initial line.

Thus the Levi-Civita parallelism ultimately is derived from a parallelism in a homoloidal space which is (or is contained in) the plenary homaloidal space for the amplitude. The primary conditions for any n-fold amplitude are

$$\frac{dX'_k}{ds_1} + \sum_i \sum_j \{ij, k\} x_i' X_j' = 0,$$

(for $k = 1, \ldots, n$), where $x_1', \ldots, x_n'$, are direction-variables of the basic curve C, and $X_1', \ldots, X_n'$, are direction-variables of the carried direction. The

simplest instance of all arises when the amplitude is a surface, the curve C is a geodesic on the surface, and the initial line is another superficial geodesic through O distinct from C. When p_2', q_2', denote the direction-variables of the carried line, the conditions of parallelism (as in § 121) are

$$\left.\begin{aligned} -\frac{dp_2'}{ds_1} &= \Gamma_{11}p_1'p_2' + \Gamma_{12}(q_1'p_2' + p_1'q_2') + \Gamma_{22}q_1'q_2' \\ -\frac{dq_2'}{ds_1} &= \Delta_{11}p_1'p_2' + \Delta_{12}(q_1'p_2' + p_1'q_2') + \Delta_{22}q_1'q_2' \end{aligned}\right\}.$$

Parallel geodesics, after Severi.

222. An essential modification, of the Levi-Civita definition and of Levi-Civita's determination of geodesic parallelism, is due to Severi * ; in effect, it is a distinct alternative. Severi associates the property with the geodesics of an amplitude by means of the geodesic surfaces of the amplitude. Adopting the Severi definition, we take two amplitudinal geodesics at O, thus determining a geodesic surface S of the amplitude at O. One of these two geodesics ORE is the basic curve C; the other of these geodesics OTF is the initial line to which the drawn geodesics are to be parallel. Through any point R on the basic curve, we take the direction RU which on the surface S is geodesically parallel to OTF; the amplitudinal geodesic in this direction is defined to be the amplitudinal geodesic which is parallel to OTF.

Let the geodesic surface of the n-fold amplitude be determined parametrically by $n-2$ equations of the type

$$\theta(x_1, \dots, x_n) = 0,$$

with $n-2$ independent functions θ. As the surface thus defined is geodesic, every geodesic of the amplitude, originating at O in the superficial orientation established by the two directions

$$x_1', \dots, x_n'; \quad z_1', \dots, z_n',$$

(the direction-variables of ORE and OTF), must lie in the surface; and therefore the intrinsic equations of every such geodesic must be satisfied in association with these parametric equations of the surface. For each equation $\theta = 0$, we have

$$\sum_k \frac{\partial\theta}{\partial x_k} x_k'' + \sum_i \sum_j \frac{\partial^2\theta}{\partial x_i \partial x_j} x_i' x_j' = 0;$$

and therefore when the regional geodesic, with the intrinsic equations

$$x_k'' = -\sum_i \sum_j \{ij, k\} x_i' x_j', \quad (k = 1, \dots n),$$

lies in the surface, we must have

$$\sum_i \sum_j \vartheta_{ij} x_i' x_j' = 0,$$

* *Rendiconti del Circ. Mat. di Palermo*, t. xlii (1917), p. 254.

where

$$\vartheta_{ij}=\frac{\partial^2\theta}{\partial x_i\,\partial x_j}-\sum_k\left[\{ij,k\}\frac{\partial\theta}{\partial x_k}\right].$$

This relation in $x_1', \ldots, x_n'$, must hold not merely for $z_1', \ldots, z_n'$, but also for all sets of variables

$$\alpha x_1'+\beta z_1',\quad \alpha x_2'+\beta z_2', \ldots,\quad \alpha x_n'+\beta z_n',$$

where α and β are parameters: that is, as in § 210, we must have three relations

$$\sum_i\sum_j\vartheta_{ij}x_i'x_j'=0,\quad \sum_i\sum_j\vartheta_{ij}x_i'z_j'=0,\quad \sum_i\sum_j\vartheta_{ij}z_i'z_j'=0,$$

satisfied in connection with each parametric equation $\theta=0$ when the surface is geodesic.

Now consider geodesics on this surface that are parallel. Because they are on the surface, the inferred property—that two parallel geodesics make the same angle in the same sense with the basic curve—can be used. Take any point R, on the basic regional geodesic ORE in the direction $x_1', \ldots, x_n'$, at any small arc-distance t from O; and through R let a geodesic lying in the surface be drawn parallel to the initial regional geodesic OTF, the direction-variables of which are denoted by $z_1', \ldots, z_n'$. If the direction-variables of this parallel geodesic at R be denoted by $\zeta_1', \ldots, \zeta_n'$, we can take

$$\zeta_i'=z_i'+t\frac{dz_i'}{ds_1}+\ldots,$$

for all values of i; and it is necessary to determine the analytical significance of the quantities $\dfrac{dz_i'}{ds_1}$. As the direction $\zeta_1', \ldots, \zeta_n'$, through the point R lies on the surface, specified by the $n-2$ parametric equations typified by $\theta=0$, the relation

$$\sum_i\left(\frac{\partial\theta}{\partial x_i}\right)_R\zeta_i'=0,$$

must be satisfied for each parametric equation $\theta=0$, the values of the derivatives $\dfrac{\partial\theta}{\partial x_i}$ being taken at R. But as the length of the geodesic arc OR is t, we have (for $i=1, \ldots, n$)

$$\left(\frac{\partial\theta}{\partial x_i}\right)_R=\frac{\partial\theta}{\partial x_i}+t\sum_j\theta_{ij}x_j'+\ldots,$$

where the unstated terms involve powers of t higher than the first; and therefore the relation at R becomes

$$\sum_i\frac{\partial\theta}{\partial x_i}\left(z_i'+t\frac{dz_i'}{ds_1}\right)+t\sum_i\sum_j\theta_{ij}z_i'x_j'+\ldots=0,$$

with the same significance as regards unstated terms. The direction $z_1', \ldots, z_n'$, at O lies in the surface, so that

$$\sum_i \frac{\partial \theta}{\partial x_i} z_i' = 0 ;$$

and therefore the θ-equation, taken so as to include only first powers of the small quantity t, yields the condition

$$\sum_i \frac{\partial \theta}{\partial x_i} \frac{dz_i'}{ds_1} + \sum_i \sum_j \theta_{ij} z_i' x_j' = 0.$$

We have seen that the relation

$$\sum_i \sum_j \vartheta_{ij} x_j' z_i' = 0$$

is satisfied, that is,

$$\sum_i \sum_j \theta_{ij} z_i' x_j' = \sum_i \sum_j \sum_k \frac{\partial \theta}{\partial x_k} \{ij, k\} z_i' x_j' ;$$

and therefore the condition becomes

$$\sum_i \frac{\partial \theta}{\partial x_i} \left[\frac{dz_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m' \right] = 0.$$

There are, in all, $n-2$ such conditions, one arising from each of the parametric equations specifying the geodesic surface.

Next, there is the permanent arc-relation at every place and for every direction in the amplitude. At R, for the direction $\zeta_1', \ldots, \zeta_n'$, this relation is

$$\sum_i \sum_j (A_{ij})_R \zeta_i' \zeta_j' = 1.$$

Now, for every value of k,

$$\zeta_k' = z_k' + t \frac{dz_k'}{ds_1} + \ldots ;$$

and as the length t, $= OR$, is measured along the geodesic ORE,

$$(A_{ij})_R = A_{ij} + t \frac{dA_{ij}}{ds_1} + \ldots ;$$

and therefore, when we write

$$\chi_m = \sum_j A_{mj} z_j'$$

in connection with the direction OTF, the permanent relation is

$$\sum A_{ij} z_i' z_j' + t \left\{ 2 \sum_m \chi_m \frac{dz_m'}{ds_1} + \sum_i \sum_j \frac{dA_{ij}}{ds_1} z_i' z_j' \right\} + O(t^2) = 1,$$

where $O(t^2)$ denotes the aggregate of terms in powers of t above the first. Also we have

$$\sum \sum A_{ij} z_i' z_j' = 1 ;$$

and the condition must be satisfied for all values of t; hence, as a necessary condition, we have

$$2\sum_m \chi_m \frac{dz_m'}{ds_1} + \sum_i \sum_j \frac{dA_{ij}}{ds_1} z_i' z_j' = 0.$$

By § 12,

$$\frac{dA_{ij}}{ds_1} = \sum_k \frac{\partial A_{ij}}{\partial x_k} x_k' = \sum_k \sum_p [\{ki, p\} A_{jp} + \{kj, p\} A_{ip}] x_k',$$

so that

$$\sum_i \sum_j \frac{dA_{ij}}{ds_1} z_i' z_j' = \sum_i \sum_j \sum_k \sum_p [\{ki, p\} A_{jp} + \{kj, p\} A_{ip}] z_i' z_j' x_k'.$$

Again

$$\sum_i \sum_j \sum_k \sum_p A_{jp} \{ki, p\} z_i' z_j' x_k' = \sum_i \sum_j \sum_k \{ki, p\} z_i' x_k' \chi_p$$
$$= \sum_i \chi_i [\sum_l \sum_m \{lm, i\} z_l' x_m'];$$

and similarly

$$\sum_i \sum_j \sum_k \sum_p A_{ip} \{kj, p\} z_i' z_j' x_k' = \sum_i \chi_i [\sum_l \sum_m \{lm, i\} z_l' x_m'].$$

Accordingly, the foregoing necessary condition, which arises out of the first power of t in the permanent arc-relation at R, is

$$\sum_i \chi_i \left[\frac{dz_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m'\right] = 0.$$

Finally, there is the condition for parallelism of the directions on the surface, represented analytically by the unchanging value of $\cos \epsilon$ along the geodesic ORE, so that

$$\frac{d}{ds_1}(\cos \epsilon) = \frac{d}{ds_1}\left\{\sum_i \sum_j A_{ij} x_j' z_i'\right\} = 0;$$

that is,

$$\sum_i \left\{\frac{dz_i'}{ds_1}\left(\sum_j A_{ij} x_j'\right)\right\} + \sum_q z_q' \left\{\frac{d}{ds_1}\left(\sum_m A_{qm} x_m'\right)\right\} = 0.$$

With the notation and the results of § 37, we have

$$u_k = \sum_m A_{km} x_m', \quad g_{pr} = \sum_t \{rt, p\} x_t',$$

$$\frac{du_k}{ds} = \sum_i g_{ik} u_i = \sum_i u_i \left(\sum_l \{kl, i\} x_l'\right);$$

thus the first term in the condition

$$= \sum_i u_i \frac{dz_i'}{ds_1},$$

and the second term in the condition

$$= \sum_q \sum_l \sum_i u_i \{ql, i\} x_l' x_q' = \sum_i u_i \left[\sum_l \sum_m \{lm, i\} x_l' z_m'\right];$$

and the condition therefore becomes

$$\sum_i u_i \left[\frac{dz_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m'\right] = 0.$$

Thus the magnitudes $Z_1, \ldots, Z_n$, where

$$Z_i = \frac{dz_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m',$$

satisfy n homogeneous linear equations in all, viz. the two equations

$$\sum_i \chi_i Z_i = 0, \quad \sum_i u_i Z_i = 0,$$

together with the $n-2$ equations

$$\sum_i \frac{\partial \theta}{\partial x_i} Z_i = 0,$$

one arising from each of the $n-2$ parametric equations $\theta = 0$ which define the geodesic surface in the amplitude. Let D denote the determinant of the coefficients of $Z_1, \ldots, Z_n$, so that

$$D = | \, u_i, \ \chi_i, \ \theta_i^{(1)}, \ \theta_i^{(2)}, \ \ldots, \ \theta_i^{(n-2)} \, |,$$

the equations of the geodesic surfaces being $\theta^{(k)} = 0$, for $k = 1, \ldots, n-2$. The determinant D does not vanish, a statement justified as follows.

The direction $x_1', \ldots, x_n'$, touching the surface, is such that the equations

$$\sum_i \frac{\partial \theta^{(k)}}{\partial x_i} x_i' = 0$$

hold for $k = 1, \ldots, n-2$. Consider the array

$$\left\| \begin{array}{cccc} \theta^{(1)}, & \theta_2^{(1)}, & \ldots\ldots, & \theta_n^{(1)} \\ \theta_1^{(2)}, & \theta_2^{(2)}, & \ldots\ldots, & \theta_n^{(2)} \\ \ldots & \ldots & \ldots & \ldots \\ \theta_1^{(n-2)}, & \theta_2^{(n-2)}, & \ldots\ldots, & \theta_n^{(n-2)} \end{array} \right\| ;$$

and let M_{ij} denote the determinant obtained by omitting the columns of rank i and j in a row. Owing to the independence of the functions θ, the quantities M_{ij} are not a vanishing set, though individual members may happen to vanish. Let M_{12} be different from zero; then the foregoing relations give

$$(-1)^{i-1} M_{12} x_i' = -M_{2i} x_1' - M_{1i} x_2',$$

for all values of i. Similarly, as the direction $z_1', \ldots, z_n'$, also touches the surface, we have

$$(-1)^{j-1} M_{12} z_j' = -M_{2j} z_1' - M_{1j} z_2',$$

for all values of j. Consequently

$$(-1)^{i+j}M_{12}^2 \begin{vmatrix} x_i', & x_j' \\ z_i', & z_j' \end{vmatrix} = \begin{vmatrix} M_{2i}, & M_{1i} \\ M_{2j}, & M_{1j} \end{vmatrix} \begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix} = -M_{12}M_{ij} \begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix},$$

or

$$\frac{(-1)^{i+j}}{M_{ij}} \begin{vmatrix} x_i', & x_j' \\ z_i', & z_j' \end{vmatrix} = -\frac{1}{M_{12}} \begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix},$$

for all values of i and j. Now

$$D = \sum_i \sum_j (-1)^{i+j-1} \begin{vmatrix} u_i, & u_j \\ \chi_i, & \chi_j \end{vmatrix} M_{ij}$$

$$= \frac{M_{12}}{\begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix}} \sum_i \sum_j \begin{vmatrix} u_i, & u_j \\ \chi_i, & \chi_j \end{vmatrix} \begin{vmatrix} x_i', & x_j' \\ z_i', & z_j' \end{vmatrix}$$

$$= \frac{M_{12}}{\begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix}} \sum_i \sum_j \sum_k \sum_l \begin{vmatrix} A_{ik}, & A_{il} \\ A_{jk}, & A_{jl} \end{vmatrix} \begin{vmatrix} x_k', & x_l' \\ z_k', & z_l' \end{vmatrix} \begin{vmatrix} x_i', & x_j' \\ z_i', & z_j' \end{vmatrix}$$

$$= \frac{M_{12} \sin^2 \epsilon}{\begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix}},$$

a quantity which does not vanish.

Accordingly, since the determinant of the coefficients of the n magnitudes Z_i in the n linear homogeneous equations does not vanish, each of those magnitudes must vanish ; and so we have the n relations

$$\frac{dz_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m' = 0, \quad (i = 1, \dots, n),$$

as the primary conditions to be satisfied for the arc-variations of the direction-variables $z_1', \dots, z_n'$, of a succession of geodesics in the amplitude, drawn (geodesically) parallel to one another along a basic geodesic ORE, the arc-variation ds_1 being taken along the geodesic *.

The formal result agrees with the result first established by Levi-Civita *, from the considerations set out in § 221 by reference to an aggregate of lines in the plenary space. The preceding process is based upon the Severi definition ; in the

* In the establishment of the result, the geodesic property of ORE has not entered : up to this stage, any curve C, in the same direction at O, could have been used without affecting the argument. But when approximations for points R along the curve C are desired, proceeding in powers of t (the length of the small arc OR), the limitation of the basic curve to geodesic quality in the amplitude secures much simplification in the necessary formulæ.

approximations including the first power of t, the two results agree. But this stage is the limit of this accordance: when we take account of powers of t beyond the first, it will be found that the values of $\zeta_1', \ldots, \zeta_n'$, the direction-variables of an amplitudinal geodesic, parallel to the initially postulated geodesic, are not the same under the Severi definition of parallelism as under the Levi-Civita definition.

Geodesics drawn from a basic curve according to any assigned law of changing inclination.

223. The preceding investigation relates to parallel geodesics. We may, however, conceive a succession of geodesics drawn through successive points along the geodesic ORE, according to an assigned law of continuous variation (instead of constant value) of the angle of inclination ϵ made with ORE at the successive points. For the purpose, the geodesic surface determined by the amplitudinal geodesics ORE and OTF is used, exactly as it was used initially by Severi. We still denote the direction-variables of the new geodesic drawn through R by

$$\zeta_i' = z_i' + t\frac{dz_i'}{ds_1} + \ldots;$$

and we still consider the conditions which emerge by taking account of the first power of the small arc t. As this geodesic at R lies in the geodesic surface, the $n-2$ equations

$$\sum_i \frac{\partial\theta}{\partial x_i}\left[\frac{dz_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m'\right] = 0$$

still hold, as also does the equation

$$\sum_i u_i \left[\frac{\partial z_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m'\right] = 0$$

deduced from the permanent arc-relation at R

$$\sum_i \sum_j (A_{ij})_R \zeta_i' \zeta_j' = 1,$$

or from its equivalent

$$\frac{d}{ds_1}\left(\sum_i \sum_j A_{ij} z_i' z_j'\right) = 0.$$

But the relation deduced from the equation

$$\cos\epsilon = \sum_i \sum_j A_{ij} x_j' z_i'$$

is changed from the earlier form, because of the law of variation of the angle ϵ; and it yields the relation

$$-\frac{d\epsilon}{ds_1}\sin\epsilon = \sum_i \chi_i \left[\frac{dz_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m'\right],$$

always with the same notation as before.

When these n equations, linear (but no longer homogeneous) in the n quantities

$$\frac{dz_i'}{ds_1} + \sum_l \sum_m \{lm, i\} x_l' z_m',$$

are resolved for these quantities, we have

$$D\left[\frac{dz_1'}{ds_1} + \sum_l \sum_m \{lm, 1\} x_l' z_m'\right] = -D_1 \frac{d\epsilon}{ds_1} \sin\epsilon,$$

with the same significance for D as before, while D_1 is the minor of u_1 in D. But

$$\begin{aligned} D_1 = & \begin{vmatrix} \chi_2, & \chi_3, & \dots, & \chi_n \\ \dfrac{\partial\theta^{(1)}}{\partial x_2}, & \dfrac{\partial\theta^{(1)}}{\partial x_3}, & \dots, & \dfrac{\partial\theta^{(1)}}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial\theta^{(n-2)}}{\partial x_2}, & \dfrac{\partial\theta^{(n-2)}}{\partial x_3}, & \dots, & \dfrac{\partial\theta^{(n-2)}}{\partial x_n} \end{vmatrix} \\ = & M_{12}\chi_2 - M_{13}\chi_3 + \dots + (-1)^n M_{1n}\chi_n \\ = & \frac{M_{12}}{\begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix}} \left\{ \begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix} \chi_2 + \begin{vmatrix} x_1', & x_3' \\ z_1', & z_3' \end{vmatrix} \chi_3 + \dots + \begin{vmatrix} x_1', & x_n' \\ z_1', & z_n' \end{vmatrix} \chi_n \right\} \\ = & \frac{M_{12}}{\begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix}} \left\{ x_1' \left(\sum_l z_l'\chi_l\right) - z_1' \left(\sum_l x_l'\chi_l\right) \right\} \\ = & \frac{M_{12}}{\begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix}} (x_1' - z_1' \cos\epsilon); \end{aligned}$$

and the value of D is (§ 222)

$$D = \frac{M_{12}}{\begin{vmatrix} x_1', & x_2' \\ z_1', & z_2' \end{vmatrix}} \sin^2\epsilon.$$

Hence

$$\frac{dz_1'}{ds_1} + \sum_l \sum_m \{lm, 1\} x_l' z_m' = -\frac{x_1' - z_1' \cos\epsilon}{\sin\epsilon} \frac{d\epsilon}{ds_1}.$$

Similarly for the other magnitudes; the aggregate result can be expressed in the form

$$\frac{dz_k}{ds_1} + \sum_l \sum_m \{lm, k\} x_l' z_m' = -\frac{x_k' - z_k' \cos\epsilon}{\sin\epsilon} \frac{d\epsilon}{ds_1},$$

for all the values $k = 1, \dots, n$.

224. Now at the point O draw an amplitudinal geodesic OQD in any direction not lying in the orientation ROT, with variables $t_1', \ldots, t_n'$; and let ds_3 denote arc-variation along OD. Through Q, where OQ is a small arc denoted by γ, let an amplitudinal geodesic QE' be drawn parallel to OE and another amplitudinal geodesic QF' be drawn parallel to OF; and, denoting the angle EOF by ϵ, let the angle $E'QF'$ be denoted by η, so that we can take

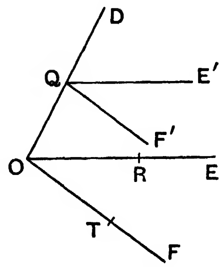

FIG. 23.

$$\eta = \epsilon + \gamma \frac{d\epsilon}{ds_3} + \ldots .$$

We proceed to find $\dfrac{d\epsilon}{ds_3}$.

We have

$$\cos \epsilon = \sum_i \sum_j A_{ij} x_i' z_j',$$

so that

$$-\sin \epsilon \frac{d\epsilon}{ds_3} = \sum_i \sum_j \left(\frac{dA_{ij}}{ds_3} x_i' z_j' + A_{ij} z_j' \frac{dx_i'}{ds_3} + A_{ij} x_i' \frac{dz_j'}{ds_3} \right).$$

Now we have (§ 12)

$$\begin{aligned} \frac{dA_{lm}}{ds_3} &= \sum_k t_k' \frac{\partial A_{lm}}{\partial x_k} \\ &= \sum_k \sum_p t_k' [A_{mp}\{kl, p\} + A_{lp}\{km, p\}]; \end{aligned}$$

also, because of parallelism,

$$\frac{dx_i'}{ds_3} = -\sum_l \sum_m \{lm, i\} t_l' x_m',$$

$$\frac{dz_j'}{ds_3} = -\sum_a \sum_b \{ab, j\} t_a' z_b'.$$

Let these values be substituted in the expression for $-\sin \epsilon \dfrac{d\epsilon}{ds_3}$, and let the complete coefficient of the magnitude $\{cd, e\}$ be selected. It is

$$\begin{aligned} &= \sum_m [A_{me} z_m' (x_d' t_c' + x_c' t_d')] + \sum_l [A_{le} x_l' (z_d' t_c' + z_c' t_d')] \\ &\qquad - \sum_j [A_{ej} z_j' (t_c' x_d' + t_d' x_c')] - \sum_i [A_{ei} x_i' (t_c' z_d' + t_d' z_c')], \end{aligned}$$

which vanishes; and therefore

$$\frac{d\epsilon}{ds_3} = 0.$$

Consequently, we have (up to the first order of small quantities)

$$\eta = \epsilon.$$

Hence at the solid angle at Q, determined by the directions QD, QE', QF', we have

$$E'QF' = EOF, \quad E'QD = EOD, \quad F'QD = FOD;$$

and therefore the solid angle at O is equal to the solid angle at Q, up to the first order of small quantities.

It follows that a solid angle, determined by three directions at any point, can be geodesically moved along any one of its determining directions without a first-order alteration of magnitude, by means of sets of geodesic parallels; and the faces of the solid angle can be regarded as remaining geodesically parallel throughout any such displacement, also up to the first order of small quantities.

Let the orientation-variables of the orientation EOF, being

$$\left\| \begin{matrix} x_1', & \ldots\ldots, & x_n' \\ z_1, & \ldots\ldots, & z_n' \end{matrix} \right\|,$$

be represented by ξ_{ij}, where $\xi_{ij} = x_i'z_j' - x_j'z_i'$; then it is easy to verify that the orientation-variables of the geodesically parallel orientation at Q are given by

$$\xi_{ij} + \gamma \frac{d\xi_{ij}}{ds_3} + \ldots,$$

where

$$\frac{d\xi_{ij}}{ds_3} = -\sum_l t_l' [\xi_{mj}\{lm, i\} + \xi_{im}\{lm, j\}].$$

Similarly for angles of higher multiplicity and for the variation of corresponding variables for geodesically parallel orientations of higher multiplicity.

Second approximation for Levi-Civita parallels in a region.

225. Returning now to the consideration of parallel geodesics in a region, we have to provide the developments in the alternative definitions of Levi-Civita and Severi. We still take a geodesic ORE, in the regional direction p_1', q_1', r_1', through O, as the basic curve C; and we still denote the geodesic arc OR by t, though now it will be necessary to include powers of t higher than the first in the approximations. The direction-variables at O of the initial geodesic OTF will be denoted by p_2', q_2', r_2'; and thus the characteristic equations of the geodesic parallelism, on both the definitions, are

$$\left.\begin{aligned} \frac{dp_2'}{ds_1} &= (\Gamma \between p_1', q_1', r_1' \between p_2', q_2', r_2') = \bar{\gamma}_{12} \\ -\frac{dq_2'}{ds_1} &= (\Delta \between p_1', q_1', r_1' \between p_2', q_2', r_2') = \bar{\delta}_{12} \\ -\frac{dr_2'}{ds_1} &= (\Theta \between p_1', q_1', r_1' \between p_2', q_2', r_2') = \bar{\theta}_{12} \end{aligned}\right\},$$

with the significance of $\bar{\gamma}$, $\bar{\delta}$, $\bar{\theta}$, as defined on p. 66.

It is clear on purely geometrical grounds that, in the developments from the alternative definitions, there must be a divergence of results. In the Levi-Civita process, the parallelism is defined by properties connected with a developable

region, tangential to the given region, and ultimately associated with the Euclidean parallelism in the developed flat as a homaloid in the plenary space. In the Severi process, the parallelism is defined by properties associated with parallelism on a geodesic surface of the region defined by means of the two geodesics ORE and OTF through O; and when once the characteristic of parallelism on this surface is imported, there is no reference to the plenary homaloidal space of the whole configuration. We may expect (and we have found) an initial agreement between the parallel directions under the two definitions; subsequent divergence, arising from the deviation of the region from the tangent flat, is also to be expected.

We begin with developments of the Levi-Civita process. The direction-variables of the regional geodesics at successive points R along ORE, which are geodesically parallel to OTF and so ultimately have relations with a fixed direction in the developed flat, depend upon the length t of the geodesic arc OR and differ from one another in expression solely through this quantity t. Let the direction-variables at R of the regional geodesic, which is parallel there to OTF, be denoted by P_2', Q_2', R_2'; so that

$$\left.\begin{aligned} P_2' &= p_2' + t\frac{dp_2'}{ds_1} + \tfrac{1}{2}t^2\frac{d^2p_2'}{ds_1^2} + \dots \\ Q_2' &= q_2' + t\frac{dq_2'}{ds_1} + \tfrac{1}{2}t^2\frac{d^2q_2'}{ds_1^2} + \dots \\ R_2' &= r_2' + t\frac{dr_2'}{ds_1} + \tfrac{1}{2}t^2\frac{d^2r_2'}{ds_1^2} + \dots \end{aligned}\right\},$$

where the second and higher derivatives of p_2', q_2', r_2', along ORE are analytical consequences of the first derivatives given by the preceding characteristic equations. For the present purpose, it will suffice to determine the second arc-derivatives of p_2', q_2', r_2'.

With the notation of § 172, the characteristic first-order equations can be written

$$\begin{aligned} -\frac{dp_2'}{ds_1} &= \alpha_1 p_2' + \beta_1 q_2' + \gamma_1 r_2', \\ -\frac{dq_2'}{ds_1} &= \xi_1 p_2' + \eta_1 q_2' + \zeta_1 r_2', \\ -\frac{dr_2'}{ds_1} &= \phi_1 p_2' + \chi_1 q_2' + \psi_1 r_2'. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d^2p_2'}{ds_1^2} = (\alpha_1^2 + \beta_1\xi_1 + \gamma_1\phi_1)p_2' + (\alpha_1\beta_1 + \beta_1\eta_1 + \gamma_1\chi_1)q_2' + (\alpha_1\gamma_1 + \beta_1\zeta_1 + \gamma_1\psi_1)r_2' \\ - \alpha_2 p_1'' - \beta_2 q_1'' - \gamma_2 r_1'' - \sum \frac{d\Gamma_{11}}{ds_1} p_1' p_2'. \end{aligned}$$

In the result obtained (§ 212) for $\sum \dfrac{d\Gamma_{11}}{ds_l} p_m' p_n'$, take $l=1$, $m=1$, $n=2$; then

noting that $\bar{\gamma}_{11} = -p_1''$, $\delta_{11} = -q_1''$, $\bar{\theta}_{11} = -r_1''$, from the definitions of $\bar{\gamma}_{ij}$, δ_{ij}, $\bar{\theta}_{ij}$, we have

$$\sum \frac{d\Gamma_{11}}{ds_1} p_1' p_2' = \alpha_1 \bar{\gamma}_{12} + \beta_1 \delta_{12} + \gamma_1 \bar{\theta}_{12} - (\alpha_2 p_1'' + \beta_2 q_1'' + \gamma_2 r_1'')$$
$$+ (\Gamma_{300} p_1'^2 p_2') + \frac{1}{3\Omega} K_1(1, 12) + \frac{1}{3\Omega} K_1(,1\ 12),$$

where (§ 212) the symbol K_1 is defined as being

$$K_1(l, mn) = \begin{vmatrix} a, & p_l', & k_{11}\xi_{mn} + k_{12}\eta_{mn} + k_{13}\zeta_{mn} \\ h, & q_l', & k_{12}\xi_{mn} + k_{22}\eta_{mn} + k_{23}\zeta_{mn} \\ g, & r_l', & k_{13}\xi_{mn} + k_{23}\eta_{mn} + k_{33}\zeta_{mn} \end{vmatrix}.$$

Thus $K_1(2, 11) = 0$. Also

$$\alpha_1 \bar{\gamma}_{12} + \beta_1 \delta_{12} + \gamma_1 \bar{\theta}_{12} = \alpha_1(\alpha_1 p_2' + \beta_1 q_2' + \gamma_1 r_2')$$
$$+ \beta_1(\xi_1 p_2' + \eta_1 q_2' + \zeta_1 r_2')$$
$$+ \gamma_1(\phi_1 p_2' + \chi_1 q_2' + \psi_1 r_2');$$

and therefore

$$-\frac{d^2 p_2'}{ds_1^2} = (\Gamma_{300} p_1'^2 p_2') + \frac{1}{3\Omega} \begin{vmatrix} a, & p_1', & k_{11}\xi_{12} + k_{12}\eta_{12} + k_{13}\zeta_{12} \\ h, & q_1', & k_{12}\xi_{12} + k_{22}\eta_{12} + k_{23}\zeta_{12} \\ g, & r_1', & k_{13}\xi_{12} + k_{23}\eta_{12} + k_{33}\zeta_{12} \end{vmatrix}$$
$$= (\Gamma_{300} p_1'^2 p_2') + \frac{1}{3\Omega} K_1(1, 12).$$

Obviously $\dfrac{d^2 p_2'}{ds_1^2}$ is a linear homogeneous function of p_2', q_2', r_2', as also is each of the arc-derivatives of p_2', q_2', r_2', of all orders. It may also be noted, as an incidental verification, that by taking the directions OTF and ORE to coincide, we obtain the customary value (§ 163) for p_1''' in a region.

Similarly for the second arc-derivatives of q_2' and r_2': they are

$$-\frac{d^2 q_2'}{ds_1^2} = (\Delta_{300} p_1'^2 p_2') + \frac{1}{3\Omega} \begin{vmatrix} h, & p_1', & k_{11}\xi_{12} + k_{12}\eta_{12} + k_{13}\zeta_{12} \\ b, & q_1', & k_{12}\xi_{12} + k_{22}\eta_{12} + k_{23}\zeta_{12} \\ f, & r_1', & k_{13}\xi_{12} + k_{23}\eta_{12} + k_{33}\zeta_{12} \end{vmatrix}$$
$$= (\Delta_{300} p_1'^2 p_2') + \frac{1}{3\Omega} K_2(1, 12);$$

$$-\frac{d^2 r_2'}{ds_1^2} = (\Theta_{300} p_1'^2 p_2') + \frac{1}{3\Omega} \begin{vmatrix} g, & p_1', & k_{11}\xi_{12} + k_{12}\eta_{12} + k_{13}\zeta_{12} \\ f & q_1', & k_{12}\xi_{12} + k_{22}\eta_{12} + k_{23}\zeta_{12} \\ c, & r_1', & k_{13}\xi_{12} + k_{23}\eta_{12} + k_{33}\zeta_{12} \end{vmatrix}$$
$$= (\Theta_{300} p_1'^2 p_2') + \frac{1}{3\Omega} K_3(1, 12).$$

Ex. Verify the relation

$$\sum (A)_R P_2'^2 = 1$$

up to the second order of small quantities inclusive; and, to the same order of small quantities, the relation

$$\sum (A)_R (P_1')_R P_2' = \cos \epsilon,$$

where $(P_1')_R$, $(Q_1')_R$, $(R_1')_R$, are the direction-variables at R of the basic geodesic ORE in the direction RE, the angle ϵ being constant as R moves along ORE.

Levi-Civita parallels do not provide regional parallelograms.

226. At the point R let a regional geodesic be drawn parallel to OTF in the Levi-Civita sense; and at the point T let a regional geodesic be drawn parallel to ORE also in the Levi-Civita sense. If the construction were effected on a free surface, the two new geodesics would meet and thus complete a geodesic parallelogram (§ 120). But the Levi-Civita definition does not take specific account even of the geodesic surface determined by OR and OT: and therefore it is to be expected that the new regional geodesics through R and T do not meet.

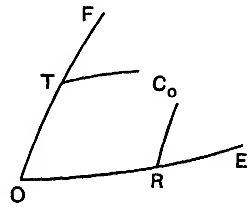

Fig. 24.

To establish this negative result, it will be sufficient to shew that the equations connected with the hypothesis of an actual intersection are inconsistent. Let this hypothetical intersection be C_0 in the diagram. For arc-lengths along the different geodesics, let

$$OR = x, \quad TC_0 = X = x + L,$$
$$OT = y, \quad RC_0 = Y = y + M;$$

where x and y are small quantities of the first order, L and M are small quantities of the second order at least. Let the direction-variables of TC_0 at T, parallel to OR, be P_1', Q_1', R_1'; and those of RC_0 at R, parallel to OT, be P_2', Q_2', R_2'. Then for these Levi-Civita parallels,

$$P_1' = p_1' + y\frac{dp_1'}{ds_2} + \tfrac{1}{2}y^2\frac{d^2p_1'}{ds_2^2} + \dots,$$

$$P_2' = p_2' + x\frac{dp_2'}{ds_1} + \tfrac{1}{2}x^2\frac{d^2p_2'}{ds_1^2} + \dots,$$

with like values for Q_1', R_1', Q_2', R_2'.

The values of the regional parameters at the hypothetical intersection C_0 must be the same, whether the intersection be approached by the broken geodesic path OR, RC_0, or by the broken geodesic path OT, TC_0. By the former path, the value at C_0 of the p-parameter is

$$= p_R + YP_2' + \tfrac{1}{2}Y^2P_2'' + \tfrac{1}{6}Y^3P_2''' + \dots$$
$$= p + xp_1' + \tfrac{1}{2}x^2p_1'' + \tfrac{1}{6}x^3p_1''' + YP_2' + \tfrac{1}{2}Y^2P_2'' + \tfrac{1}{6}Y^3P_2''' + \dots,$$

neglecting powers of small quantities higher than the third; and by the latter path, the value of the same parameter is

$$=p+yp_2'+\tfrac{1}{2}y^2p_2''+\tfrac{1}{6}y^3p_2'''+XP_1'+\tfrac{1}{2}X^2P_1''+\tfrac{1}{6}X^3P_1'''+\dots,$$

up to the same order. The equality of these values requires a relation

$$XP_1'-xp_1'+\tfrac{1}{2}(X^2P_1''-x^2p_1'')+\tfrac{1}{6}(X^3P_1'''-x^3p_1''')$$
$$=YP_2'-yp_2'+\tfrac{1}{2}(Y^2p_2''-y^2p_2'')+\tfrac{1}{6}(Y^3P_2'''-y^3p_2'''),$$

neglecting higher powers of small quantities; and there are similar relations arising out of the q-parameter and the r-parameter at C_0.

In these relations, we substitute the values of P_1', Q_1', R_1'; P_2', Q_2', R_2'; and proceed by approximations.

The terms of the first order of small quantities balance without leaving any residuary condition.

When L and M (both of order higher than the first) are retained in the approximation of the second order, we have

$$XP_1'=(x+L)\left(p_1'+y\frac{dp_1'}{ds_2}\right),$$

and therefore

$$XP_1'-xp_1'=Lp_1'+xy\frac{dp_1'}{ds_2}=Lp_1'-xy\sum\Gamma_{11}p_1'p_2',$$

neglecting a term in yL as being of higher order. Similarly, up to the second order,

$$YP_2'-yp_2'=Mp_2'-xy\sum\Gamma_{11}p_1'p_2'.$$

Also, up to the second order,

$$X^2P_1''=x^2P_1''=-x^2\sum(\Gamma_{11})_T P_1'^2=-x^2\sum\Gamma_{11}p_1'^2=x^2p_1'',$$

so that, to this order,

$$X^2P_1''=x^2p_1'';$$

and similarly, to the same order,

$$Y^2P_2''=y^2p_2''.$$

The remaining terms of the relation are of order higher than the second: hence, in this second-order approximation, there is a residuary condition

$$Lp_1'=Mp_2'$$

from the p-parameter general relation. Similarly, from the other two relations,

$$Lq_1'=Mq_2', \quad Lr_1'=Mr_2'.$$

It follows that $L=0$, $M=0$: that is, within the second order of small quantities. Accordingly, if the equations can coexist, the magnitudes L and M must be of the third order at least.

Proceeding now to the third-order approximation, we can (for this purpose) take

$$X^2=x^2,\quad X^3=x^3,\quad Y^2=y^2,\quad Y^3=y^3.$$

We now have

$$XP_1'=(x+L)\left(p_1'+y\frac{dp_1'}{ds_2}+\tfrac{1}{2}y^2\frac{d^2p_1'}{ds_2^2}\right)$$

$$=xp_1'+xy\frac{dp_1'}{ds_2}+\tfrac{1}{2}xy^2\frac{d^2p_1'}{ds_2^2}+Lp_1',$$

to this order; and therefore

$$XP_1'-xp_1'=-xy\sum\Gamma_{11}p_1'p_2'+\tfrac{1}{2}xy^2\frac{d^2p_1'}{ds_2^2}+Lp_1'.$$

Similarly, to the third order,

$$YP_2'-yp_2'=-xy\sum\Gamma_{11}p_1'p_2'+\tfrac{1}{2}x^2y\frac{d^2p_2}{ds_1^2}+Mp_2'.$$

Hence

$$(XP_1'-xp_1')-(YP_2'-yp_2')=Lp_1'-Mp_2'+\tfrac{1}{2}\left(xy^2\frac{d^2p_1}{ds_2^2}-x^2y\frac{d^2p_2}{ds_1^2}\right).$$

Again, we have

$$P_1''=-\sum(\Gamma_{11})_T P_1'^2,$$

and P_1'' has a factor X^2, $=x^2$; so that, in the developed value of P_1'', terms of the first order must be retained. Now

$$(\Gamma_{ij})_T=\Gamma_{ij}+y\frac{d\Gamma_{ij}}{ds_2},$$

up to the first order; and therefore

$$P_1''=-\sum\Gamma_{11}p_1'^2-y\sum\frac{d\Gamma_{11}}{ds_2}p_1'^2-2y\left(\alpha_1\frac{dp_1'}{ds_2}+\beta_1\frac{dq_1'}{ds_2}+\gamma_1\frac{dr_1'}{ds_2}\right)$$

$$=p_1''-y\sum\frac{d\Gamma_{11}}{ds_2}p_1'^2-2y\left(\alpha_1\frac{dp_1'}{ds_2}+\beta_1\frac{dq_1'}{ds_2}+\gamma_1\frac{dr_1'}{ds_2}\right).$$

Hence

$$X^2P_1''-x^2p_1''=-x^2y\sum\frac{d\Gamma_{11}}{ds_2}p_1'^2-2x^2y\left(\alpha_1\frac{dp_1'}{ds_2}+\beta_1\frac{dq_1'}{ds_2}+\gamma_1\frac{dr_1'}{ds_2}\right).$$

But, by the result in § 212 when we take $l=2$, $m=1$, $n=1$,

$$\sum\frac{d\Gamma_{11}}{ds_2}p_1'^2=2(\alpha_1\bar{\gamma}_{12}+\beta_1\delta_{12}+\gamma_1\bar{\theta}_{12})+(\Gamma_{300}p_1'^2p_2')+\frac{2}{3\Omega}K_1(1,21);$$

and, as in §§ 221, 225,

$$\frac{dp_1'}{ds_2}=-\bar{\gamma}_{12},\quad \frac{dq_1'}{ds_2}=-\delta_{12},\quad \frac{dr_1'}{ds_2}=-\bar{\theta}_{12};$$

therefore

$$X^2P_1''-x^2p_1''=-x^2y(\Gamma_{300}p_1'^2p_2')-\frac{2}{3\Omega}x^2yK_1(1,21).$$

In the same way, we find

$$Y^2P_2'' - y^2p_2'' = -xy^2(\Gamma_{300}p_1'p_2'^2) - \frac{2}{3\Omega}xy^2K_1(2, 12).$$

Further, in the remaining terms up to the third order (inclusive, but not to higher orders), we have

$$X^3P_1''' = x^3p_1''', \quad Y^3P_2''' = y^3p_2'''.$$

Consequently, the third-order residuary condition arising through the p-parameter becomes

$$Lp_1' - Mp_2' + \tfrac{1}{2}\left(xy^2\frac{d^2p_1}{ds_2^2} - x^2y\frac{d^2p_2}{ds_1^2}\right)$$
$$= \tfrac{1}{2}x^2y(\Gamma_{300}p_1'^2p_2') + \frac{1}{3\Omega}x^2yK_1(1, 21) - \tfrac{1}{2}xy^2(\Gamma_{300}p_1'p_2'^2) - \frac{1}{3\Omega}xy^2K_1(2, 12).$$

But (§ 225)

$$\frac{d^2p_2'}{ds_1^2} = -(\Gamma_{300}p_1'^2p_2') - \frac{1}{3\Omega}K_1(1, 12),$$
$$\frac{d^2p_1'}{ds_2^2} = -(\Gamma_{300}p_1'p_2'^2) - \frac{1}{3\Omega}K_1(2, 21);$$

when these values are substituted, and terms are collected, the condition is found to be

$$Lp_1' - Mp_2' = \frac{1}{2\Omega}\{x^2yK_1(1, 21) - xy^2K_1(2, 12)\}.$$

When treated in the same way, the q-parameter equality and the r-parameter equality provide the respective residuary conditions

$$Lq_1' - Mq_2' = \frac{1}{2\Omega}\{x^2yK_2(1, 21) - xy^2K_2(2, 12)\},$$

$$Lr_1' - Mr_2' = \frac{1}{2\Omega}\{x^2yK_3(1, 21) - xy^2K_3(2, 12)\}.$$

Thus, in the third-order approximation, there are apparently three conditions which must be satisfied by the two magnitudes. For a general region, the magnitudes $K_i(l, mn)$, for $i=1, 2, 3$, do not vanish in general; while x and y, small arc-lengths, are independent of one another. Thus we cannot have L and M both vanishing: that is, the alternative of fourth-order magnitude for L and M is excluded.

If the three equations could coexist for these non-vanishing magnitudes L and M, then the relation

$$x^2y[\xi_{12}K_1(1, 21) + \eta_{12}K_2(1, 21) + \zeta_{12}K_3(1, 21)]$$
$$= xy^2[\xi_{12}K_1(2, 12) + \eta_{12}K_2(2, 12) + \zeta_{12}K_3(2, 12)]$$

would have to be satisfied for all values of x and y, and for arbitrarily assumed directions ORE, OTF, at O—a requirement manifestly not satisfied.

Thus the three equalities, arising necessarily from the values of the parameters at the hypothetical intersection C_0, do not hold. Consequently the Levi-Civita regional geodesic, through R parallel to OT, does not meet the Levi-Civita regional geodesic, through T parallel to OR.

Second approximation for Severi parallels.

227. We now pass to the consideration of the same approximations for the direction of a Severi regional geodesic through R, parallel to OTF, as have been effected for the direction of a Levi-Civita regional geodesic parallel to the same initial geodesic. After the investigation of § 222 which applies to any n-fold configuration, we shall assume that, for the Severi geodesic through R, parallel to OTF, we may take its direction-variables at R as given by

$$p_2' - x\bar{\gamma}_{12}, \quad q_2' - x\bar{\delta}_{12}, \quad r_2' - x\bar{\theta}_{12},$$

up to the first power of the small arc-length; and we denote those direction-variables, to the next order of approximation, by

$$\bar{P}_2' = p_2' - x\bar{\gamma}_{12} + \tfrac{1}{2}x^2 P, \quad \bar{Q}_2' = q_2' - x\bar{\delta}_{12} + \tfrac{1}{2}x^2 Q, \quad \bar{R}_2' = r_2' - x\bar{\theta}_{12} + \tfrac{1}{2}x^2 R.$$

Thus, for the required approximation, the three quantities P, Q, R, must be obtained; and, serving to determine them, there are three data. We have (i), the permanent arc-relation

$$\sum (A)_R \bar{P}_2'^2 = 1$$

at R; (ii), the requirement of geodesic parallelism between the geodesic through R and the geodesic OTF, lying on the same geodesic surface, in the form

$$\sum (A)_R \bar{P}_2' (p_1')_R = \cos\epsilon = \sum A p_2' p_1',$$

where $(p_1')_R$, $(q_1')_R$, $(r_1')_R$, are the direction-variables of ORE at R in the direction RE; and (iii), the requirement of lying in that geodesic surface at R, expressed by the condition

$$(\theta_1)_R \bar{P}_2' + (\theta_2)_R \bar{Q}_2' + (\theta_3)_R \bar{R}_2' = 0.$$

The only small quantity, which occurs in this stage of the enquiry, is the magnitude x of the small arc of the geodesic ORE.

We shall begin with the third of these three conditions, the surface-requirement. Here, up to the second order of small quantities,

$$(\theta_1)_R = \theta_1 + x\frac{d\theta_1}{ds_1} + \tfrac{1}{2}x^2 \frac{d^2\theta_1}{ds_1^2}.$$

Now

$$\begin{aligned} \frac{d\theta_1}{ds_1} &= \theta_{11} p_1' + \theta_{12} q_1' + \theta_{13} r_1' \\ &= \vartheta_{11} p_1' + \vartheta_{12} q_1' + \vartheta_{13} r_1' + \theta_1 \alpha_1 + \theta_2 \xi_1 + \theta_{31}\phi. \end{aligned}$$

Because the surface at O is geodesic to the region, certain second-order conditions (p. 61) are satisfied; when these are used, we find

$$\vartheta_{11}p_1' + \vartheta_{12}q_1' + \vartheta_{22}r_1' = \vartheta_{11}p_1' + \tfrac{1}{2}\left(\frac{\vartheta_{11}}{\theta_1^2} + \frac{\vartheta_{22}}{\theta_2^2}\right)\theta_1\theta_2 q_1' + \tfrac{1}{2}\left(\frac{\vartheta_{11}}{\theta_1^2} + \frac{\vartheta_{33}}{\theta_3^2}\right)\theta_1\theta_3 r_1'$$

$$= \tfrac{1}{2}\frac{\vartheta_{11}}{\theta_1}(\theta_1 p_1' + \theta_2 q_1' + \theta_3 r_1') + \tfrac{1}{2}\theta_1\left(\frac{\vartheta_{11}}{\theta_1}p_1' + \frac{\vartheta_{22}}{\theta_2}q_1' + \frac{\vartheta_{33}}{\theta_3}r_1'\right) = \tfrac{1}{2}\theta_1 T_1,$$

where

$$T_1 = \frac{\vartheta_{11}}{\theta_1}p_1' + \frac{\vartheta_{22}}{\theta_2}q_1' + \frac{\vartheta_{33}}{\theta_3}r_1';$$

and therefore

$$\frac{d\theta_1}{ds_1} = \theta_1\alpha_1 + \theta_2\xi_1 + \theta_3\phi_1 + \tfrac{1}{2}\theta_1 T_1.$$

Similarly

$$\frac{d\theta_2}{ds_1} = \theta_1\beta_1 + \theta_2\eta_1 + \theta_3\chi_1 + \tfrac{1}{2}\theta_2 T_1,$$

$$\frac{d\theta_3}{ds_1} = \theta_1\gamma_1 + \theta_2\zeta_1 + \theta_3\psi_1 + \tfrac{1}{2}\theta_3 T_1.$$

Also, denoting by T_2 the result of substituting p_2', q_2', r_2', in T_1, we have the three corresponding relations of the type

$$\frac{d\theta_1}{ds_2} = \theta_1\alpha_2 + \theta_2\xi_2 + \theta_3\phi_2 + \tfrac{1}{2}\theta_1 T_2.$$

(i) When substitution of the values of $(\theta_1)_R$, $(\theta_2)_R$, $(\theta_3)_R$, and of the postulated values of $\bar{P}_2'$, $\bar{Q}_2'$, $\bar{R}_2'$, is effected in the surface-condition, the terms free from the small quantity x are

$$\theta_1 p_2' + \theta_2 q_2' + \theta_3 r_2',$$

which vanishes: no residuary condition is left. In the aggregate of terms, involving the first power of x, the coefficient of that first power

$$= p_2'\frac{d\theta_1}{ds_1} + q_2'\frac{d\theta_2}{ds_1} + r_2'\frac{d\theta_3}{ds} - (\theta_1\bar{\gamma}_{12} + \theta_2\bar{\delta}_{12} + \theta_3\bar{\theta}_{13}),$$

which also vanishes identically: again, no residuary condition is left *. Finally, for the present approximation, the aggregate of terms involving x^2 must disappear, so that the coefficient of $\frac{1}{2}x^2$ must vanish; and this demand leaves the residuary condition $C=0$, where

$$C = p_2'\frac{d^2\theta_1}{ds_1^2} + q_2'\frac{d^2\theta_2}{ds_1^2} + r_2'\frac{d^2\theta_3}{ds_1^2}$$
$$- 2\left(\bar{\gamma}_{12}\frac{d\theta_1}{ds_1} + \bar{\delta}_{12}\frac{d\theta_2}{ds_1} + \bar{\theta}_{12}\frac{d\theta_3}{ds_1}\right)$$
$$+ \theta_1 P + \theta_2 Q + \theta_3 R.$$

* This result is, in effect, merely a verification of the general result of § 222, when the n-fold amplitude is specialised into a region.

As regards the terms in the first line in C, we have

$$\frac{d^2\theta_1}{ds_1^2}=\theta_{11}p_1''+\theta_{12}q_1''+\theta_{13}r_1''$$
$$+\theta_{300}p_1'^2+2\theta_{210}p_1'q_1'+2\theta_{201}p_1'r_1'+\theta_{120}q_1'^2+2\theta_{111}q_1'r_1'+\theta_{102}r_1'^2.$$

To modify the right-hand side, we substitute for the quantities θ_{ijk} their values from the general relation (p. 58)

$$\theta_{ijk}=\vartheta_{ijk}+\theta_1\Gamma_{ijk}+\theta_2\Delta_{ijk}+\theta_3\Theta_{ijk}$$
$$+(\theta_{1i}\Gamma_{jk}+\theta_{1j}\Gamma_{ki}+\theta_{1k}\Gamma_{ij})+(\theta_{2i}\Delta_{jk}+\theta_{2j}\Delta_{ki}+\theta_{2k}\Delta_{ij})+(\theta_{3i}\Theta_{jk}+\theta_{3j}\Theta_{ki}+\theta_{3k}\Theta_{ij}),$$

and we also substitute the values of p_1'', q_1'', r_1'', in terms of p_1', q_1', r_1'.

Further, we use (merely as a temporary abbreviation) a symbol ϕ_{ijk} defined by

$$\phi_{ijk}=\vartheta_{ijk}+\theta_1\Gamma_{ijk}+\theta_2\Delta_{ijk}+\theta_3\Theta_{ijk};$$

and after some re-arrangement, we find

$$\frac{d^2\theta_1}{ds_1^2}=\phi_{300}p_1'^2+2\phi_{210}p_1'q_1'+2\phi_{201}p_1'r_1'+\phi_{120}q_1'^2+2\phi_{111}q_1'r_1'+\phi_{102}r_1'^2$$
$$+2\left(\frac{d\theta_1}{ds_1}\alpha_1+\frac{d\theta_2}{ds_1}\xi_1+\frac{d\theta_3}{ds_1}\phi_1\right).$$

Similarly,

$$\frac{d^2\theta_2}{ds_1^2}=\phi_{210}p_1'^2+2\phi_{120}p_1'q_1'+2\phi_{111}p_1'r_1'+\phi_{030}q_1'^2+2\phi_{021}q_1'r_1'+\phi_{012}r_1'^2$$
$$+2\left(\frac{d\theta_1}{ds_1}\beta_1+\frac{d\theta_2}{ds_1}\eta_1+\frac{d\theta_3}{ds_1}\chi_1\right),$$

$$\frac{d^2\theta_3}{ds_1^2}=\phi_{201}p'^2+2\phi_{111}p_1'q_1'+2\phi_{102}p_1'r_1'+\phi_{021}q_1'^2+2\phi_{012}q_1'r_1'+\phi_{003}r_1'^2$$
$$+2\left(\frac{d\theta_1}{ds_1}\gamma_1+\frac{d\theta_2}{ds_1}\zeta_1+\frac{d\theta_3}{ds_1}\psi_1\right).$$

When these values are substituted in the first line of the expression for C, that first line

$$=\sum\phi_{300}p_1'^2p_2'+2\left(\bar{\gamma}_{12}\frac{d\theta_1}{ds_1}+\bar{\delta}_{12}\frac{d\theta_2}{ds_1}+\bar{\theta}_{12}\frac{d\theta_3}{ds_1}\right);$$

and therefore, as the whole expression for C, we have

$$C=\sum\phi_{300}p_1'^2p_2'+\theta_1P+\theta_2Q+\theta_3R.$$

We now separate the quantities ϕ_{ijk} into the two parts, one involving the quantities ϑ_{ijk} and the other involving the quantities of the type Γ_{ijk}; and we write

$$P_0=P+(\Gamma_{300}p_1'^2p_2'),$$
$$Q_0=Q+(\Delta_{300}p_1'^2p_2'),$$
$$R_0=R+(\Theta_{300}p_1'^2p_2').$$

Thus the residuary condition $C=0$ becomes

$$\sum\vartheta_{300}p_1'^2p_2'+\theta_1P_0+\theta_2Q_0+\theta_3R_0=0.$$

In this reduction, the completely geodesic quality of the surface at O relative to the domain has not yet been used. There are certain third-order conditions consisting of relations between the quantities ϑ_{ijk}, as obtained in § 211; and they can be used, either in the form as given on p. 63, or in the umbral form as given on p. 65.

In the full literal form, they lead to the equation

$$3\sum \vartheta_{300} p_1'^2 p_2' = (\theta_1 p_2' + \theta_2 q_2' + \theta_3 r_2') W_{11} + 2(\theta_1 p_1' + \theta_2 q_1' + \theta_3 r_1') W_{12},$$

where, for $ij = 11, 12$,

$$W_{ij} = \frac{\vartheta_{300}}{\theta_1} p_i' p_j' + \frac{\vartheta_{030}}{\theta_2} q_i' q_j' + \frac{\vartheta_{003}}{\theta_3} r_i' r_j'$$
$$+ \mathfrak{a}(q_i' r_j' + q_j' r_i') + \mathfrak{b}(r_i' p_j' + r_j' p_i') + \mathfrak{c}(p_i' q_j' + p_j' q_i').$$

Because the two directions lie in the geodesic surface, we have

$$\theta_1 p_1' + \theta_2 q_1' + \theta_3 r_1' = 0, \quad \theta_1 p_2' + \theta_2 q_2' + \theta_3 r_2' = 0;$$

and therefore

$$\sum \vartheta_{300} p_1'^2 p_2' = 0.$$

In the umbral form of the third-order conditions, with the umbral notation of p. 64, we have

$$\sum \vartheta_{300} p_1'^2 p_2' = a_{p_1'}{}^2 a_{p_2'},$$

where, because the two directions lie in the surface,

$$a_{p_1'} = \Gamma_1 p_1' + \Gamma_2 q_1', \quad a_{p_2'} = \Gamma_1 p_2' + \Gamma_2 q_2';$$

and therefore

$$\begin{aligned}\sum \vartheta_{300} p_1'^2 p_2' &= (\Gamma_1 p_1' + \Gamma_2 q_1')^2 (\Gamma_1 p_2' + \Gamma_2 q_2') \\ &= \Gamma_1^3 p_1'^2 p_2' + \Gamma_1^2 \Gamma_2 (2p_1' q_1' p_2' + p_1'^2 q_2') \\ &\quad + \Gamma_1 \Gamma_2^2 (2p_1' q_1' q_2' + q_1'^2 p_2') + \Gamma_2^3 q_1'^2 q_2'.\end{aligned}$$

The third-order relations of geodesic quality are

$$\Gamma_1^3 = 0, \quad \Gamma_1^2 \Gamma_2 = 0, \quad \Gamma_1 \Gamma_2^2 = 0, \quad \Gamma_2^3 = 0;$$

and therefore, as before,

$$\sum \vartheta_{300} p_1'^2 p_2' = 0.$$

The final form of the surface-relation at R thus becomes

$$\theta_1 P_0 + \theta_2 Q_0 + \theta_3 R_0 = 0.$$

(ii) Next, we use the permanent arc-relation at R which, for the parallel geodesic, is

$$\sum (A)_R P_2'^2 = 1;$$

and we require approximations of like order. We have

$$(A)_R = A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2 \frac{d^2 A}{ds_1^2},$$
$$P_2'^2 = p_2'^2 - 2xp_2' \bar{\gamma}_{12} + x^2(\bar{\gamma}_{12}^2 + p_2' P),$$

up to the order at present necessary, with corresponding values for the other primary magnitudes at R, and for the other quadratic combinations of the direction-variables; and the successive approximations must be framed.

The terms independent of the small arc x require the relation

$$\sum A p_2'^2 = 1:$$

it is satisfied, without imposing any residuary condition.

The terms involving the first power of x require the relation

$$\sum \frac{dA}{ds_1} p_2'^2 - 2\sum A p_2' \bar{\gamma}_{12} = 0.$$

When the derivatives of the primary magnitudes are inserted by taking the relation in § 212 for $l=1$, $m=2$, $n=2$, the first term

$$= 2\{u_1^{(2)}\bar{\gamma}_{12} + u_2^{(2)}\bar{\delta}_{12} + u_3^{(2)}\bar{\theta}_{12}\}:$$

that is, the required relation is satisfied without imposing any residuary condition.

The terms involving the second power of x, if they are to disappear, impose the relation

$$\tfrac{1}{2}\sum \frac{d^2A}{ds_1^2} p_2'^2 - 2\sum \frac{dA}{ds_1} p_2'\bar{\gamma}_{12} + \sum A(\bar{\gamma}_{12}^2 + p_2'P) = 0.$$

The value of the first term on the left-hand side is derivable from the result in § 212, by taking $l=1$, $m=2$, $n=2$; thus

$$\begin{aligned}\tfrac{1}{2}\sum \frac{d^2A}{ds_1^2} p_2'^2 = & -\tfrac{1}{3}\sum k_{11}\xi_{12}^2 \\ & + u_1^{(2)}(\Gamma_{300}p_1'^2p_2') + u_2^{(2)}(\Delta_{300}p_1'^2p_2') + u_3^{(2)}(\Theta_{300}p_1'^2p_2') \\ & + \sum A\bar{\gamma}_{12}^2 \\ & + 2u_1^{(2)}(\alpha_1\bar{\gamma}_{12} + \beta_1\bar{\delta}_{12} + \gamma_1\bar{\theta}_{12}) \\ & \qquad + 2u_2^{(2)}(\xi_1\bar{\gamma}_{12} + \eta_1\bar{\delta}_{12} + \zeta_1\bar{\theta}_{12}) \\ & \qquad\qquad + 2u_3^{(2)}(\phi_1\bar{\gamma}_{12} + \chi_1\bar{\delta}_{12} + \psi_1\bar{\theta}_{12}).\end{aligned}$$

For the second term on the left-hand side, we have

$$\begin{aligned}\sum \frac{dA}{ds_1} p_2'\bar{\gamma}_{12} = & \bar{\gamma}_{12}\left(p_2'\frac{dA}{ds_1} + q_2'\frac{dH}{ds_1} + r_2'\frac{dG}{ds_1}\right) \\ & + \bar{\delta}_{12}\left(p_2'\frac{dH}{ds_1} + q_2'\frac{dB}{ds_1} + r_2'\frac{dF}{ds_1}\right) \\ & + \bar{\theta}_{12}\left(p_2'\frac{dG}{ds_1} + q_2'\frac{dF}{ds_1} + r_2'\frac{dC}{ds_1}\right) \\ = & \sum A\bar{\gamma}_{12}^2 + \bar{\gamma}_{12}(u_1^{(2)}\alpha_1 + u_2^{(2)}\xi_1 + u_3^{(2)}\phi_1) \\ & + \bar{\delta}_{12}(u_1^{(2)}\beta_1 + u_2^{(2)}\eta_1 + u_3^{(2)}\chi_1) \\ & + \bar{\theta}_{12}(u_1^{(2)}\gamma_1 + u_2^{(2)}\zeta_1 + u_3^{(2)}\psi_1),\end{aligned}$$

from the result in § 212. Also

$$\sum Ap_2'P = u_1^{(2)}P + u_2^{(2)}Q + u_3^{(2)}R.$$

When the terms are gathered together, the relation becomes

$$-\tfrac{1}{3}\sum k_{11}\xi_{12}^2 + u_1^{(2)}(\Gamma_{300}p'^2p_2') + u_2^{(2)}(\Delta_{300}p_1'^2p_2') + u_3^{(2)}(\Theta_{300}p_1'^2p_2')$$
$$+ u_1^{(2)}P + u_2^{(2)}Q + u_3^{(2)}R = 0.$$

Now

$$\sum k_{11}\xi_{12}^2 = K \sin^2 \epsilon,$$

where K denotes the sphericity of the region in the orientation determined by the two geodesics ORE, OTF. Hence, with the former significance for P_0, Q_0, R_0, the residuary condition from the second-order approximation in the permanent arc-relation at R is

$$u_1^{(2)}P_0 + u_2^{(2)}Q_0 + u_3^{(2)}R_0 = \tfrac{1}{3}K \sin^2\epsilon.$$

(iii) The remaining condition, being the condition of parallelism of geodesics as expressed by the constancy in the value of ϵ as R moves along the basic geodesic ORE, is

$$\sum (A)_R P_2'(p_1')_R = \cos \epsilon = \sum Ap_2'p_1'.$$

Here, $(p_1')_R$, $(q_1')_R$, $(r_1')_R$, are the direction-variables of the geodesic ORE at R in the direction RE, so that there are expressions of the type

$$(p_1')_R = p_1' + xp_1'' + \tfrac{1}{2}x^2p_1'''.$$

As before,

$$(A)_R = A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2},$$
$$P_2' = p_2' - x\bar{\gamma}_{12} + \tfrac{1}{2}x^2P,$$

with similar expressions for the other primary magnitudes at R, and for Q_2', R_2'. We take the ordered approximations in turn, after these values have been substituted.

The relation, arising out of the terms free from the small arc x, is satisfied identically.

The relation, arising out of the terms which involve the first power of x, is

$$\sum Ap_2'p_1'' + \sum\frac{dA}{ds_1}p_1'p_2' - \sum Ap_1'\bar{\gamma}_{12} = 0.$$

By the result in § 212, we have

$$\sum\frac{dA}{ds_1}p_1'p_2' = u_1^{(2)}\bar{\gamma}_{11} + u_2^{(2)}\bar{\delta}_{11} + u_3^{(2)}\bar{\theta}_{11} + u_1^{(1)}\bar{\gamma}_{12} + u_2^{(1)}\bar{\delta}_{12} + u_3^{(1)}\bar{\theta}_{12}$$
$$= -u_1^{(2)}p_1'' - u_2^{(2)}q_1'' - u_3^{(2)}r_1'' + u_1^{(1)}\bar{\gamma}_{12} + u_2^{(1)}\bar{\delta}_{12} + u_3^{(1)}\bar{\theta}_{12};$$

also

$$\sum Ap_2'p_1'' = u_1^{(2)}p_1'' + u_2^{(2)}q_1'' + u_3^{(2)}r_1'',$$
$$\sum Ap_1'\bar{\gamma}_{12} = u_1^{(1)}\bar{\gamma}_{12} + u_2^{(1)}\bar{\delta}_{12} + u_3^{(1)}\bar{\theta}_{12}:$$

the relation is satisfied without leaving any residuary condition.

The relation, arising out of the terms which involve the second power of x in the condition of parallelism, is

$$\tfrac{1}{2}\sum Ap_1'P+\tfrac{1}{2}\sum Ap_2'p_1'''+\tfrac{1}{2}\sum\frac{d^2A}{ds_1^2}p_1'p_2'$$
$$+\sum\frac{dA}{ds_1}p_2'p_1''-\sum Ap_1''\bar{\gamma}_{12})-\sum\frac{dA}{ds_1}p_1'\bar{\gamma}_{12}=0.$$

The first term in the first line

$$=\tfrac{1}{2}\{u_1^{(1)}P+u_2^{(1)}Q+u_3^{(1)}R\};$$

and the second term in that line

$$=-\tfrac{1}{2}\{u_1^{(2)}(\Gamma_{300}p_1'^3)+u_2^{(2)}(\Delta_{300}p_1'^3)+u_3^{(2)}(\Theta_{300}p_1'^3)\}.$$

The third term in the first line, by the result in § 212,

$$=\tfrac{1}{2}\{u_1^{(1)}(\Gamma_{300}p_1'^2p_2')+u_2^{(1)}(\Delta_{300}p_1'^2p_2')+u_3^{(1)}(\Theta_{300}p_1'^2p_2')\}$$
$$+\tfrac{1}{2}\{u_1^{(2)}(\Gamma_{300}p_1'^3)+u_2^{(2)}(\Delta_{300}p_1'^3)+u_3^{(2)}(\Theta_{300}p_1'^3)\}$$
$$+(A, B, C, F, G, H\between\bar{\gamma}_{11}, \bar{\delta}_{11}, \bar{\theta}_{11}\between\bar{\gamma}_{12}, \bar{\theta}_{12}, \bar{\delta}_{12})$$
$$+u_1^{(1)}(\alpha_1\bar{\gamma}_{12}+\beta_1\bar{\delta}_{12}+\gamma_1\bar{\theta}_{12})+u_1^{(2)}(\alpha_1\bar{\gamma}_{11}+\beta_1\bar{\delta}_{11}+\gamma_1\bar{\theta}_{11})$$
$$+u_2^{(1)}(\xi_1\bar{\gamma}_{12}+\eta_1\bar{\delta}_{12}+\zeta_1\bar{\theta}_{12})+u_2^{(2)}(\xi_1\bar{\gamma}_{11}+\eta_1\bar{\delta}_{11}+\zeta_1\bar{\theta}_{11})$$
$$+u_3^{(1)}(\phi_1\bar{\gamma}_{12}+\chi_1\bar{\delta}_{12}+\psi_1\bar{\theta}_{12})+u_3^{(2)}(\phi_1\bar{\gamma}_{11}+\chi_1\bar{\delta}_{11}+\psi_1\bar{\theta}_{11}),$$

in which

$$\bar{\gamma}_{11}=-p_1'', \quad \bar{\delta}_{11}=-q_1'', \quad \bar{\theta}_{11}=-r_1''.$$

The first term in the second line

$$-p_1''\left(p_2'\frac{dA}{ds_1}+q_2'\frac{dH}{ds_1}+r_2'\frac{dG}{ds_1}\right)$$
$$+q_1''\left(p_2'\frac{dH}{ds_1}+q_2'\frac{dB}{ds_1}+r_2'\frac{dF}{ds_1}\right)+r_1''\left(p_2'\frac{dG}{ds_1}+q_2'\frac{dF}{ds_1}+r_2'\frac{dC}{ds_1}\right)$$
$$=p_1''(A\bar{\gamma}_{12}+H\bar{\delta}_{12}+G\bar{\theta}_{12}+u_1^{(2)}\alpha_1+u_2^{(2)}\xi_1+u_3^{(2)}\phi_1)$$
$$+q_1''(H\bar{\gamma}_{12}+B\bar{\delta}_{12}+F\bar{\theta}_{12}+u_1^{(2)}\beta_1+u_2^{(2)}\eta_1+u_3^{(2)}\chi_1)$$
$$+r_1''(G\bar{\gamma}_{12}+F\bar{\delta}_{12}+C\bar{\theta}_{12}+u_1^{(2)}\gamma_1+u_2^{(2)}\zeta_1+u_3^{(2)}\psi_1),$$

one-half of which is the same (save for sign) as the second term in the second line. Finally, the third term in the second line

$$=-\bar{\gamma}_{12}\left(p_1'\frac{dA}{ds_1}+q_1'\frac{dH}{ds_1}+r_1'\frac{dG}{ds_1}\right)$$
$$-\bar{\delta}_{12}\left(p_1'\frac{dH}{ds_1}+q_1'\frac{dB}{ds_1}+r_1'\frac{dF}{ds_1}\right)-\bar{\theta}_{12}\left(p_1'\frac{dG}{ds_1}+q_1'\frac{dF}{ds_1}+r_1'\frac{dC}{ds_1}\right)$$
$$=-\bar{\gamma}_{12}(A\bar{\gamma}_{11}+H\bar{\delta}_{11}+G\bar{\theta}_{11}+u_1^{(1)}\alpha_1+u_2^{(1)}\xi_1+u_3^{(1)}\phi_1)$$
$$-\bar{\delta}_{12}(H\bar{\gamma}_{11}+B\bar{\delta}_{11}+F\bar{\theta}_{11}+u_1^{(1)}\beta_1+u_2^{(1)}\eta_1+u_3^{(1)}\chi_1)$$
$$-\bar{\theta}_{12}(G\bar{\gamma}_{11}+F\bar{\delta}_{11}+C\bar{\theta}_{11}+u_1^{(1)}\gamma_1+u_2^{(1)}\zeta_1+u_3^{(1)}\psi_1).$$

When these respective values are substituted, and reduction is effected, the condition is found to take the form

$$\tfrac{1}{2}u_1^{(1)}\{P+(\Gamma_{300}p_1'^2p_2')\}+\tfrac{1}{2}u_2^{(1)}\{Q+(\Delta_{300}p_1'^2p_2')\}+\tfrac{1}{2}u_3^{(1)}\{R+(\Theta_{300}p_1'^2p_2')\}=0:$$

that is, in effect,

$$u_1^{(1)}P_0+u_2^{(1)}Q_0+u_3^{(1)}R_0=0,$$

with the former significance for P_0, Q_0, R_0.

(iv) Hence, for the determination of these quantities P_0, Q_0, R_0, we have the three equations

$$\begin{aligned}\theta_1P_0+\quad\theta_2Q_0+\quad\theta_3R_0&=0,\\ u_1^{(1)}P_0+u_2^{(1)}Q_0+u_3^{(1)}R_0&=0,\\ u_1^{(2)}P_0+u_2^{(2)}Q_0+u_3^{(2)}R_0&=\tfrac{1}{3}K\sin^2\epsilon,\end{aligned}$$

and therefore

$$\begin{aligned}P_0&=\tfrac{1}{3}K(p_2'-p_1'\cos\epsilon),\\ Q_0&=\tfrac{1}{3}K(q_2'-q_1'\cos\epsilon),\\ R_0&=\tfrac{1}{3}K(r_2'-r_1'\cos\epsilon).\end{aligned}$$

Accordingly, the direction-variables of the Severi geodesic, through R parallel to the geodesic OTF, are

$$\left.\begin{aligned}P_2'&=p_2'-x\bar{\gamma}_{12}+\tfrac{1}{2}x^2\{\tfrac{1}{3}K(p_2'-p_1'\cos\epsilon)-(\Gamma_{300}p_1'^2p_2')\}\\ Q_2'&=q_2'-x\delta_{12}+\tfrac{1}{2}x^2\{\tfrac{1}{3}K(q_2'-q_1'\cos\epsilon)-(\Delta_{300}p_1'^2p_2')\}\\ R_2'&=r^{2'}-x\bar{\gamma}_{12}+\tfrac{1}{2}x^2\{\tfrac{1}{3}K(r_2'-r_1'\cos\epsilon)-(\Theta_{300}p_1'^2p_2')\}\end{aligned}\right\}.$$

Divergence between the directions of a Levi-Civita parallel and a Severi parallel.

228. It was proved (§ 225) that the direction-variables for the Levi-Civita geodesic through R, parallel to the initial geodesic OTF, are

$$\begin{aligned}p'&=p_2'+x\frac{dp_2'}{ds_1}+\tfrac{1}{2}x^2\frac{d^2p_2'}{ds_1^2}+\dots\\ &=p_2'-x\bar{\gamma}_{12}\quad-\tfrac{1}{2}x^2\{(\Gamma_{300}p_1'^2p_2')+\frac{1}{3\Omega}K_1(1,12)\},\\ q'&=q_2'-x\delta_{12}\quad-\tfrac{1}{2}x^2\{(\Delta_{300}p_1'^2p_2')+\frac{1}{3\Omega}K_2(1,12)\},\\ r'&=r_2'-x\bar{\theta}_{12}\quad-\tfrac{1}{2}x^2\{(\Theta_{300}p_1'^2p_2')+\frac{1}{3\Omega}K_3(1,12)\},\end{aligned}$$

up to the second power of x inclusive.

As the quantity $-\frac{1}{\Omega}K_1(1,12)$ is not equal to $K(p_2'-p_1'\cos\epsilon)$ in general, nor $-\frac{1}{\Omega}K_2(1,12)$ to $K(q_2'-q_1'\cos\epsilon)$, nor $-\frac{1}{\Omega}K_3(1,12)$ to $K(r_2'-r_1'\cos\epsilon)$, it follows

that the direction-variables of the Levi-Civita parallel differ from the direction-variables of the Severi parallel. The two parallels are distinct, in general.

But if the region be of constant sphericity, the two parallels coincide up to the second order of small quantities—a remark due to Severi himself*. Let K_0 denote the constant sphericity of the region; then the relation

$$\sum k_{11}\xi_{12}^2 = K_0 \sum a\xi_{12}^2$$

is satisfied for all orientations in the region, and therefore

$$k_{11}=K_0 a, \quad k_{22}=K_0 b, \quad k_{33}=K_0 c,$$
$$k_{23}=K_0 f, \quad k_{31}=K_0 g, \quad k_{12}=K_0 h.$$

With these values,

$$K_1(1, 12) = \begin{vmatrix} a, & p_1', & a\xi_{12}+h\eta_{12}+g\zeta_{12} \\ h, & q_1', & h\xi_{12}+b\eta_{12}+f\zeta_{12} \\ g, & r_1', & g\xi_{12}+f\eta_{12}+c\zeta_{12} \end{vmatrix} K_0.$$

When K_1 is expanded, so as to exhibit its linearity in ξ_{12}, η_{12}, ζ_{12}, the coefficient of ξ_{12} is zero; the coefficient of η_{12}

$$= \begin{vmatrix} a, & p_1', & h \\ h, & q_1', & b \\ g, & r_1', & f \end{vmatrix} = -\Omega u_3^{(1)}:$$

and the coefficient of ζ_{12}

$$= \begin{vmatrix} a, & p_1', & g \\ h, & q_1', & f \\ g, & r_1', & c \end{vmatrix} = \Omega u_2^{(1)}.$$

Hence

$$\begin{aligned} K_1(1, 12) &= \Omega K_0\{\zeta_{12}u_2^{(1)} - \eta_{12}u_3^{(1)}\} \\ &= \Omega K_0\{(p_1'q_2' - q_1'p_2')u_2^{(1)} - (r_1'p_2' - p_1'r_2')u_3^{(1)}\} \\ &= \Omega K_0[p_1'\{p_2'u_1^{(1)} + q_2'u_2^{(1)} + r_2'u_3^{(1)}\} - p_2'\{p_1'u_1^{(1)} + q_1'u_2^{(1)} + r_1'u_3^{(1)}\}] \\ &= \Omega K_0(p_1'\cos\epsilon - p_2'). \end{aligned}$$

Similarly

$$K_2(1, 12) = \Omega K_0(q_1'\cos\epsilon - q_2'), \quad K_3(1, 12) = \Omega K_0(r_1'\cos\epsilon - r_2').$$

When these values are inserted, we have (up to the second power of x)

$$p' = P_2', \quad q' = Q_2', \quad r' = R_2':$$

that is, up to the second order of small quantities, the direction-variables of the two parallels are the same for a region of constant sphericity; and Severi's remark is verified.

* *Rend. d. Circ. Mat. di Palermo*, t. xlii (1917), p. 256.

It is obvious that, for regions of variable sphericity, the angular divergence between the Levi-Civita regional parallel to OTF and the Severi regional parallel to OTF is a quantity of the second order in x, the length of the geodesic arc OR; and an expression for this angular divergence, to be denoted by ∇, can be obtained * as follows. We have

$$P_2' - p' = \tfrac{1}{6}x^2 K(p_2' - p_1' \cos\epsilon) + \frac{x^2}{6\Omega} K_1(1, 12),$$

with similar values for $Q_2' - q'$ and $R_2' - r'$. We write

$$\overline{K}_1(1, 12) = \begin{vmatrix} a, & p_1', & (k_{11} - Ka)\xi_{12} + (k_{12} - Kh)\eta_{12} + (k_{13} - Kg)\zeta_{12} \\ h, & q_1', & (k_{12} - Kh)\xi_{12} + (k_{22} - Kb)\eta_{12} + (k_{23} - Kf)\zeta_{12} \\ g, & r_1', & (k_{13} - Kg)\xi_{12} + (k_{23} - Kf)\eta_{12} + (k_{33} - Kc)\zeta_{12} \end{vmatrix} :$$

or, if

$$\left.\begin{aligned} D_1 &= (k_{11} - Ka)\xi_{12} + (k_{12} - Kh)\eta_{12} + (k_{13} - Kg)\zeta_{12} \\ D_2 &= (k_{12} - Kh)\xi_{12} + (k_{22} - Kb)\eta_{12} + (k_{23} - Kf)\zeta_{12} \\ D_3 &= (k_{13} - Kg)\xi_{12} + (k_{23} - Kf)\eta_{12} + (k_{33} - Kc)\zeta_{12} \end{aligned}\right\},$$

with

$$\lambda = q_1' D_3 - r_1' D_2, \quad \mu = r_1' D_1 - p_1' D_3, \quad \nu = p_1' D_2 - q_1' D_1,$$

then

$$\overline{K}_1(1, 12) = a\lambda + h\mu + g\nu;$$

and similarly, with corresponding definitions of $\overline{K}_2(1, 12)$, $\overline{K}_3(1, 12)$,

$$\overline{K}_2(1, 12) = h\lambda + b\mu + f\nu,$$
$$\overline{K}_3(1, 12) = g\lambda + f\mu + c\nu.$$

Now, with these magnitudes,

$$K_1(1, 12) - \overline{K}_1(1, 12) = K \begin{vmatrix} a, & p_1', & a\xi_{12} + h\eta_{12} + g\zeta_{12} \\ h, & q_1', & h\xi_{12} + b\eta_{12} + f\zeta_{12} \\ g, & r_1', & g\xi_{12} + f\eta_{12} + c\zeta_{12} \end{vmatrix}$$
$$= -K\Omega(p_2' - p_1' \cos\epsilon),$$

by the preceding analysis. Hence

$$P_2' - p' = \frac{x^2}{6\Omega} \overline{K}_1(1, 12);$$

and similarly

$$Q_2' - q' = \frac{x^2}{6\Omega} \overline{K}_2(1, 12),$$

$$R_2' - r' = \frac{x^2}{6\Omega} \overline{K}_3(1, 12).$$

* Such an expression, for the general amplitude and not solely for a region, was first obtained by Bompiani (*l.c.* § 119).

Accordingly, we have

$$\sum A(P_2' - p')^2 = \frac{x^4}{36\Omega^2} \sum A\bar{K}_1^2,$$

that is,

$$2 - 2\cos\phi = \frac{x^4}{36\Omega^2} \sum A(a\lambda + h\mu + g\nu)^2.$$

Consequently, as ϕ is a small magnitude,

$$\begin{aligned}
\phi^2 &= \frac{x^4}{36\Omega} \sum a\lambda^2 \\
&= \frac{x^4}{36\Omega} \sum a(q_1'D_3 - r_1'D_2)^2 \\
&= \frac{x^4}{36\Omega} \{(\sum Ap_1'^2)(\sum AD_1^2) - (\sum Ap_1'D_1)^2\} \\
&= \frac{x^4}{36\Omega} \{\sum AD_1^2 - (\sum Ap_1'D_1)^2\},
\end{aligned}$$

the expression in question.

Manifestly, ϕ is of the order x^2; also, as D_1, D_2, D_3, vanish for a region of constant sphericity, ϕ then also vanishes.

Pérès parallelogram on a geodesic surface of the region.

229. Now consider a Pérès parallelogram on the surface at O geodesic to the region, with OA and OB as adjacent geodesic sides; and let AC', BC', meeting in C', be the geodesics parallel to OB, OA. As arc-lengths, let

$$OA = x, \quad BC' = x + X, \quad OB = y, \quad BC'' = y + Y,$$

where x and y are accurate, while X and Y will be found to the desired degree of approximation.

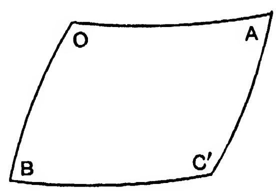

Fig. 25.

With p_1', q_1', r_1', and p_2', q_2', r_2', as direction-variables of OA and OB respectively, the direction-variables of BC' at B in the direction BC' are P_1', Q_1', R_1', where

$$P_1' = p_1' - y\bar{\gamma}_{12} + \tfrac{1}{2}y^2\{\tfrac{1}{3}K(p_1' - p_2'\cos\epsilon) - (\Gamma_{300}p_1'p_2'^2)\},$$

up to the second order of small quantities, with like expressions for Q_1' and R_1'; and the direction-variables of AC' at A in the direction AC' are P_2', Q_2', R_2', where

$$P_2' = p_2' - x\gamma_{12} + \tfrac{1}{2}x^2\{\tfrac{1}{3}K(p_2' - p_1'\cos\epsilon) - (\Gamma_{300}p_1'^2p_2')\},$$

also up to the second order, and with like expressions for Q_2' and R_2'.

At the intersection C', the values of the regional parameters must be the same,

attained by the path OAC' and by the path OBC'. By the equality in value for the p-parameter, there is the relation

$$p+xp_1'+\tfrac{1}{2}x^2p_1''+\tfrac{1}{6}x^3p_1'''+(y+Y)P_2'+\tfrac{1}{2}(y+Y)^2P_2''+\tfrac{1}{6}(y+Y)^3P_2'''$$
$$=p+yp_2'+\tfrac{1}{2}y^2p_2''+\tfrac{1}{6}y^3p_2'''+(x+X)P_1'+\tfrac{1}{2}(x+X)^2P_1''+\tfrac{1}{6}(x+X)^3P_1''',$$

up to the third order, the quantities P_2'', P_2''', being taken at A and the quantities P_1'', P_1''', being taken at B; and the terms of the successive orders of small quantities in the relation must balance. There are similar relations arising out of the q-parameter and the r-parameter.

In previous investigations of the kind, it has appeared that magnitudes such as X and Y are of the third degree at least in small quantities; this characteristic will be assumed here initially, being verified incidentally in the course of the analysis. Thus, up to the order retained in the relations,

$$(x+X)^2=x^2,\quad (x+X)^3=x^3,\quad (y+Y)^2=y^2,\quad (y+Y)^3=y^3.$$

Now P_1''', at B along BC', differs from p_1''', at O along OA, by small quantities of the first order at least; and p_1''' is the same for the geodesic surface at O as for the region: hence, to the retained order,

$$(x+X)^3P_1'''=x^3p_1'''.$$

Similarly, to the retained order,

$$(y+Y)^3P_2'''=y^3p_2'''.$$

But the terms involving P_1'' and P_2'' have quantities of the second order as coefficients of those magnitudes; hence P_1'' and P_2'' must be evaluated up to the first order inclusive. Now, as the geodesic AC' at A lies in the geodesic surface * under consideration, we have

$$P_2''+\sum(\Gamma_{11})_A P_2'^2=\left(\frac{1}{\gamma}\frac{dp}{dn}\right)_A,$$

where, on the right-hand side, we must take the values at A; and, for the immediate purpose, quantities of the first order in the value of P_2'' must be retained. Accordingly,

$$\sum(\Gamma_{11})_A P_2'^2=\sum\left(\Gamma_{11}+x\frac{d\Gamma_{11}}{ds_1}\right)(p_2'^2-2xp_2'\bar{\gamma}_{12})$$
$$=\sum\Gamma_{11}p_2'^2+x\left\{\sum\frac{d\Gamma_{11}}{ds_1}p_2'^2-2(a_2\bar{\gamma}_{12}+\beta_2\bar{\delta}_{12}+\gamma_2\bar{\theta}_{12})\right\},$$

up to this order. But, by the result of § 212, I,

* The surface is geodesic to the region at O, and the regional geodesic OA (with its continuation) lies entirely in the surface; but unless the surface is geodesic to the region everywhere and not at O solely, a regional geodesic at A, in the initial direction of the superficial geodesic AC' at A, does not necessarily coincide with that superficial geodesic. In the general event of non-coincidence, the quantity P_2'' will not be the ame for the surface and the region; and similarly for the quantity P_1'' at B.

$$\sum \frac{d\Gamma_{11}}{ds_1} p_2'^2 - 2(\alpha_2\bar{\gamma}_{12} + \beta_2\bar{\delta}_{12} + \gamma_2\bar{\theta}_{12}) + \frac{2}{3\Omega} K_1(2, 12) + (\Gamma_{300} p_1' p_2'^2),$$

with the significance of K_1 there given; consequently

$$\sum (\Gamma_{11})_A P_2'^2 = -p_2'' + x\left\{(\Gamma_{300} p_1' p_2'^2) + \frac{2}{3\Omega} K_1(2, 12)\right\}.$$

Again, the general value of the geodesic flexure of any surface in the region is given by

$$\frac{1}{\gamma} = -\frac{1}{\theta_n} \sum \vartheta_{11} p'^2,$$

a magnitude which vanishes at O for the geodesic surface. At A, where $OA = x$, we have to evaluate the right-hand side up to the first order, that is, we require

$$\sum (\vartheta_{11})_A P_2'^2.$$

Now, to this order,

$$(\vartheta_{11})_A = \vartheta_{11} + x\frac{d\vartheta_{11}}{ds_1}, \quad P_2'^2 = p_2'^2 - 2xp_2'\bar{\gamma}_{12},$$

and therefore

$$\begin{aligned}\sum (\vartheta_{11})_A P_2'^2 &= \sum \vartheta_{11} p_2'^2 - 2x\sum \bar{\gamma}_{12}(\vartheta_{11} p_2' + \vartheta_{12} q_2' + \vartheta_{13} r_2') + x\sum \frac{d\vartheta_{11}}{ds_1} p_2'^2 \\ &= x\sum \frac{d\vartheta_{11}}{ds_1} p_2'^2 - 2x\sum \bar{\gamma}_{12}(\theta_{11} p_2' + \theta_{12} q_2' + \theta_{13} r_2') \\ &\qquad + 2x\sum \bar{\gamma}_{12}(\theta_1 \alpha_2 + \theta_2 \xi_2 + \theta_3 \phi_2),\end{aligned}$$

with the notation of § 172 for the quantities of the type α, ξ, ϕ.

For the evaluation of the initial term on the right-hand side, we have

$$\begin{aligned}\frac{d\vartheta_{11}}{ds_1} &= \frac{d}{ds_1}(\theta_{11} - \theta_1\Gamma_{11} - \theta_2\Delta_{11} - \theta_3\Theta_{11}) \\ &= \theta_{300} p_1' + \theta_{210} q_1' + \theta_{201} r_1' \\ &\qquad - \left(\frac{d\theta_1}{ds_1}\Gamma_{11} + \frac{d\theta_2}{ds_1}\Delta_{11} + \frac{d\theta_3}{ds_1}\Theta_{11}\right) \\ &\qquad - \left(\frac{d\Gamma_{11}}{ds_1}\theta_1 + \frac{d\Delta_{11}}{ds_1}\theta_2 + \frac{d\Theta_{11}}{ds_1}\theta_3\right).\end{aligned}$$

As in § 227, we substitute, for the quantities θ_{ijk}, their values (p. 58) in terms of the quantities ϑ_{ijk}; that is, again using the symbols ϕ_{ijk} under the definition

$$\phi_{ijk} = \vartheta_{ijk} + \theta_1\Gamma_{ijk} + \theta_2\Delta_{ijk} + \theta_3\Theta_{ijk},$$

we take

$$\begin{aligned}\theta_{ijk} = \phi_{ijk} &+ (\theta_{1i}\Gamma_{jk} + \theta_{1j}\Gamma_{ki} + \theta_{1k}\Gamma_{ij}) + (\theta_{2i}\Delta_{jk} + \theta_{2j}\Delta_{ki} + \theta_{2k}\Delta_{ij}) \\ &+ (\theta_{3i}\Theta_{jk} + \Theta_{3j}\Theta_{ki} + \Theta_{3k}\Theta_{ij}).\end{aligned}$$

When the first two lines of the expression for $\frac{d\vartheta_{11}}{ds_1}$ are combined after these substitutions, reduction leads to the result

$$\theta_{300}p_1' + \theta_{210}q_1' + \theta_{201}r_1' - \left(\frac{d\theta_1}{ds_1}\Gamma_{11} + \frac{d\theta_2}{ds_1}\Delta_{11} + \frac{d\theta_3}{ds_1}\Theta_{11}\right)$$
$$= \phi_{300}p_1' + \phi_{210}q_1' + \phi_{201}r_1' + 2(\theta_{11}\alpha_1 + \theta_{12}\xi_1 + \theta_{13}\phi_1);$$

and we thus have a value of $\frac{d\vartheta_{11}}{ds_1}$.

Similar analysis leads to the result

$$\frac{d\vartheta_{12}}{ds_1} = \phi_{210}p_1' + \phi_{120}q_1' + \phi_{111}r_1' - \left(\frac{d\Gamma_{12}}{ds_1}\theta_1 + \frac{d\Delta_{12}}{ds_1}\theta_2 + \frac{d\Theta_{12}}{ds_1}\theta_3\right)$$
$$+ (\theta_{11}\beta_1 + \theta_{12}\eta_1 + \theta_{13}\chi_1) + (\theta_{21}\alpha_1 + \theta_{22}\xi_1 + \theta_{23}\phi_1);$$

and there are corresponding results for the arc-derivatives of ϑ_{22}, ϑ_{13}, ϑ_{23}, ϑ_{33}.

When all these values are inserted in the magnitude $\sum \frac{d\vartheta_{11}}{ds_1} p_2'^2$, the resulting equation can be arranged in the form

$$\sum \frac{d\vartheta_{11}}{ds_1} p_2'^2 = \sum \phi_{ijk} p_1' p_2'^2$$
$$+ 2\sum \bar{\gamma}_{12}(\theta_{11}p_2' + \theta_{12}q_2' + \theta_{13}r_2')$$
$$- \left\{\theta_1 \sum \frac{d\Gamma_{11}}{ds_1} p_2'^2 + \theta_2 \sum \frac{d\Delta_{11}}{ds_1} p_2'^2 + \theta_3 \sum \frac{d\Theta_{11}}{ds_1} p_2'^2\right\}.$$

Now using the formulæ in § 212, I, with the assumptions $l=1$, $m=2$, $n=2$, we have successively

$$\sum \frac{d\Gamma_{11}}{ds_1} p_2'^2 = 2(\bar{\gamma}_{12}\alpha_2 + \bar{\delta}_{12}\beta_2 + \bar{\theta}_{12}\gamma_2) + (\Gamma_{300}p_1'p_2'^2) + \frac{2}{3\Omega} K_1(1, 12),$$

$$\sum \frac{d\Delta_{11}}{ds_1} p_2'^2 = 2(\bar{\gamma}_{12}\xi_2 + \bar{\delta}_{12}\eta_2 + \bar{\theta}_{12}\zeta_2) + (\Delta_{300}p_1'p_2'^2) + \frac{2}{3\Omega} K_2(1, 12),$$

$$\sum \frac{d\Theta_{11}}{ds_1} p_2'^2 = 2(\bar{\gamma}_{12}\phi_2 + \bar{\delta}_{12}\chi_2 + \bar{\theta}_{12}\psi_2) + (\Theta_{300}p_1'p_2'^2) + \frac{2}{3\Omega} K_3(1, 12),$$

as values to be inserted in the foregoing equation; and their terms involving the magnitudes of the type Γ_{ijk} can be combined with the terms involving the magnitudes ϕ_{ijk}.

After complete reduction, we find

$$\sum (\vartheta_{11})_A P_2'^2 = xZ,$$

where

$$Z = \sum \vartheta_{300} p_1' p_2'^2 - \frac{2}{3\Omega}\{\theta_1 K_1(1, 12) + \theta_2 K_2(1, 12) + \theta_3 K_3(1, 12)\}.$$

Thus far, the analytical restrictions required by the geodesic quality of the surface have not been imposed. In the present approximation, these restrictions are the

third-order relations among the quantities ϑ_{ijk}, set out in § 211. We use the umbral form (p. 64) of these relations; and thus, as on p. 119,

$$\sum \vartheta_{300} p_1' p_2'^2 = a_{p_1'} a^2_{p_2'}.$$

Because the directions, with the variables p_1', q_1', r_1', and p_2', q_2', r_2', lie in the superficial orientation at O, we have

$$a_{p_1'} = \Gamma_1 p_1' + \Gamma_2 q_1', \quad a_{p_2'} = \Gamma_1 p_2' + \Gamma_2 q_2';$$

and therefore

$$\begin{aligned} a_{p_1'} a^2_{p_2'} &= (\Gamma_1 p_1' + \Gamma_2 q_1')(\Gamma_1 p_2' + \Gamma_2 q_2')^2 \\ &= \Gamma_1^3 p_1' p_2'^2 + \Gamma_1^2 \Gamma_2 (2p_1' p_2' q_2' + p_2'^2 q_1') \\ &\quad + \Gamma_1 \Gamma_2^2 (p_1' q_2'^2 + 2p_2' q_1' q_2') + \Gamma_2^3 q_1' q_2'^2. \end{aligned}$$

The third-order relations, required by the geodesic quality of the surface at O, are

$$\Gamma_1^3 = 0, \quad \Gamma_1^2 \Gamma_2 = 0, \quad \Gamma_1 \Gamma_2^2 = 0, \quad \Gamma_2^3 = 0.$$

Hence

$$\sum \vartheta_{300} p_1' p_2'^2 = 0,$$

so that the first term in Z vanishes; and we have

$$\sum (\vartheta_{11})_A P_2'^2 = -\frac{2x}{3\Omega} \{\theta_1 K_1(1, 12) + \theta_2 K_2(1, 12) + \theta_3 K_3(1, 12)\}.$$

It follows that the regional flexure of the superficial geodesic through A, drawn parallel to the (regional and superficial) geodesic OB on the geodesic surface, is given by

$$\frac{1}{\gamma_A} = \frac{2x}{3\Omega\theta_n} \{\theta_1 K_1(2, 12) + \theta_2 K_2(2, 12) + \theta_3 K_3(2, 12)\}.$$

230. The various terms in the equation for P_2'' have now been evaluated; the result is

$$\begin{aligned} P_2'' = p_2'' - x &\left\{(\Gamma_{300} p_1' p_2'^2) + \frac{2}{3\Omega} K_1(2, 12)\right\} \\ &+ \frac{2x}{3\Omega\theta_n} \frac{dp}{dn} \{\theta_1 K_1(2, 12) + \theta_2 K_2(2, 12) + \theta_3 K_3(2, 12)\}, \end{aligned}$$

an expression which admits of simplification in the terms involving K_1, K_2, K_3. We have

$$\theta_n = \theta_1 \frac{dp}{dn} + \theta_2 \frac{dq}{dn} + \theta_3 \frac{dr}{dn};$$

and therefore the coefficient of

$$\frac{2x}{3\Omega}$$

in this expression for P_2''

$$=\frac{\theta_2}{\theta_n}\left\{K_2(2,\,12)\frac{dp}{dn}-K_1(2,\,12)\frac{dq}{dn}\right\}+\frac{\theta_3}{\theta_n}\left\{K_3(2,\,12)\frac{dp}{dn}-K_1(2,\,12)\frac{dr}{dn}\right\}.$$

Now (p. 66)

$$K_1(2,\,12)=\begin{vmatrix} k_{11}\xi_{12}+k_{12}\eta_{12}+k_{13}\zeta_{12}, & a, & p_2' \\ k_{21}\xi_{12}+k_{22}\eta_{12}+k_{23}\zeta_{12}, & h, & q_2' \\ k_{31}\xi_{12}+k_{32}\eta_{12}+k_{33}\zeta_{12}, & g, & r_2' \end{vmatrix};$$

when we substitute h, b, f, for the constituents of the second column, we obtain the value of $K_2(2, 12)$; and when we substitute g, f, c, for those constituents, we obtain the value of $K_3(2, 12)$. Hence, in the foregoing coefficient, the total coefficient of the constituent $k_{11}\xi_{12}+k_{12}\eta_{12}+k_{13}\zeta_{12}$ arising out of all the terms in K_1, K_2, K_3,

$$=\frac{\theta_2}{\theta_n}\left\{\frac{dp}{dn}(br_2'-fq_2')-\frac{dq}{dn}(hr_2'-gq_2')\right\}$$
$$+\frac{\theta_3}{\theta_n}\left\{\frac{dp}{dn}(fr_2'-cq_2')-\frac{dr}{dn}(hr_2'-gq_2')\right\}.$$

But

$$\Omega\theta_n\frac{dp}{dn}=a\theta_1+h\theta_2+g\theta_3,$$

$$\Omega\theta_n\frac{dq}{dn}=h\theta_1+b\theta_2+f\theta_3,$$

$$\Omega\theta_n\frac{dr}{dn}=g\theta_1+f\theta_2+c\theta_3,$$

so that

$$\begin{aligned}
&\frac{dp}{dn}(br_2'-fq_2')-\frac{dq}{dn}(hr_2'-gq_2')\\
&=\frac{1}{\Omega\theta_n}[r_2'\{b(a\theta_1+h\theta_2+g\theta_3)-h(h\theta_1+b\theta_2+f\theta_3)\}\\
&\qquad -q_2'\{f(a\theta_1+h\theta_2+g\theta_3)-g(h\theta_1+b\theta_2+f\theta_3)\}]\\
&=\frac{1}{\theta_n}\{r_2'(C\theta_1-G\theta_3)-q_2'(G\theta_2-F\theta_1)\}\\
&=-\frac{\theta_1}{\theta_n}(Gp_2'+Fq_2'+Cr_2'),
\end{aligned}$$

because $\theta_1 p_2'+\theta_2 q_2'+\theta_3 r_2'=0$. Similarly, we find

$$\frac{dp}{dn}(fr_2'-cq_2')-\frac{dr}{dn}(hr_2'-gq_2')=-\frac{\theta_1}{\theta_n}(Hp_2'+Bq_2'+Fr_2').$$

Thus the aforesaid total coefficient of the constituent $k_{11}\xi_{12}+k_{12}\eta_{12}+k_{13}\zeta_{12}$

$$=\frac{\theta_1}{\theta_n^2}\{\theta_2(Gp_2'+Fq_2'+Cr_2')-\theta_3(Hp_2'+Bq_2'+Fr_2')\}.$$

Further, we have

$$\frac{\theta_1}{\xi_{12}}=\frac{\theta_2}{\eta_{12}}=\frac{\theta_3}{\zeta_{12}}=\Omega^{\frac{1}{2}}\frac{\theta_n}{\sin\epsilon};$$

and

$$\eta_{12}(Gp_2'+Fq_2'+Cr_2')-\zeta_{12}(Hp_2'+Bq_2'+Fr_2')$$
$$=-p_1'\sum Ap_2'^2+p_2'\sum Ap_1'p_2'=-p_1'+p_2'\cos\epsilon,$$

so that the total coefficient of the constituent $k_{11}\xi_{12}+k_{12}\eta_{12}+k_{13}\zeta_{12}$

$$=\frac{\Omega}{\sin^2\epsilon}(-p_1'+p_2'\cos\epsilon)\,\xi_{12}.$$

Similarly the coefficients of $k_{12}\xi_{12}+k_{22}\eta_{12}+k_{23}\zeta_{12}$ and $k_{13}\xi_{12}+k_{23}\eta_{12}+k_{33}\zeta_{12}$ are

$$\frac{\Omega}{\sin^2\epsilon}(-p_1'+p_2'\cos\epsilon)\,\eta_{12},\quad \frac{\Omega}{\sin^2\epsilon}(-p_1'+p_2'\cos\epsilon)\,\zeta_{12},$$

respectively, while

$$\sum k_{11}\xi_{12}^2=K\sin^2\epsilon,$$

K being the sphericity of the geodesic surface. Accordingly, the coefficient of $2x/3\Omega$ in the expression for P_2''

$$=\Omega K(-p_1'+p_2'\cos\epsilon);$$

and the equation for P_2'' thus becomes

$$P_2''=p_2''-x\{(\Gamma_{300}p_1'p_2'^2)+\tfrac{2}{3}K(p_1'-p_2'\cos\epsilon)\},$$

up to the first order of small quantities.

Similarly, up to the same order, we have

$$P_1''=p_1''-y\{(\Gamma_{300}p_1'^2p_2')+\tfrac{2}{3}K(p_2'-p_1'\cos\epsilon)\}.$$

We now return to the parameter-relations. The p-relation has become

$$xp_1'+\tfrac{1}{2}x^2p_1''+(y+Y)P_2'+\tfrac{1}{2}y^2P_2''$$
$$=yp_2'+\tfrac{1}{2}y^2p_2''+(x+X)P_1'+\tfrac{1}{2}x^2P_1'';$$

all other third-order terms, which initially were retained, have cancelled each other. In this form, after the values of P_1' and P_2' have been substituted, the first-order terms cancel; and, after the values obtained for P_1'' and P_2'' have

been substituted, the second-order terms cancel. The requirement, that the third-order terms shall balance, is met by the relation

$$\begin{aligned} &Yp_2' + \tfrac{1}{2}x^2y\{\tfrac{1}{3}K(p_2' - p_1'\cos\epsilon) - (\Gamma_{300}p_1'^2p_2')\} \\ &\quad - \tfrac{1}{2}xy^2\{\tfrac{2}{3}K(p_1' - p_2'\cos\epsilon) + (\Gamma_{300}p_1'p_2'^2)\} \\ &= Xp_1' + \tfrac{1}{2}xy^2\{\tfrac{1}{3}K(p_1' - p_2'\cos\epsilon) - (\Gamma_{300}p_1'p_2'^2)\} \\ &\quad - \tfrac{1}{2}x^2y\{\tfrac{2}{3}K(p_2' - p_1'\cos\epsilon) + (\Gamma_{300}p_1'^2p_2')\}, \end{aligned}$$

that is,

$$\{Y + \tfrac{1}{2}Kxy(x + y\cos\epsilon)\}p_2' = \{X + \tfrac{1}{2}Kxy(y + x\cos\epsilon)\}p_1'.$$

Similarly, the relation arising from the equality in value of the q-parameter at C', by the alternative paths OAC' and OBC', is

$$\{Y + \tfrac{1}{2}Kxy(x + y\cos\epsilon)\}q_2' = \{X + \tfrac{1}{2}Kxy(y + x\cos\epsilon)\}q_1';$$

and the corresponding relation, from the similar equality in value at C' in the r-parameter, is

$$\{Y + \tfrac{1}{2}Kxy(x + y\cos\epsilon)\}r_2' = \{X + \tfrac{1}{2}Kxy(y + x\cos\epsilon)\}r_1'.$$

The three relations are consistent, partly in virtue of the two conditions $\theta_1p_1' + \theta_2q_1' + \theta_3r_1' = 0 = \theta_1p_2' + \theta_2q_2' + \theta_3r_2'$, and of the two results

$$\left.\begin{aligned} X &= -\tfrac{1}{2}Kxy(y + x\cos\epsilon) \\ Y &= -\tfrac{1}{2}Kxy(x + y\cos\epsilon) \end{aligned}\right\}.$$

These contain the promised verification that X and Y are of the third order of small quantities; and they are in accord with the result in § 124, K now being the measure of sphericity of the (geodesic) surface.

Further, the values of the regional parameters at the angle C' of the Pérès parallelogram on the geodesic surface can be deduced at once. Thus the value of p is

$$\begin{aligned} p_{C'} &= p + xp_1' + \tfrac{1}{2}x^2p_1'' + \tfrac{1}{6}x^3p_1''' \\ &\quad + (y + Y)P_2' + \tfrac{1}{2}y^2P_2'' + \tfrac{1}{6}y^3P_2''', \end{aligned}$$

up to the third order: that is, on substitution of the values of Y, P_2', P_2'', and P_2''', in their respective terms,

$$\begin{aligned} p_{C'} &= p + xp_1' + yp_2' + \tfrac{1}{2}(x^2p_1'' - 2xy\bar{\gamma}_{12} + y^2p_2'') \\ &\quad + \tfrac{1}{6}x^3p_1''' - \tfrac{1}{2}x^2y(\Gamma_{300}p_1'^2p_2') - \tfrac{1}{2}xy^2(\Gamma_{300}p_1'p_2'^2) + \tfrac{1}{6}y^3p_2''' \\ &\quad - \tfrac{1}{3}xy^2K(p_1' - p_2'\cos\epsilon) + \tfrac{1}{6}x^2yK(p_2' - p_1'\cos\epsilon) \\ &\quad - \tfrac{1}{2}Kxy(x + y\cos\epsilon)p_2'. \end{aligned}$$

Let a small quantity z_0 and direction-variables P', Q', R', be taken such that

$$z_0P' = xp_1' + yp_2', \quad z_0Q' = xq_1' + yq_2', \quad z_0R' = xr_1' + yr_2', \quad \sum AP'^2 = 1;$$

then the foregoing value becomes

$$p_{C'} = p + z_0 P' + \tfrac{1}{2} z_0^2 P'' + \tfrac{1}{6} z_0^3 P'''$$
$$- xy\{x(\tfrac{1}{3}p_2' + \tfrac{1}{6}p_1' \cos\epsilon) + y(\tfrac{1}{3}p_1' + \tfrac{1}{6}p_2' \cos\epsilon)\}K,$$

up to the third order. Similarly

$$q_{C'} = q + z_0 Q' + \tfrac{1}{2} z_0^2 Q'' + \tfrac{1}{6} z_0^3 Q'''$$
$$- xy\{x(\tfrac{1}{3}q_2' + \tfrac{1}{6}q_1' \cos\epsilon) + y(\tfrac{1}{3}q_1' + \tfrac{1}{6}q_2' \cos\epsilon)\}K,$$
$$r_{C'} = r + z_0 R' + \tfrac{1}{2} z_0^2 R'' + \tfrac{1}{6} z_0^3 R'''$$
$$- xy\{x(\tfrac{1}{3}r_2' + \tfrac{1}{6}r_1' \cos\epsilon) + y(\tfrac{1}{3}r_1' + \tfrac{1}{6}r_2' \cos\epsilon)\}K,$$

results which agree with the former results (§ 125) for any surface, and now are obtained for the surface that is geodesic to the region with K as the measure of sphericity.

Regional geodesic diagonal of the Pérès parallelogram.

231. The analysis in §§ 210-216 can be adapted to the discussion of another cognate question: when the regional geodesic UV is drawn from a point U in a regional geodesic OA to a point V in a regional geodesic OB, so that its points U and V are on the geodesic surface AOB, does the regional geodesic UV lie wholly in the geodesic surface? It will be sufficient to determine whether the direction of the regional geodesic UV at U lies in the geodesic surface; manifestly, the geodesic cannot be contained in the surface, if its direction at U is not so contained.

It has been proved (§ 216) that, if $OU = x$, $OV = y$, the direction-variables p', q', r', at U of the regional geodesic UV are given by

$$p' = p_0' - x\bar{\gamma}_{01} + \tfrac{1}{2} x^2 P,$$

where

$$z p_0' = y p_2' - x p_1', \quad z q_0' = y q_2' - x q_1', \quad z r_0' = y r_2' - x r_1',$$
$$P = -\tfrac{1}{2} x^2 (\Gamma_{300} p_0 p_1'^2) + \tfrac{1}{6} p_0' \frac{x^2 y^2}{z^2} K \sin^2\epsilon + \frac{xz}{3\Omega} K_1(0, 10),$$

the value of $K_1(0, 10)$ being defined in § 212. There are corresponding expressions for q' and r'.

The condition that this direction at U should lie in the geodesic surface $\theta = 0$ is, by § 227 (i),

$$(\theta_1)_U p' + (\theta_2)_U q' + (\theta_3)_U r' = 0.$$

As approximations of the second order in small quantities have been made, when determining the values of p', q', r', by means of magnitudes at O, and when obtaining the values of θ_1, θ_2, θ_3, at U also in terms of magnitudes at O, this

equation would certainly have to be satisfied up to that second order inclusive. We have

$$(\theta_1)_U = \theta_1 + x(\theta_1 \alpha_1 + \theta_2 \xi_1 + \theta_3 \phi_1) + \tfrac{1}{2}x^2 \frac{d^2\theta_1}{ds_1^{\,2}},$$

up to that order, and similarly for $(\theta_2)_U$ and for $(\theta_3)_U$; when these values are substituted, the aggregates of terms of successive orders must vanish, each in its own order.

The finite terms in the condition

$$\begin{aligned} &= \theta_1 p_0' + \theta_2 q_0' + \theta_3 r_0' \\ &\quad - \frac{1}{z}\{y(\theta_1 p_2' + \theta_2 q_2' + \theta_3 r_2') - x(\theta_1 p_1' + \theta_2 q_1' + \theta_3 r_1')\} = 0, \end{aligned}$$

because of the constitution of the geodesic surface. This part of the condition is therefore satisfied.

The aggregate of terms of the first order of small quantities

$$\begin{aligned} = \quad & \theta_1 \bar{\gamma}_{01} - p_0' (\theta_1 \alpha_1 + \theta_2 \xi_1 + \theta_3 \phi_1) \\ + \, & \theta_2 \delta_{01} - q_0' (\theta_1 \beta_1 + \theta_2 \eta_1 + \theta_3 \chi_1) \\ + \, & \theta_3 \bar{\theta}_{01} - r_0' (\theta_1 \gamma_1 + \theta_2 \zeta_1 + \theta_3 \psi_1). \end{aligned}$$

Now

$$\begin{aligned} p_0' \alpha_1 + q_0' \beta_1 + r_0' \gamma_1 &= \bar{\gamma}_{01}, \\ p_0' \xi_1 + q_0' \eta_1 + r_0' \zeta_1 &= \delta_{01}, \\ p_0' \phi_1 + q_0' \chi_1 + r_0' \psi_1 &= \bar{\theta}_{01}. \end{aligned}$$

Hence the aggregate of terms of the first order vanishes; and this part of the necessary condition is therefore satisfied.

The aggregate of terms of the second order of small quantities has, for the coefficient of $\tfrac{1}{2}x^2$, a quantity which

$$\begin{aligned} &= p_0' \frac{d^2\theta_1}{ds_1^{\,2}} + q_0' \frac{d^2\theta_2}{ds_1^{\,2}} + r_0' \frac{d^2\theta_3}{ds_1^{\,2}} \\ &\quad - 2\{\bar{\gamma}_{01}(\theta_1 \alpha_1 + \theta_2 \xi_1 + \theta_3 \phi_1) + \delta_{01}(\theta_1 \beta_1 + \theta_2 \eta_1 + \theta_3 \chi_1) + \bar{\theta}_{01}(\theta_1 \gamma_1 + \theta_2 \zeta_1 + \theta_3 \psi_1)\} \\ &\quad + \theta_1 P + \theta_2 Q + \theta_3 R. \end{aligned}$$

When we insert the values of $\dfrac{d^2\theta_1}{ds_1^{\,2}}, \dfrac{d^2\theta_2}{ds_1^{\,2}}, \dfrac{d^2\theta_3}{ds_1^{\,2}}$, obtained in § 227 as connected with the surface which is geodesic at O, and when reductions take place, this coefficient can be expressed in the form

$$\theta_1\{P + (\Gamma_{300} p_0' p_1'^2)\} + \theta_2\{Q + (\Delta_{300} p_0' p_1'^2)\} + \theta_3\{R + (\Theta_{300} p_0' p_1'^2)\}.$$

After the values of P, Q, R, are inserted, two sets of terms arise. One set involves K in the form

$$\tfrac{1}{6}(\theta_1 p_0' + \theta_2 q_0' + \theta_3 r_0') \frac{x^2 y^2}{z^2} K \sin^2 \epsilon,$$

which vanishes, as in the finite terms of the central condition. The other set of terms

$$=\theta_1 K_1(0, 10)+\theta_2 K_2(0, 10)+\theta_3 K_3(0, 10)$$

$$=\begin{vmatrix} a\theta_1+h\theta_2+g\theta_3, & p_0', & k_{11}\xi_{10}+k_{12}\eta_{10}+k_{13}\zeta_{10} \\ h\theta_1+b\theta_2+f\theta_3, & q_0', & k_{12}\xi_{10}+k_{22}\eta_{10}+k_{23}\zeta_{10} \\ g\theta_1+f\theta_2+c\theta_3, & r_0', & k_{13}\xi_{10}+k_{23}\eta_{10}+k_{33}\zeta_{10} \end{vmatrix},$$

and does not vanish. Consequently the aggregate of terms of the second order does not vanish; and the condition (even up to the second order of small quantities), that the regional geodesic UV should lie in the geodesic surface UOV, is not satisfied.

It follows that the geodesic surface, determined by the set of regional geodesics originating at O in the orientation through the directions of OA and OB at O, is geodesic towards the region for all directions at O in the orientation. At a point on OA, other than O, the surface is geodesic to the region only for the continuation of OA; and similarly at a point on OB, other than O, the surface is geodesic to the region only for the continuation of OB. Thus, in § 229, an estimate is obtained for the regional flexure of the superficial geodesic at U drawn in the direction which, under the Severi definition, provides a parallel to OB through U; this flexure is not zero; and therefore the regional geodesic, drawn at U in the specified direction, does not lie in the surface which is geodesic to the region at O.

CHAPTER XIX

Geodesic Parallelograms, Geodesic Cells, in a Region

Geodesic parallelograms in a region not practicable on preceding definitions.

232. The geodesics in the preceding investigation in § 229 are superficial geodesics, not regional geodesics, the limitation to the surface being introduced on p. 128; and we have seen (§ 231) that the regional geodesic UV does not lie in the surface that is geodesic to the region at O. Accordingly, it is not to be expected (and it is not the fact) that the regional geodesics at U and at V, drawn at those points in the directions of the Severi parallels at those points, lie in the surface that is geodesic to the region at O; it remains to prove that these regional geodesics, thus drawn at U and at V, do not intersect.

On an assumption that they intersect in a point Z, let $x+X$ and $y+Y$ denote the lengths of VZ and UZ respectively where (after earlier experiences) it may be assumed that X and Y are of the third order of small quantities: the last assumption will incidentally be verified. We write (§ 227)

$$P_2'=p_2'-x\bar{\gamma}_{12}-\tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2')+\tfrac{1}{6}Kx^2(p_2'-p_1'\cos\epsilon),$$

with like expressions for Q_2' and R_2', as the direction-variables at U of the Severi direction parallel to OV, and

$$P_1'=p_1'-y\bar{\gamma}_{12}-\tfrac{1}{2}y^2(\Gamma_{300}p_1'p_2'^2)+\tfrac{1}{6}Ky^2(p_1'-p_2'\cos\epsilon),$$

with like expressions for Q_1' and R_1', as the direction-variables at V of the Severi direction parallel to OU. Owing to the assumed intersection of the regional geodesics VZ and UZ, the regional parameters at Z have the same value by the OUZ approach as by the OVZ approach. Consequently, as arising out of the common value of p at Z, there is a relation (up to the third order of small quantities)

$$\begin{aligned}p+xp_1'+\tfrac{1}{2}x^2p_1''+\tfrac{1}{6}x^3p_1'''+(y+Y)P_2'+\tfrac{1}{2}(y+Y)^2P_2''+\tfrac{1}{6}(y+Y)^3P_2'''\\ =p+yp_2'+\tfrac{1}{2}y^2p_2''+\tfrac{1}{6}y^3p_2'''+(x+X)P_1'+\tfrac{1}{2}(x+X)^2P_1''+\tfrac{1}{6}(x+X)^3P_1'''.\end{aligned}$$

As small quantities of order higher than three are not retained,

$$(x+X)^2=x^2,\quad (x+X)^3=x^3,\quad (y+Y)^2=y^2,\quad (y+Y)^3=y^3,$$

to the retained order of approximation; so that

$$(y+Y)^3P_2'''=-y^3(\Gamma_{300}P_2'^3)=-y^3(\Gamma_{300}p_2'^3)=y^3p_2''',$$

to this order; and similarly

$$(x+X)^3P_1'''=x^3p_1'''.$$

It is necessary to retain the values of P_2'' at U and P_1'' at V, up to the first order inclusive, connected with regional geodesics ; and, as on p. 129, we thus have

$$P_2'' = -\sum(\Gamma_{11})_U P_2'^2 = p_2'' - x(\Gamma_{300} p_1' p_2'^2) - \frac{2}{3\Omega} x K_1(2, 12),$$

$$P_1'' = -\sum(\Gamma_{11})_V P_1'^2 = p_1'' - y(\Gamma_{300} p_1'^2 p_2') - \frac{2}{3\Omega} y K_1(1, 21).$$

When these values are substituted, the terms involving the quantities Γ_{ijk} cancel ; and the relation becomes

$$Yp_2' + \tfrac{1}{6}Kx^2y(p_2' - p_1' \cos\epsilon) - \frac{xy^2}{3\Omega}K_1(2, 12)$$
$$= Xp_1' + \tfrac{1}{6}Kxy^2(p_1' - p_2' \cos\epsilon) - \frac{x^2y}{3\Omega}K_2(1, 21):$$

that is,

$$p_2'\{Y + \tfrac{1}{6}Kxy(x + y\cos\epsilon)\} - p_1'\{X + \tfrac{1}{6}Kxy(y + x\cos\epsilon)\}$$
$$= \frac{xy^2}{3\Omega}K_1(2, 12) - \frac{x^2y}{3\Omega}K_2(1, 12).$$

There are similar relations arising from the q-parameter and the r-parameter, in the form

$$q_2'\{Y + \tfrac{1}{6}Kxy(x + y\cos\epsilon)\} - q_1'\{X + \tfrac{1}{6}Kxy(y + x\cos\epsilon)\}$$
$$= \frac{xy^2}{3\Omega}K_2(2, 12) - \frac{x^2y}{3\Omega}K_2(1, 12),$$
$$r_2'\{Y + \tfrac{1}{6}Kxy(x + y\cos\epsilon)\} - r_1'\{X + \tfrac{1}{6}Kxy(y + x\cos\epsilon)\}$$
$$= \frac{xy^2}{3\Omega}K_3(2, 12) - \frac{x^2y}{3\Omega}K_3(1, 12).$$

These three conditions cannot simultaneously be satisfied in general * by values of X and Y. Accordingly the regional geodesics, drawn in the Severi parallel directions, do not intersect.

It thus appears, for parallel geodesics in a region, when at U a regional geodesic is drawn in a direction defined to be parallel at U to the geodesic OV at O, and when at V a regional geodesic is drawn in a direction defined to be parallel at V to the geodesic OU at O, these regional geodesics do not intersect, whether the Levi-Civita definition or the Severi definition be adopted. Hence, on neither definition, can a regional geodesic parallelogram be constructed ; the Pérès geodesic parallelogram in § 229 is a figure in the geodesic surface, and its two new sides are not regional geodesics.

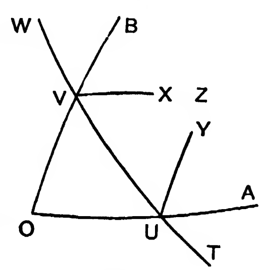

Fig. 26.

* An exception would arise if the sphericity of the region were constant for all orientations, but this case is tacitly excluded.

Other definitions can be propounded. Thus at U, there are two regional geodesics UA and UV; and they determine a surface geodesic to the region at U, distinct from the surface that is geodesic to the region at O. Similarly at V, the two regional geodesics VB and VU determine a surface, there geodesic to the region and distinct from the surface that is geodesic to the region at O. Accordingly, it can be proposed that, in this geodesic orientation at U a direction UY shall be taken, and in this geodesic orientation at V a direction VX shall be taken, to provide parallels, under specified definitions two of which may be noted.

I. By one definition, we make the parallelism in each instance revert to the initial point O, by requiring each of the angles AUY and XVB to be equal to UOV. If we write

$$\mathrm{p}_1' = p_1' + xp_1'' + \tfrac{1}{2}x^2 p_1''' + \dots$$

with quantities q_1' and r_1', to denote the direction-variables of the regional geodesic UA at U in the direction UA, and retain p', q', r', to denote the direction-variables of the regional geodesic UV at U in the direction UV, then

$$p' = p_0' - x\bar{\gamma}_{01} + P_0,$$

where (§ 216)

$$P_0 = -\tfrac{1}{2}x^2(\Gamma_{300}p_0'p_1'^2) + \tfrac{1}{6}p_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon + \frac{xy}{3\Omega}K_1(0, 12),$$

with like expressions for q' and r'; and the direction-variables of the suggested parallel UY at U in the direction UY are

$$\mathrm{P}_2' = \frac{\sin(\epsilon + U)}{\sin U}\mathrm{p}_1' + \frac{\sin\epsilon}{\sin U}p',$$

and similar expressions for Q_2' and R_2'. As (§ 217)

$$U - U_0 = Kxy(\tfrac{1}{3}\sin\epsilon - \tfrac{1}{6}\cos U_0 \sin V_0),$$

we can obtain the values of P_2', Q_2', R_2', up to the second order of small quantities inclusive. In particular, up to the first order of small quantities, we find

$$\mathrm{P}_2' = p_2' - x\bar{\gamma}_{12}, \quad \mathrm{Q}_2' = q_2' - x\bar{\delta}_{12}, \quad \mathrm{R}_2' = r_2' - x\bar{\theta}_{12}:$$

so that the definition, up to this order, agrees with the common portion of the direction-variables of the Levi-Civita parallel and the Severi parallel; but, in the terms of the second order of small quantities, there is divergence from the variables for both these parallels. Similarly for the direction-variables of VX at V under this definition.

But, when regional geodesics are drawn through U in the suggested direction UY at U and through V in the suggested direction VX at V, it is found * that such regional geodesics do not intersect.

* The calculations are similar to those on the preceding page.

The suggested definition does not therefore render practicable the construction of a regional geodesic parallelogram.

II. By another definition, the parallelism of the regional geodesics ultimately to be drawn through U and through V (though still drawn in the superficial orientations at those places) is made to depend upon VU as the basic geodesic for this purpose, thus making a partial reversion to the initial definition of parallels. Under this definition of parallelism, the parallel UY through U would be such that the angle YUV is equal to the angle BVW, and the parallel VX through V would be such that the angle XVU is equal to the angle AUT. Then the direction-variables of this suggested parallel UY at U in the direction UY are

$$\mathrm{P}_2' = \frac{\sin V}{\sin U}\,\mathrm{p}_1' + \frac{\sin (U+V)}{\sin U}\,p',$$

with similar expressions for Q_2', R_2'. The value of $U - U_0$ is as cited before; and, up to the order of approximation retained,

$$V - V_0 = Kxy(\tfrac{1}{3}\sin\epsilon - \tfrac{1}{6}\cos V_0 \sin U_0).$$

Thus the values of P_2', Q_2', R_2', can be obtained up to the second order of small quantities retained; and in particular, up to the first order inclusive, it is found that

$$\mathrm{P}_2' = p_2' - x\bar{\gamma}_{12}, \quad \mathrm{Q}_2' = q_2' - x\bar{\delta}_{12}, \quad \mathrm{R}_2' = r_2' - x\bar{\theta}_{12},$$

so that, up to this first order, the direction-variables of the suggested parallel are the same as the corresponding portions of the direction-variables under the Levi-Civita definition and the Severi definition.

Similarly for the suggested direction VX at V.

But, as before, when regional geodesics are drawn through U in the suggested direction at U and through V in the suggested direction at V, it is found * that they do not intersect.

Consequently this definition also fails to render practicable the construction of a regional geodesic parallelogram.

Alternative selection of parallels; regional parallelograms.

233. Accordingly, as all the four definitions preclude the construction of a regional geodesic parallelogram, we seek other values of the direction-variables at a place such as U along OA and the analytically symmetrical direction-variables at a place such as V along OB, for respective directions through those places, which shall conform to the primary conditions of parallelism in the four definitions and shall admit of the intersection of regional geodesics drawn in those directions. We therefore postulate direction-variables P'_U, Q'_U, R'_U, at U for the direction UY, and direction-variables P'_V, Q'_V, R'_V, at V for the direction VX, of regional

* Again, the calculations are similar to those on p. 139.

geodesics which are to satisfy the imposed conditions; and, as initial forms, we take

$$P'_U = p_2' - x\bar{\gamma}_{12} - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2 p_2') + P_U,$$
$$P'_V = p_1' - y\bar{\gamma}_{12} - \tfrac{1}{2}y^2(\Gamma_{300}p_1' p_2'^2) + P_V,$$

with corresponding expressions for Q'_U, R'_U, and Q'_V, R'_V, the sets of magnitudes P_U and P_V being of the second order in the small quantities. Moreover, the magnitudes P_U and P_V must change, each into the other, by interchange of the direction-variables p_1', q_1', r_1', and the small arc x, with the direction-variables p_2', q_2', r_2', and the small arc y. In P_U there can exist no term in y^2 alone because, when x vanishes, P'_U must become p_2'; and similarly in P_V there can exist no term in x^2 alone because, when y vanishes, P'_V must become p_1'—results in accordance with the interchange property of P_U and P_V as stated.

(i) In the first place, the permanent arc-relations of the region

$$\sum A_U (P'_U)^2 = 1, \quad \sum A_V (P'_V)^2 = 1,$$

at U and at V respectively, must be satisfied; but there is no precedent value set upon the angle AUY or upon the angle BVX, as no one of the four definitions of parallelism in § 232 has been adopted. The first of these two arc-relations is

$$\sum\left[\left(A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1{}^2}\right)\{p_2'^2 - 2xp_2'\bar{\gamma}_{12} - x^2 p_2'(\Gamma_{300}p_1'^2 p_2') + x^2\bar{\gamma}_{12}{}^2 + 2p_2' P_U\}\right] = 1.$$

The finite terms cancel, owing to the relation $\sum Ap_2'^2 = 1$. The aggregate terms, of the first order of small quantities,

$$= x\left\{\sum\frac{dA}{ds_1}p_2'^2 - 2\sum Ap_2'\bar{\gamma}_{12}\right\} = 0,$$

when the values (§ 212) of the first arc-derivatives of the primary magnitudes are inserted. The aggregate terms, of the second order of small quantities, must vanish; and therefore we must have

$$\begin{aligned} &2\{u_1^{(2)}P_U + u_2^{(2)}Q_U + u_3^{(2)}R_U\} + x^2\sum A\bar{\gamma}_{12}{}^2 \\ &- x^2\{u_1^{(2)}(\Gamma_{300}p_1'^2 p_2') + u_2^{(2)}(\Delta_{300}p_1'^2 p_2') + u_3^{(2)}(\Theta_{300}p_1'^2 p_2')\} \\ &- 2x^2\sum\frac{dA}{ds_1}p_2'\bar{\gamma}_{12} + \tfrac{1}{2}x^2\sum\frac{d^2A}{ds_1{}^2}p_2'^2 = 0. \end{aligned}$$

When substitution for the last two terms involving the first and the second arc-derivatives of the primary magnitudes is made and reduction is effected, the relation becomes

$$u_1^{(2)}P_U + u_2^{(2)}Q_U + u_3^{(2)}R_U = \tfrac{1}{6}x^2 K \sin^2\epsilon,$$

where K is the sphericity of the region in the orientation at O.

In the same way, the permanent arc-relation at V is satisfied, unconditionally

for finite terms and for terms of the first order of small quantities; as regards terms of the second order of small quantities, the relation

$$u_1^{(1)}P_V + u_2^{(1)}Q_V + u_3^{(1)}R_V = \tfrac{1}{6}y^2K\sin^2\epsilon$$

must be satisfied.

(ii) The regional geodesics, drawn through U in the direction P_U', Q_U', R_U', and through V in the direction P_V', Q_V', R_V', are to intersect. Let Z be their intersection; and denote the lengths of UZ and VZ by $y+Y$ and $x+X$ respectively, where X and Y may be expected to be of the third order—an expectation that will be verified incidentally in the analysis. The regional parameters at Z must acquire the same values, when Z is attained by the path OUZ as when it is attained by the path OVZ; and, therefore, when the corresponding values of p are equated, there is the relation

$$\begin{aligned} p + xp_1' + \tfrac{1}{2}x^2p_1'' + \tfrac{1}{6}x^3p_1''' + (y+Y)P_U' + \tfrac{1}{2}(y+Y)^2P_U'' + \tfrac{1}{6}(y+Y)^3P_U''' \\ = p + yp_2' + \tfrac{1}{2}y^2p_2'' + \tfrac{1}{6}y^3p_2''' + (x+X)P_V' + \tfrac{1}{2}(x+X)^2P_V'' + \tfrac{1}{6}(x+X)^3P_V''', \end{aligned}$$

up to the third order of small quantities.

We have

$$(x+X)^2 = x^2, \quad (x+X)^3 = x^3, \quad (y+Y)^2 = y^2, \quad (y+Y)^3 = y^3,$$

up to this order inclusive: so that, to this order,

$$(y+Y)^3P_U''' = y^3p_U''', \quad (x+X)^3P_V''' = x^3p_V''',$$

while the values of P_1'' at V and P_2'' at U are required up to the first order of small quantities inclusive. Proceeding as in § 229, we find

$$P_U'' = p_2'' - x(\Gamma_{300}p_1'p_2'^2) - \frac{2}{3\Omega}xK_1(2, 12),$$

$$P_V'' = p_1'' - y(\Gamma_{300}p_1'^2p_2') - \frac{2}{3\Omega}yK_1(1, 21),$$

up to the required order. When the values are substituted, it appears that the finite terms cancel, the terms of the first order of small quantities cancel, and the terms of the second order of small quantities cancel. In order that the terms of the third order shall cancel, it is necessary that the equation

$$\begin{aligned} Yp_2' + yP_U - \tfrac{1}{2}x^2y(\Gamma_{300}p_1'^2p_2') - \tfrac{1}{2}xy^2(\Gamma_{300}p_1'p_2'^2) - \frac{1}{3\Omega}xy^2K_1(2, 12) \\ = Xp_1' + xP_V - \tfrac{1}{2}xy^2(\Gamma_{300}p_1'p_2'^2) - \tfrac{1}{2}x^2y(\Gamma_{300}p_1'^2p_2') - \frac{1}{3\Omega}x^2yK_1(1, 21): \end{aligned}$$

that is, the equation

$$Yp_2' + y\{P_U - \frac{1}{3\Omega}xyK_1(2, 12)\} = Xp_1' + x\{P_V - \frac{1}{3\Omega}xyK_1(1, 21)\},$$

shall be satisfied, together with the corresponding equations arising from the q-parameter and the r-parameter. Now write

$$P_U - \frac{1}{3\Omega} xyK_1(2, 12) = \bar{P}_2', \quad P_V - \frac{1}{3\Omega} xyK_1(1, 21) = \bar{P}_1',$$

$$Q_U - \frac{1}{3\Omega} xyK_2(2, 12) = \bar{Q}_2', \quad Q_V - \frac{1}{3\Omega} xyK_2(1, 21) = \bar{Q}_1',$$

$$R_U - \frac{1}{3\Omega} xyK_3(2, 12) = \bar{R}_2', \quad R_V - \frac{1}{3\Omega} xyK_3(1, 21) = \bar{R}_1',$$

so that the relations become

$$Yp_2' + y\bar{P}_2' = Xp_1' + x\bar{P}_1',$$
$$Yq_2' + y\bar{Q}_2' = Xq_1' + x\bar{Q}_1',$$
$$Yr_2' + y\bar{R}_2' = Xr_1' + x\bar{R}_1'.$$

In the first place, the quantities P_U and P_V are to interchange when the set of direction-variables p_2', q_2', r_2', and the arc y, are interchanged with the set of direction-variables p_1', q_1', r_1', and the arc x, each into the other; and this property actually holds of the magnitudes $xyK_i(2, 12)$ and $xyK_i(1, 21)$, for $i=1, 2, 3$, in the respective pairs. Hence the same property holds concerning $\bar{P}_1'$ and $\bar{P}_2'$, $\bar{Q}_1'$ and $\bar{Q}_2'$, $\bar{R}_1'$ and $\bar{R}_2'$.

In the second place, we have

$$u_1^{(2)}K_1(2, 12) + u_2^{(2)}K_2(2, 12) + u_3^{(2)}K_3(2, 12)$$
$$= \begin{vmatrix} au_1^{(2)} + hu_2^{(2)} + gu_3^{(2)}, & p_2', & k_{11}\xi_{12} + k_{12}\eta_{12} + k_{13}\zeta_{12} \\ hu_1^{(2)} + bu_2^{(2)} + fu_3^{(2)}, & q_2', & k_{21}\xi_{12} + k_{22}\eta_{12} + k_{23}\zeta_{12} \\ gu_1^{(2)} + fu_2^{(2)} + cu_3^{(2)}, & r_2', & k_{31}\xi_{12} + k_{32}\eta_{12} + k_{33}\zeta_{12} \end{vmatrix}.$$

But the three constituents in the first column $=\Omega p_2'$, $\Omega q_2'$, $\Omega r_2'$, respectively, so that the determinant vanishes, and

$$u_1^{(2)}K_1(2, 12) + u_2^{(2)}K_2(2, 12) + u_3^{(2)}K_3(2, 12) = 0.$$

Hence the second-order condition from the arc-relation at U becomes

$$u_1^{(2)}\bar{P}_2' + u_2^{(2)}\bar{Q}_2' + u_3^{(2)}\bar{R}_2' = \tfrac{1}{6}x^2 K \sin^2 \epsilon.$$

Similarly, it is proved that

$$u_1^{(1)}K_1(1, 21) + u_2^{(1)}K_2(1, 21) + u_3^{(1)}K_3(1, 21) = 0;$$

and therefore the second-order condition from the arc-relation at V becomes

$$u_1^{(1)}\bar{P}_1' + u_2^{(1)}\bar{Q}_1' + u_3^{(1)}\bar{R}_1' = \tfrac{1}{6}y^2 K \sin^2 \epsilon.$$

(iii) Having regard to the forms of the direction-variables of the parallels under the Severi definition (and also under the third and the fourth definitions in

§ 232), we postulate the forms

$$\bar{P}_2' = Kx^2(\alpha p_1' + \beta p_2') + Kxy(\gamma p_1' + \delta p_2'),$$
$$\bar{Q}_2' = Kx^2(\alpha q_1' + \beta q_2') + Kxy(\gamma q_1' + \delta q_2'),$$
$$\bar{R}_2' = Kx^2(\alpha r_1' + \beta r_2') + Kxy(\gamma r_1' + \delta r_2'),$$

where α, β, γ, δ, are quantities independent of u and v, and are symmetrical in the two sets of direction-variables p_1', q_1', r_1', and p_2', q_2', r_2', should they involve these variables. The term in y^2 alone is omitted because, when x vanishes, P_2' becomes p_2' and therefore $\bar{P}_2'$ vanishes; and likewise for $\bar{Q}_2'$ and $\bar{R}_2'$. Owing to the interchangeable property between $\bar{P}_1'$, $\bar{Q}_1'$, $\bar{R}_1'$, and $\bar{P}_2'$, $\bar{Q}_2'$, $\bar{R}_2'$, when the variables of the arcs OU and OV are interchanged, the postulation for $\bar{P}_2'$, $\bar{Q}_2'$, $\bar{R}_2'$, admits the further postulation

$$\bar{P}_1' = Ky^2(\alpha p_2' + \beta p_1') + Kxy(\gamma p_2' + \delta p_1'),$$
$$\bar{Q}_1' = Ky^2(\alpha q_2' + \beta q_1') + Kxy(\gamma q_2' + \delta q_1'),$$
$$\bar{R}_1' = Ky^2(\alpha r_2' + \beta r_1') + Kxy(\gamma r_2' + \delta r_1'),$$

derivable by the indicated interchanges. Let these values be substituted in the two second-order conditions surviving from the arc-relations at U and at V respectively; then we find

$$\tfrac{1}{6}\sin^2\epsilon = \beta + \alpha\cos\epsilon, \quad 0 = \delta + \gamma\cos\epsilon,$$

as equations to be satisfied by α, β, γ, δ.

Again, the three second-order relations (on p. 144), which emerge as the conditions for the required intersection of the two regional geodesics drawn through U and V, lead to the values

$$X = Kxy\{x(\alpha - \delta) + y(\gamma - \beta)\}, \quad Y = Kxy\{x(\gamma - \beta) + y(\alpha - \delta)\}.$$

Thus far in the construction of the geodesic quadrilateral, the explicit demands have exacted, beyond the necessity of actual intersection of the two geodesics, the requirement of satisfying merely the primary conditions of parallelism at U and V. Two parametric quantities remain at our disposal. The deduced values of X and Y manifestly admit a descriptive demand affecting the sides of the quadrilateral, viz. that the opposite sides shall be equal in length: which, moreover, is a property as characteristic of a plane rectilinear parallelogram as is the postulated property relating to the exact equality of angles. In accordance with this postulate of equal sides, we take $X=0$, $Y=0$. The lengths x and y are independent of one another; and therefore

$$\alpha = \delta, \quad \gamma = \beta,$$

so that

$$\beta = \tfrac{1}{6} = \gamma, \quad \alpha = -\tfrac{1}{6}\cos\epsilon = \delta.$$

The values of $\bar{P}_2'$, $\bar{Q}_2'$, $\bar{R}_2'$, and of $\bar{P}_1'$, $\bar{Q}_1'$, $\bar{R}_1'$, now are determinate.

The geodesic quadrilateral, thus framed, has its opposite sides OU and VZ equal, and likewise the opposite sides OV and UZ. The primary conditions of parallelism, at U and V, of these respective pairs of sides are satisfied. It will be proved (§ 234) that, up to an included second order of small quantities, the angles AUZ and BVZ are equal to one another. We therefore regard the quadrilateral $OUZVO$, with regional geodesics for sides, as a *regional parallelogram*.

The direction-variables of the sides UZ at U and VZ at V are P_2', Q_2', R_2', and P_1', Q_1', R_1', respectively, where

$$\left.\begin{aligned} P_2' &= p_2' - x\bar{\gamma}_{12} - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2') + \frac{1}{3\Omega}xyK_1(2, 12) \\ &\qquad + \tfrac{1}{6}Kx\{x(p_2' - p_1'\cos\epsilon) + y(p_1' - p_2'\cos\epsilon)\} \\ Q_2' &= q_2' - x\bar{\delta}_{12} - \tfrac{1}{2}x^2(\Delta_{300}p_1'^2p_2') + \frac{1}{3\Omega}xyK_2(2, 12) \\ &\qquad + \tfrac{1}{6}Kx\{x(q_2' - q_1'\cos\epsilon) + y(q_1' - q_2'\cos\epsilon)\} \\ R_2' &= r_2' - x\bar{\theta}_{12} - \tfrac{1}{2}x^2(\Theta_{300}p_1'^2p_2') + \frac{1}{3\Omega}xyK_3(2, 12) \\ &\qquad + \tfrac{1}{6}Kx\{x(r_2' - r_1'\cos\epsilon) + y(r_1' - r_2'\cos\epsilon)\} \end{aligned}\right\},$$

$$\left.\begin{aligned} P_1' &= p_1' - y\bar{\gamma}_{12} - \tfrac{1}{2}y^2(\Gamma_{300}p_1'p_2'^2) + \frac{1}{3\Omega}xyK_1(1, 21) \\ &\qquad + \tfrac{1}{6}Ky\{y(p_1' - p_2'\cos\epsilon) + x(p_2' - p_1'\cos\epsilon)\} \\ Q_1' &= q_1' - y\bar{\delta}_{12} - \tfrac{1}{2}y^2(\Delta_{300}p_1'p_2'^2) + \frac{1}{3\Omega}xyK_2(1, 21) \\ &\qquad + \tfrac{1}{6}Ky\{y(q_1' - q_2'\cos\epsilon) + x(q_2' - q_1'\cos\epsilon)\} \\ R_1' &= r_1' - y\bar{\theta}_{12} - \tfrac{1}{2}y^2(\Theta_{300}p_1'p_2'^2) + \frac{1}{3\Omega}xyK_3(1, 21) \\ &\qquad + \tfrac{1}{6}Ky\{y(r_1' - r_2'\cos\epsilon) + x(r_2' - r_1'\cos\epsilon)\} \end{aligned}\right\}.$$

The values of the parameters at Z, the point of intersection of the regional geodesics UY and VX, are

$$\begin{aligned} &p + (xp_1' + yp_2') - \tfrac{1}{2}\textstyle\sum\{\Gamma_{11}(xp_1' + yp_2')^2\} - \tfrac{1}{6}\sum\{\Gamma_{300}(xp_1' + yp_2')^3\} \\ &\qquad\qquad + \tfrac{1}{6}Kxy\{x(p_2' - p_1'\cos\epsilon) + y(p_1' - p_2'\cos\epsilon)\}, \\ &q + (xq_1' + yq_2') - \tfrac{1}{2}\textstyle\sum\{\Delta_{11}(xp_1' + yp_2')^2\} - \tfrac{1}{6}\sum\{\Delta_{300}(xp_1' + yp_2')^3\} \\ &\qquad\qquad + \tfrac{1}{6}Kxy\{x(q_2' - q_1'\cos\epsilon) + y(q_1' - q_2'\cos\epsilon)\}, \\ &r + (xr_1' + yr_2') - \tfrac{1}{2}\textstyle\sum\{\Theta_{11}(xp_1' + yp_2')^2\} - \tfrac{1}{6}\sum\{\Theta_{300}(xp_1' + yp_2')^3\} \\ &\qquad\qquad + \tfrac{1}{6}Kxy\{x(r_2' - r_1'\cos\epsilon) + y(r_1' - r_2'\cos\epsilon)\}, \end{aligned}$$

up to the third order of small quantities inclusive.

Length and direction of diagonal of the parallelogram.

234. To find the length of the geodesic diagonal OZ, and the direction-variables p', q', r', at O of that diagonal, we proceed as before. Let a small arc-length t and direction-variables p_4', q_4', r_4', at O be chosen, so that

$$tp_4' = xp_1' + yp_2', \quad tq_4' = xq_1' + yq_2', \quad tr_4' = xr_1' + yr_2';$$

and therefore

$$t^2 = x^2 + y^2 + 2xy \cos \epsilon.$$

We denote the direction-variables of the geodesic OZ at O by

$$p' = p_4' + P, \quad q' = q_4' + Q, \quad r' = r_4' + R,$$

where P, Q, R, are small quantities, which will be found to be of the second order of small quantities; and we denote the geodesic length OZ by $t+T$, where T will be found to be of the third order. Then the value of the p-parameter at Z

$$= p + (t+T)p' + \tfrac{1}{2}(t+T)^2 p'' + \tfrac{1}{6}(t+T)^3 p''',$$

up to the third order of small quantities; and this is necessarily equal to the value just obtained, which may be written

$$p + tp_4' - \tfrac{1}{2}\sum\{\Gamma_{11}(tp_4')^2\} - \tfrac{1}{6}\sum\{\Gamma_{300}(tp_4')^3\}$$
$$+ \tfrac{1}{6}Kxy\{x(p_2' - p_1' \cos \epsilon) + y(p_1' - p_2' \cos \epsilon)\}.$$

Now, for this approximation,

$$\begin{aligned} p'' &= -\sum \Gamma_{11} p'^2 \\ &= -\sum\{\Gamma_{11}(p_4'^2 + 2p_4'P)\} \\ &= p_4'' - 2\sum \Gamma_{11} p_4' P; \end{aligned}$$

and therefore

$$tP + Tp_4' - t^2 \sum \Gamma_{11} p_4' P = \tfrac{1}{6}Kxy\{x(p_2' - p_1' \cos \epsilon) + y(p_1' - p_2' \cos \epsilon)\}.$$

Manifestly the third term on the left-hand side, involving quantities of the order t^2P, is negligible compared with the first term on the left-hand side; and thus the equation is

$$tP + Tp_4' = \tfrac{1}{6}Kxy\{x(p_2' - p_1' \cos \epsilon) + y(p_1' - p_2' \cos \epsilon)\}.$$

Similarly, by using values of the q-parameter and the r-parameter at Z, we have relations

$$tQ + Tq_4' = \tfrac{1}{6}Kxy\{x(q_2' - q_1' \cos \epsilon) + y(q_1' - q_2' \cos \epsilon)\},$$
$$tR + Tr_4' = \tfrac{1}{6}Kxy\{x(r_2' - r_1' \cos \epsilon) + y(r_1' - r_2' \cos \epsilon)\}.$$

The arc-relation at O for the geodesic diagonal OZ is

$$\sum Ap'^2 = 1:$$

that is,

$$\sum A(p_4'^2 + 2p_4'P) = 1,$$

and therefore

$$u_1^{(4)}P+u_2^{(4)}Q+u_3^{(4)}R=0.$$

There are four equations to determine P, Q, R, T. When the first three equations are multiplied by $u_1^{(4)}$, $u_2^{(4)}$, $u_3^{(4)}$, respectively, and the results are added, then, by means of the fourth equation and of the relation $\sum Ap_4'^2=1$, we have

$$T=\tfrac{1}{6}Kxy\sum[u_1^{(4)}\{x(p_2'-p_1'\cos\epsilon)+y(p_1'-p_2'\cos\epsilon)\}],$$

shewing that T is of the third order of small quantities. The first three equations then shew that P, Q, R, are of the second order of small quantities. Also

$$\sum u_1^{(4)}(p_2'-p_1'\cos\epsilon)=\frac{1}{t}\sum\{(xu_1^{(1)}+yu_2^{(1)})(p_2'-p_1'\cos\epsilon)\}$$

$$=\frac{y}{t}\sin^2\epsilon,$$

$$\sum u_1^{(4)}(p_1'-p_2'\cos\epsilon)=\frac{x}{t}\sin^2\epsilon;$$

and therefore

$$T=\frac{1}{3t}Kx^2y^2\sin^2\epsilon.$$

Thus the length of the geodesic diagonal OZ

$$=t+\frac{1}{3t}Kx^2y^2\sin^2\epsilon,$$

with the value of t equal to $(x^2+y^2+2xy\cos\epsilon)^{\frac{1}{2}}$.

The values of P, Q, R, are obtained by substituting the value of T; and, after reduction, we find

$$P=\frac{xy}{6t^3}(x^2-y^2)K\{x(p_2'-p_1'\cos\epsilon)-y(p_1'-p_2'\cos\epsilon)\},$$

$$Q=\frac{xy}{6t^3}(x^2-y^2)K\{x(q_2'-q_1'\cos\epsilon)-y(q_1'-q_2'\cos\epsilon)\},$$

$$R=\frac{xy}{6t^3}(x^2-y^2)K\{x(r_2'-r_1'\cos\epsilon)-y(r_1'-r_2'\cos\epsilon)\}.$$

The direction-variables of the geodesic diagonal OZ at O are given by

$$p'=p_4'+P,\quad q'=q_4'+Q,\quad r'=r_4'+R.$$

The approximations for these direction-variables are (as usual) taken up to the second order of small quantities inclusive, while the approximation to the length of the geodesic diagonal is taken up to the third order of small quantities.

Angles of the regional parallelogram.

235. The direction-variables of the regional geodesic UZ at Z in the continuation of the direction UZ are

$$\begin{aligned}&=(P_2')_U+y(P_2'')_U+\tfrac{1}{2}y^2(P_2''')_U\\&=p_2'-\sum[\Gamma_{11}\{p_2'(xp_1'+yp_2')\}]-\tfrac{1}{2}\sum[\Gamma_{300}\{p_2'(xp_1'+yp_2')^2\}]-\frac{1}{3\Omega}xyK_1(2,12)\\&\qquad+\tfrac{1}{6}Kx\{x(p_2'-p_1'\cos\epsilon)+y(p_1'-p_2'\cos\epsilon)\},\end{aligned}$$

with like expressions for the q-direction-variable and the r-direction-variable; and the direction-variables of the regional geodesic VZ at Z in the continuation of the direction VZ are

$$\begin{aligned}&-p_1'-\sum[\Gamma_{11}\{p_1'(xp_1'+yp_2')\}]-\tfrac{1}{2}\sum[\Gamma_{300}\{p_1'(xp_1'+yp_2')^2\}]-\frac{1}{3\Omega}xyK_1(1,21)\\&\qquad+\tfrac{1}{6}Ky\{y(p_1'-p_2'\cos\epsilon)+x(p_2'-p_1'\cos\epsilon)\},\end{aligned}$$

with like expressions for the q-direction-variable and the r-direction-variable.

In the construction of the parallelogram, the direction-variables at U, for the direction UZ of the regional geodesic, were made to agree with the first-order portion of the corresponding direction-variables of the Levi-Civita parallel and the Severi parallel. A more precise relation between the angle AUZ and the angle AOB is found by merely calculating the angle AUZ. At U, the value of A is

$$A+x\frac{dA}{ds_1}+\tfrac{1}{2}x^2\frac{d^2A}{ds_1^2};$$

the direction-variables of UA at U in the direction UA are

$$p_1'+xp_1''+\tfrac{1}{2}x^2p_1''',$$

with two similar expressions; and the direction-variables of UZ at U in the direction UZ are the foregoing magnitudes P_2', Q_2', R_2', on p. 146. Hence

$$\cos AUZ=\sum\left\{\left(A+x\frac{dA}{ds_1}+\tfrac{1}{2}x^2\frac{d^2A}{ds_1^2}\right)(p_1'+xp_1''+\tfrac{1}{2}x^2p_1''')P_2'\right\}.$$

When substitution is made for P_2', Q_2', R_2', and the terms of various orders in the small quantities are collected, the terms of finite order

$$=\sum Ap_1'p_2'=\cos\epsilon;$$

and the aggregate of terms of the first order of small quantities

$$\begin{aligned}&=x\sum\frac{dA}{ds_1}p_1'p_2'-x\sum Ap_1'\gamma_{12}+x\sum Ap_2'p_1''\\&=0,\end{aligned}$$

when the value of the first term is inserted from § 212.

The aggregate of terms of the second order of small quantities

$$= \tfrac{1}{2}x^2 \sum \frac{d^2A}{ds_1^{\,2}} p_1'p_2' + \tfrac{1}{2}x^2 \sum Ap_2p_1'''$$

$$+ \sum Ap_1' \left[-\tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2^2) + \frac{1}{3\Omega} xyK_1(2, 12) \right.$$

$$\left. + \tfrac{1}{6}Kx\{x(p_2' - p_1' \cos\epsilon) + y(p_1' - p_2' \cos\epsilon)\} \right]$$

$$- x^2 \sum Ap_1''\bar{\gamma}_{12} - x^2 \sum \frac{dA}{ds_1} p_1'\bar{\gamma}_{12} + x^2 \sum \frac{dA}{ds_1} p_2'p_1''.$$

When substitution for the magnitudes involving the first and the second of the arc-derivatives of the primary magnitudes is made from § 212, and when reduction is effected, the aggregate becomes

$$\sum u_1^{(1)} \left[\frac{1}{3\Omega} xyK_1(2, 12) + \tfrac{1}{6}Kx\{x(p_2' - p_1' \cos\epsilon) + y(p_1' - p_2' \cos\epsilon)\} \right].$$

But

$$\sum u_1^{(1)}K_1(2, 12) = \begin{vmatrix} au_1^{(1)} + hu_2^{(1)} + gu_3^{(1)}, & p_2', & k_{11}\xi_{12} + k_{12}\eta_{12} + k_{13}\zeta_{12} \\ hu_1^{(1)} + bu_2^{(1)} + fu_3^{(1)}, & q_2', & k_{12}\xi_{12} + k_{22}\eta_{12} + k_{23}\zeta_{12} \\ gu_1^{(1)} + fu_2^{(1)} + cu_3^{(1)}, & r_2', & k_{13}\xi_{12} + k_{23}\eta_{12} + k_{33}\zeta_{12} \end{vmatrix}$$

$$= \Omega \begin{vmatrix} p_1', & p_2', & k_{11}\xi_{12} + k_{12}\eta_{12} + k_{13}\zeta_{12} \\ q_1', & q_2', & k_{12}\xi_{12} + k_{22}\eta_{12} + k_{23}\zeta_{12} \\ r_1', & r_2', & k_{13}\xi_{12} + k_{23}\eta_{12} + k_{33}\zeta_{12} \end{vmatrix}$$

$$= \Omega \sum k_{11}\xi_{12}^{\,2} = \Omega K \sin^2\epsilon\,;$$

while

$$\sum u_1^{(1)}p_2' = \cos\epsilon, \quad \sum u_1^{(1)} = 1.$$

Consequently, the aggregate of terms of the second order in question

$$= \tfrac{1}{3}Kxy \sin^2\epsilon + \tfrac{1}{6}Kxy \sin^2\epsilon = \tfrac{1}{2}Kxy \sin^2\epsilon.$$

We therefore have, up to the second order inclusive,

$$\cos AUZ = \cos\epsilon + \tfrac{1}{2}Kxy \sin^2\epsilon,$$

and therefore also, up to that order,

$$AUZ = \epsilon - \tfrac{1}{2}Kxy \sin\epsilon.$$

Similarly we have

$$BVZ = \epsilon - \tfrac{1}{2}Kxy \sin\epsilon.$$

236. Next, the angles of the geodesic triangle UZV must be found. For the angle ZUV, the direction-variables of UZ at U in the direction UZ are the

quantities P_2', Q_2', R_2', of § 233; and those of UV at U in the direction UV are the quantities p', q', r', of p. 82. Hence

$$\cos ZUV = \sum A_U P_2' p'$$
$$= \sum \left(A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2 \frac{d^2A}{ds_1^2}\right) P_2' p',$$

where

$$P_2' = p_2' - x\bar{\gamma}_{12} - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2 p_2) + \frac{1}{3\Omega} xyK_1(2, 12)$$
$$+ \tfrac{1}{6}Kx\{x(p_2' - p_1' \cos\epsilon) + y(p_1' - p_2' \cos\epsilon)\},$$
$$p' = p_0' - x\bar{\gamma}_{01} - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2 p_0') + \tfrac{1}{6}p_0' \frac{x^2y^2}{z^2} K \sin^2\epsilon + \frac{1}{3\Omega} xyK_1(0, 12),$$

with like values for Q_2', R_2', and q_0', r_0'.

In this expression for $\cos ZUV$, the finite terms

$$= \sum A p_2' p_0' = \cos V_0.$$

The aggregate of terms of the first order

$$= x\left\{\sum \frac{dA}{ds_1} p_2' p_0' - \sum A p_2' \bar{\gamma}_{01} - \sum A p_0' \bar{\gamma}_{12}\right\}$$
$$= 0,$$

when the values of the arc-derivatives of the primary magnitudes are substituted.

Denoting the aggregate of terms of the second order by T_2, we have

$$T_2 = \tfrac{1}{2}x^2 \sum \frac{d^2A}{ds_1^2} p_2' p_0'$$
$$- \tfrac{1}{2}x^2 \sum \{A p_0' (\Gamma_{300}p_1'^2 p_2')\} + \frac{1}{3\Omega} xy \sum \{A p_0' K_1(2, 12)\}$$
$$+ \tfrac{1}{6}Kx \sum [A p_0' \{x(p_2' - p_1' \cos\epsilon) + y(p_1' - p_2' \cos\epsilon)\}]$$
$$- \tfrac{1}{2}x^2 \sum \{A p_2' (\Gamma_{300}p_1'^2 p_0')\} + \frac{1}{3\Omega} xy \{\sum A p_2' K_1(0, 12)\}$$
$$+ \tfrac{1}{6}\sum \left(A p_0' p_2' \frac{x^2y^2}{z^2} K \sin^2\epsilon\right)$$
$$- x^2 \sum \left\{\frac{dA}{ds_1}(p_0' \bar{\gamma}_{12} + p_2' \bar{\gamma}_{01})\right\} + x^2 \sum A \bar{\gamma}_{01} \bar{\gamma}_{12}.$$

In this expression, the total coefficient of $\tfrac{1}{2}x^2$ in the term in the first line, on substitution for second arc-derivatives of the primary magnitudes (§ 212), becomes

$$-\tfrac{2}{3}\sum k_{11}\xi_{10}\xi_{12} + \sum \{u_1^{(2)}(\Gamma_{300}p_1'^2 p_0')\} + \sum \{u_1^{(0)}(\Gamma_{300}p_1'^2 p_2')\}$$
$$+ 2\sum A \bar{\gamma}_{10} \bar{\gamma}_{12}$$
$$+ 2\sum [u_1^{(2)}\{\alpha_1 \bar{\gamma}_{10} + \beta_1 \bar{\delta}_{10} + \gamma_1 \bar{\theta}_{10}\}] + 2\sum [u_1^{(0)}\{\alpha_1 \bar{\gamma}_{12} + \beta_1 \bar{\delta}_{12} + \gamma_1 \bar{\theta}_{12}\}],$$

the first term of which

$$= -\tfrac{2}{3}\frac{y}{z}\sum k_{11}\xi_{12}^2$$

$$= -\tfrac{2}{3}\frac{y}{z}K\sin^2\epsilon\,;$$

and, in the last line of the expression for T_2,

$$\sum\left\{\frac{dA}{ds_1}(p_0'\bar{\gamma}_{12}+p_2'\bar{\gamma}_{01})\right\}$$

$$=2\sum(A\bar{\gamma}_{01}\bar{\gamma}_{12})+\sum\bar{\gamma}_{12}\{a_1u_1^{(0)}+\xi_1u_2^{(0)}+\phi_1u_3^{(0)}\}+\sum\bar{\gamma}_{10}\{a_1u_1^{(2)}+\xi_1u_2^{(2)}+\phi_1u_3^{(0)}\}.$$

When these values are substituted, we find, after some reduction,

$$\begin{aligned} T_2 = &-\tfrac{1}{3}\frac{x^2y}{z}K\sin^2\epsilon \\ &+\sum u_1^{(0)}\left[\frac{1}{3\Omega}xyK_1(2,\,12)+\tfrac{1}{6}Kx\{x(p_2'-p_1'\cos\epsilon)+y(p_1'-p_2'\cos\epsilon)\}\right] \\ &+\sum u_1^{(2)}\left[\tfrac{1}{6}p_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon+\frac{1}{3\Omega}xyK_1(0,\,12)\right]. \end{aligned}$$

Now

$$\sum u_1^{(0)}p_2'=\cos V_0,\quad \sum u_1^{(0)}p_1'=-\cos U_0,\quad \sum u_1^{(2)}p_0'=\cos V_0\,;$$

also

$$\begin{aligned} \sum u_1^{(0)}K_1(2,\,12) &= \begin{vmatrix} au_1^{(0)}+hu_2^{(0)}+gu_3^{(0)}, & p_2', & k_{11}\xi_{12}+k_{12}\eta_{12}+k_{13}\zeta_{12} \\ hu_1^{(0)}+bu_2^{(0)}+fu_3^{(0)}, & q_2', & k_{12}\xi_{12}+k_{22}\eta_{12}+k_{23}\zeta_{12} \\ gu_1^{(0)}+fu_2^{(0)}+cu_3^{(0)}, & r_2', & k_{13}\xi_{12}+k_{23}\eta_{12}+k_{33}\zeta_{12} \end{vmatrix} \\ &= \Omega\,|\; p_0',\; p_2',\; k_{11}\xi_{12}+k_{12}\eta_{12}+k_{13}\zeta_{12}\;| \\ &= \Omega\sum k_{11}\xi_{02}\xi_{12} \\ &= \Omega\frac{x}{z}\sum k_{11}\xi_{12}^2=\Omega\frac{x}{z}K\sin^2\epsilon\,; \end{aligned}$$

and similarly

$$\sum u_1^{(2)}K_1(0,\,12)=\Omega\sum k_{11}\xi_{20}\xi_{12}=-\Omega\frac{x}{z}K\sin^2\epsilon.$$

Hence

$$\begin{aligned} T_2=-\tfrac{1}{3}\frac{x^2y}{z}K\sin^2\epsilon+\tfrac{1}{6}Kx\{x(\cos V_0+\cos U_0\cos\epsilon)-y(\cos U_0+\cos V_0\cos\epsilon)\} \\ +\tfrac{1}{6}\frac{x^2y^2}{z^2}K\sin^2\epsilon\cos V_0. \end{aligned}$$

But, as $U_0+V_0+\epsilon=\pi$, we have

$$\cos V_0+\cos U_0\cos\epsilon=\sin U_0\sin\epsilon,\quad \cos U_0+\cos V_0\cos\epsilon=\sin V_0\sin\epsilon\,;$$

also

$$x\sin U_0=y\sin V_0\,;$$

and therefore

$$T_2 = -\tfrac{1}{3}xyK \sin\epsilon \sin V_0 + \tfrac{1}{6}xyK \sin U_0 \sin V_0 \cos V_0.$$

Consequently

$$\cos ZUV - \cos V_0 = T_2,$$

and therefore, up to the second order of small quantities inclusive,

$$ZUV = V_0 + xyK(\tfrac{1}{3}\sin\epsilon - \tfrac{1}{6}\sin U_0 \cos V_0) = V\,;$$

and, similarly, we have

$$ZVU = U_0 + xyK(\tfrac{1}{3}\sin\epsilon - \tfrac{1}{6}\sin V_0 \cos U_0) = U,$$

with the former (§§ 217, 232) significance of U and V.

As $ZV=x$, $ZU=y$, $ZUV=OVU$, $ZVU=OUV$, as the side UV is common to the two geodesic triangles OUV, ZUV, and as the sphericity can be taken as measured at U or at V in connection with the area of the triangle OUV, the angle UZV would appear to be equal to ϵ, up to the second order of small quantities at least. This inference we establish as follows.

237. The direction-variables $\bar{P}_1'$, $\bar{Q}_1'$, $\bar{R}_1'$, of the regional geodesic VZ at Z, in the direction which is the continuation of VZ, are

$$\bar{P}_1' = P_1' + xP_1'' + \tfrac{1}{2}x^2 P_1''',$$

as the arc VZ is of length x: that is, as (p. 129)

$$P_1'' = p_1'' - y(\Gamma_{300}p_1'^2 p_2') + \frac{2}{3\Omega} yK_1(1,12)$$

for the regional geodesic, we have

$$\bar{P}_1' = p_1' - \sum \Gamma_{11} p_1'(xp_1' + yp_2') - \tfrac{1}{2}\sum \{\Gamma_{300} p_1'(xp_1' + yp_2')^2\}$$
$$+ \frac{xy}{3\Omega} K_1(1,12) + \tfrac{1}{6}KyT_p,$$

where

$$T_p = x(p_2' - p_1'\cos\epsilon) + y(p_1' - p_2'\cos\epsilon)\,;$$

and there are corresponding values for $\bar{Q}_1'$, $\bar{R}_1'$, involving quantities T_q, T_r, with variables q', r', instead of p' in T_p. Similarly, the direction-variables of UZ at Z, in the direction which is the continuation of UZ, are

$$\bar{P}_2' = p_2' - \sum \Gamma_{11} p_2'(xp_1' + yp_2') - \tfrac{1}{2}\sum \{\Gamma_{300} p_2'(xp_1' + yp_2')^2\}$$
$$+ \frac{xy}{3\Omega} K_1(2,21) + \tfrac{1}{6}KxT_p,$$

with corresponding values for $\bar{Q}_2'$, $\bar{R}_2'$.

The values, at Z, of the primary magnitudes are required. We have

$$A_Z = A_U + y\frac{dA_U}{ds_2} + \tfrac{1}{2}y^2 \frac{d^2 A_U}{ds_2^2},$$

up to the second order, y being the accurate length of UZ; and

$$A_U = A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2},$$

to the same order. For approximation to A_Z, we retain the first-order terms in the first derivative of A_U, and only the term free from small quantities in the second derivative. Thus we take

$$\frac{d^2A_U}{ds_2^2} = \frac{d^2A}{ds_2^2}, \quad \frac{dA_U}{ds_2} = \frac{dA}{ds_2} + x\frac{d}{ds_2}\left(\frac{dA}{ds_1}\right) = \frac{dA}{ds_2} + x\frac{d^2A}{ds_1\,ds_2},$$

under the convention of §§ 213, 309, as regards this second-order differentiation of a function solely of position, effected along directions ds_1 and ds_2 independent of one another. We therefore have

$$A_Z = A + x\frac{dA}{ds_1} + y\frac{dA}{ds_2} + \tfrac{1}{2}\left(x^2\frac{d^2A}{ds_1^2} + 2xy\frac{d^2A}{ds_1\,ds_2} + y^2\frac{d^2A}{ds_2^2}\right),$$

up to the second order inclusive.

Now $$\cos UZV = \sum A_Z \bar{P}_1' \bar{P}_2';$$

and the value of $\cos UZV$ can be derived, up to the second order inclusive.

In this expression for $\cos UZV$, the finite terms

$$= \sum A p_1' p_2' - \cos\epsilon.$$

The aggregate of terms of the first order of small quantities

$$\begin{aligned}
&= \sum\left\{\left(x\frac{dA}{ds_1} + y\frac{dA}{ds_2}\right)p_1'p_2'\right\} \\
&\qquad - \sum[u_1^{(2)}\{\sum \Gamma_{11}p_1'(xp_1' + yp_2')\}] - \sum[u_1^{(1)}\{\sum \Gamma_{11}p_2'(xp_1' + yp_2')\}] \\
&\quad - x\sum\left(\frac{dA}{ds_1}p_1'p_2'\right) + y\sum\left(\frac{dA}{ds_2}p_1'p_2'\right) \\
&\qquad - \sum\{u_1^{(2)}(x\bar{\gamma}_{11} + y\bar{\gamma}_{12})\} - \sum\{u_1^{(1)}(x\bar{\gamma}_{12} + y\bar{\gamma}_{22})\} \\
&= 0,
\end{aligned}$$

on substitution from § 212 of the values of the first arc-derivatives of the primary magnitudes.

Let E_2 denote the aggregate of terms of the second order, so that

$$\begin{aligned}
E_2 = \tfrac{1}{2}\sum&\left\{\left(x^2\frac{d^2A}{ds_1^2} + 2xy\frac{d^2A}{ds_1ds_2} + y^2\frac{d^2A}{ds_2^2}\right)p_1'p_2'\right\} \\
&+ \sum u_1^{(2)}[-\tfrac{1}{2}\{\Gamma_{300}p_1'(xp_1' + yp_2')^2\} + \frac{xy}{3\Omega}K_1(1, 12) + \tfrac{1}{6}KyT_p] \\
&+ \sum u_1^{(1)}[-\tfrac{1}{2}\{\Gamma_{300}p_2'(xp_1' + yp_2')^2\} + \frac{xy}{3\Omega}K_1(2, 21) + \tfrac{1}{6}KxT_p] \\
&- \sum\left[\left(x\frac{dA}{ds_1} + y\frac{dA}{ds_2}\right)\{(x\bar{\gamma}_{11} + y\bar{\gamma}_{12})p_2' + (x\bar{\gamma}_{12} + y\bar{\gamma}_{22})p_1'\}\right] \\
&+ \sum\{A(x\bar{\gamma}_{11} + y\bar{\gamma}_{12})(x\bar{\gamma}_{12} + y\bar{\gamma}_{22})\}.
\end{aligned}$$

When we substitute, from §§ 212, 213, the values of the first arc-derivatives and of the second arc-derivatives of the primary magnitudes, and collect terms, we find, after some reductions,

$$E_2 = \tfrac{1}{3}xy\sum k_{11}\xi_{12}{}^2 + \frac{xy}{3\Omega}[\{\sum u_1{}^{(2)}K_1(1, 12)\} + \{\sum u_1{}^{(1)}K_1(2, 21)\}]$$
$$+\tfrac{1}{6}K[y\{\sum u_1{}^{(2)}T_p\} + x\{\sum u_1{}^{(1)}T_p\}].$$

Now, as in § 236,

$$\sum u_1{}^{(2)}K_1(1, 12) = -\Omega\sum k_{11}\xi_{12}{}^2, \quad \sum u_1{}^{(1)}K_1(2, 21) = -\Omega\sum k_{11}\xi_{12}{}^2;$$

and also

$$\sum u_1{}^{(2)}T_p = x\sin^2\epsilon, \quad \sum u_1{}^{(1)}T_p = y\sin^2\epsilon;$$

consequently

$$E_2 = -\tfrac{1}{3}xy\sum k_{11}\xi_{12}{}^2 + \tfrac{1}{3}Kxy\sin^2\epsilon = 0.$$

It follows that, certainly up to the second order of small quantities inclusive,

$$\cos UZV = \cos\epsilon:$$

or the inference, that the angle UZV is equal to ϵ, is verified up to the second order of small quantities inclusive.

It follows that the two regional geodesic triangles OUV and ZVU, when ZVU is constructed by drawing the geodesics UZ and VZ in the respectively assigned directions, are such that the sides and the angles of the geodesic triangle ZVU are respectively equal to the sides and the angles of the geodesic triangle OUV, to the retained order of approximation.

Hence a small geodesic parallelogram can be constructed in the region. To this end, we take two regional geodesics OU and OV in any regional directions at O, the lengths of the arcs $OU(=x)$ and $OV(=y)$ being small. Then we draw regional geodesics through U and V in specific directions; and we measure lengths x and y along the geodesics through V and U respectively. We assign (as is possible) the specific directions so that, not merely are the primary conditions of parallelism at U and at V obeyed, but also the two new geodesics intersect, in some point Z. In the geodesic parallelogram $OUZVO$ thus formed, opposite sides are actually equal; and as regards angles, it has been proved that

$$ZUV = OVU, \quad ZVU = OUV, \quad UZV = VOU,$$

the equalities subsisting up to the second order of small quantities inclusive.

Relation of regional geodesics to the various superficial orientations.

238. There remains a double question as to whether the direction UZ at U lies in the superficial orientation AUV of the region at U, and whether the direction VZ at V lies in the superficial orientation BVU at V. The direction UZ will lie as suggested, if quantities λ and μ exist such that

$$P_2' = \lambda p' + \mu(p_1' + xp_1'' + \tfrac{1}{2}x^2 p_1'''),$$

with like expressions for Q_2' and R_2', where the coefficients of μ are the direction-

variables of the regional geodesic UA at U, and p', q', r', are the direction-variables of the regional geodesic UV at U, so that (§ 216)

$$p' = p_0' - x\bar{\gamma}_{01} - \tfrac{1}{2}x^2(\Gamma_{300}p_0'p_1'^2) + \tfrac{1}{6}p_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon + \frac{xy}{3\Omega}K_1(0, 12),$$

with like values for q' and r'. As approximations up to the second order of small quantities have been made, the expressions for P_2', Q_2', R_2', must be satisfied up to that order; we therefore take

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2, \quad \mu = \mu_0 + \mu_1 + \mu_2,$$

where κ_i (for $\kappa = \lambda$, μ, and $i = 0, 1, 2$) is of order i in the small quantities.

Let the expressions for P_2', Q_2', R_2', be substituted. In order that finite terms may balance, we have

$$p_2' = \lambda_0 p_0' + \mu_0 p_1',$$

and therefore

$$\lambda_0 = \frac{z}{y}, \quad \mu_0 = \frac{x}{y},$$

with the former significance (§ 214) for z. In order that terms of the first order may balance, there are three relations of the type

$$-x\bar{\gamma}_{12} = \lambda_1 p_0' + \mu_1 p' - \lambda_0 x\bar{\gamma}_{01} + \mu_0 x p_1''.$$

Now

$$z\bar{\gamma}_{01} = y\bar{\gamma}_{12} - x\bar{\gamma}_{11} = y\bar{\gamma}_{12} + xp_1'',$$

so that

$$x\bar{\gamma}_{12} = \lambda_0 x\bar{\gamma}_{01} - \mu_0 x p_1'',$$

and therefore

$$\lambda_1 p_0' + \mu_1 p' = 0.$$

Similarly

$$\lambda_1 q_0' + \mu_1 q' = 0, \quad \lambda_1 r_0' + \mu_1 r' = 0;$$

consequently we must have

$$\lambda_1 = 0, \quad \mu_1 = 0.$$

In order that the terms of the second order may balance, there are three relations of the type

$$\begin{aligned}
&-\tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2') + \frac{xy}{3\Omega}K_1(2, 12) + \tfrac{1}{6}Kx\{x(p_2' - p_1'\cos\epsilon) + y(p_1' - p_2'\cos\epsilon)\} \\
&\quad = \lambda_2 p_0' + \mu_2 p_1' \\
&\qquad + \frac{z}{y}\left\{-\tfrac{1}{2}x^2(\Gamma_{300}p_0'p_1'^2) + \tfrac{1}{6}p_0'\frac{x^2y^2}{z^2}K\sin^2\epsilon + \frac{xy}{3\Omega}K_1(0, 12)\right\} + \tfrac{1}{2}\frac{x^3}{y}p_1'''.
\end{aligned}$$

When substitution is made for the value of p_1''', the terms in this typical relation involving the magnitudes Γ_{ijk} cancel; and the relation can then be modified so as to become

$$\begin{aligned}
&\lambda_2 p_0' + \mu_2 p_1' \\
&\quad = \frac{x^2}{3\Omega}K_1(1, 12) + \tfrac{1}{6}Kx\{x(p_2' - p_1'\cos\epsilon) + y(p_1' - p_2'\cos\epsilon)\} - \tfrac{1}{6}p_0'\frac{x^2y}{z}K\sin^2\epsilon;
\end{aligned}$$

and the other two relations, similarly modified, become

$$\lambda_2 q_0' + \mu_2 q_1'$$
$$= \frac{x^2}{3\Omega} K_2(1, 12) + \tfrac{1}{6} K x \{x(q_2' - q_1' \cos\epsilon) + y(q_1' - q_2' \cos\epsilon)\} - \tfrac{1}{6} q_0' \frac{x^2 y}{z} K \sin^2\epsilon,$$

$$\lambda_2 r_0' + \mu_2 r_1'$$
$$= \frac{x^2}{3\Omega} K_3(1, 12) + \tfrac{1}{6} K x \{x(r_2' - r_1' \cos\epsilon) + y(r_1' - r_2' \cos\epsilon)\} - \tfrac{1}{6} r_0' \frac{x^2 y}{z} K \sin^2\epsilon.$$

Now $$p_1'\xi_{12} + q_1'\eta_{12} + r_1'\zeta_{12} = 0, \quad p_2'\xi_{12} + q_2'\eta_{12} + r_2'\zeta_{12} = 0,$$

and therefore $$p_0'\xi_{12} + q_0'\eta_{12} + r_0'\zeta_{12} = 0.$$

Multiply the three modified relations by ξ_{12}, η_{12}, ζ_{12}, respectively, and add the products; then the result requires that

$$\xi_{12} K_1(1, 12) + \eta_{12} K_2(1, 12) + \zeta_{12} K_3(1, 12)$$

shall vanish, contrary to easily verified fact.

Hence the three relations do not coexist; simultaneous values of the two quantities λ_2 and μ_2 cannot be found. In other words, there do not exist quantities λ and μ, such that the three conditions of the type

$$P_2' = \lambda p' + \mu(p_1' + x p_1'' + \tfrac{1}{2} x^2 p_1''')$$

can be satisfied. Consequently, the direction UZ at U does not lie in the superficial orientation at U determined by the two regional geodesics UA and UV.

Similarly we infer that the direction VZ at V does not lie in the superficial orientation at V determined by the two regional geodesics VB and VU.

239. As an indication of the deviation between the foregoing regional parallelogram and the earlier Pérès parallelogram (§§ 229, 230) in the surface which is geodesic to the region, consider the inclination of the two regional geodesic diagonals, OZ of the former, OC' of the latter.

The respective sets of the direction-variables at O of the two diagonals are required, up to the second order of small quantities. With the notation of § 234, the direction-variables of OZ at O are

$$p_4' + P, \quad q_4' + Q, \quad r_4' + R,$$

with the values of P, Q, R, as given on p. 148.

With the same notation, let the length of the regional geodesic OC' of § 231 be denoted by $t + \bar{T}$, to the third order of small quantities: it appears that $\bar{T}$ itself is of that third order. Let the direction-variables of OC' at O be denoted by

$$p_4' + P_4, \quad q_4' + Q_4, \quad r_4' + R_4,$$

to the required second order of small quantities: it appears that P_4, Q_4, R_4,

themselves are of that second order. By analysis of the kind used in similar investigations, we obtain the three equations

$$tP_4+p_4'\bar{T}= -Kxy\{x(\tfrac{1}{3}p_2'+\tfrac{1}{6}p_1'\cos\epsilon)+y(\tfrac{1}{3}p_1'+\tfrac{1}{6}p_2'\cos\epsilon)\},$$
$$tQ_4+q_4'\bar{T}= -Kxy\{x(\tfrac{1}{3}q_2'+\tfrac{1}{6}q_1'\cos\epsilon)+y(\tfrac{1}{3}q_1'+\tfrac{1}{6}q_2'\cos\epsilon)\},$$
$$tR_4+r_4'\bar{T}= -Kxy\{x(\tfrac{1}{3}r_2'+\tfrac{1}{6}r_1'\cos\epsilon)+y(\tfrac{1}{3}r_1'+\tfrac{1}{6}r_2'\cos\epsilon)\},$$

from the property that the values of the parameters at C' are the same, by the diagonal path OC' as by the lateral path OUC'. There is also the equation

$$u_1^{(4)}P_4+u_2^{(4)}Q_4+u_3^{(4)}R_4=0,$$

as a residuary condition from the permanent arc-relation at O along OC'.

Multiplying the three parameter-equations by $u_1^{(4)}$, $u_2^{(4)}$, $u_3^{(4)}$, adding the products, and using the residuary arc-condition, we find

$$\bar{T}= -Kxy\{x(\tfrac{1}{3}\cos\beta+\tfrac{1}{6}\cos\alpha\cos\epsilon)+y(\tfrac{1}{3}\cos\alpha+\tfrac{1}{6}\cos\beta\cos\epsilon)\},$$

where α denotes the angle UOC' and β the angle VOC'. When this value of $\bar{T}$ is inserted in the parameter-equations, they give

$$P_4=K\frac{xy(x^2-y^2)}{3t^3}\{y(p_1'-p_2'\cos\epsilon)-x(p_2'-p_1'\cos\epsilon)\},$$

with like values for Q_4 and R_4: that is,

$$P_4=-2P,\quad Q_4=-2Q,\quad R_4=-2R.$$

Now let ϕ denote the inclination at O between the two regional diagonals OZ and OC': thus

$$\begin{aligned}\sin^2\phi&=\sum a\{(q_4'+Q_4)(r_4'+R)-(r_4'+R_4)(q_4'+Q)\}^2\\ &=9\sum a(q_4'R-r_4'Q)^2\\ &=9\{(\sum Ap_4'^2)(\sum AP^2)-(\sum Ap_4'P)^2\}\\ &=9\sum AP^2=K^2\frac{x^2y^2(x^2-y^2)^2}{4t^4}\sin^2\epsilon,\end{aligned}$$

on reduction. Consequently

$$\phi=K\frac{xy(x^2-y^2)}{2t^2}\sin\epsilon,$$

shewing that the inclination is a small quantity of the second order.

Ex. 1. Prove that the length of the small geodesic arc $C'Z$ is $\frac{1}{2}Kxyt$; and find its direction-variables.

Ex. 2. Prove that, to the second order of small quantities inclusive,

$$ZOU=\alpha-\tfrac{1}{3}\phi,\quad ZOV=\beta-\tfrac{1}{3}\phi.$$

Regional cells.

240. We now can construct a geodesic parallelepiped (or *geodesic cell*) in a region, the definition being the obvious extension of that of the geodesic parallelogram in § 233.

Take any three non-complanar directions at a point O in the region; and along the regional geodesics drawn in these directions, measure small arc-lengths $OX=x$, $OY=y$, $OZ=z$, the direction-variables at O of the respective directions being denoted by p_1', q_1', r_1': p_2', q_2', r_2'; p_3', q_3', r_3'; respectively. Complete the geodesic parallelogram (in the sense of § 237) having OX and OY as adjacent geodesic sides; in this parallelogram $OYZ'XO$, the small arc $YZ'=x$ and the small arc $XZ'=y$. The geodesic parallelogram $OXY'ZO$, with the side $XY'=z$ and the side $ZY'=x$, is completed; and likewise the parallelogram $OZX'YO$, with the side $ZX'=y$ and the side $YX'=z$. The sides of all these parallelograms, in their respective pairs, have direction-variables which conform to the primary conditions (§ 221) of parallelism, up to the first order inclusive in each instance; and there are the established second-order terms, in conformity with the results of § 233.

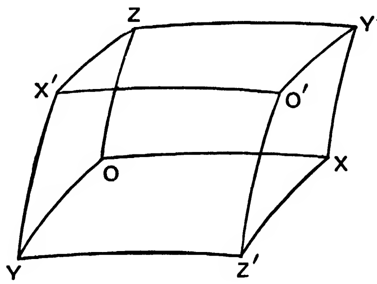

FIG. 27.

To complete the geodesic parallelepiped, we take a regional geodesic $X'O'$ parallel (in the foregoing sense, that is, up to the first order) to OX and of length x; a regional geodesic $Y'O'$ similarly parallel to OY and of length y; and a regional geodesic $Z'O'$ similarly parallel to OZ and of length z. It will, of course, be necessary to take account of the positions X', Y', Z', as the respective initial points of $X'O'$, $Y'O'$, $Z'O'$; and we take O' to be the point of intersection of these three geodesics, O' being at geodesic distances x, y, z, respectively from X', Y', Z'. The parallelepiped, thus formed in the region, with regional geodesics for its edges, and having regional parallelograms for its faces, will be termed a *cell* or (where necessary) a *regional cell.*

Earlier investigations (p.146) shew that the direction-variables of the regional geodesic YZ' at Y are

$$p_1' - y\gamma_{12} - \tfrac{1}{2}y^2(\Gamma_{300}p_1'p_2'^2) + P_{12},$$

with two other expressions, the second-order quantities P_{12}, Q_{12}, R_{12}, depending on the four-index symbols, and that the direction-variables of XZ' at X are

$$p_2' - x\gamma_{12} - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2') + P_{21},$$

with two other expressions, the second-order quantities P_{21}, Q_{21}, R_{21}, depending on the four-index symbols. Similarly, the direction-variables of XY' at X are three magnitudes of the type

$$p_3' - x\gamma_{13} - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_3') + P_{31},$$

and the direction-variables of YX' at Y are three magnitudes of the type

$$p_3' - y\gamma_{23} - \tfrac{1}{2}y^2(\Gamma_{300}p_2'^2p_3') + P_{32}.$$

Similarly for the direction-variables of ZY' at Z, and of ZX' at Z.

We therefore, as the direction-variables of a regional geodesic $X'O'$ at X' in the direction $X'O'$, take* quantities P_1', Q_1', R_1', defined by

$$P_1' = p_1' - [\Gamma_{11}p_1'(yp_2' + zp_3')] - \tfrac{1}{2}[\Gamma_{300}p_1'(yp_2' + zp_3')^2] + P_1,$$
$$Q_1' = q_1' - [\Delta_{11}p_1'(yp_2' + zp_3')] - \tfrac{1}{2}[\Delta_{300}p_1'(yp_2' + zp_3')^2] + Q_1,$$
$$R_1' = r_1' - [\Theta_{11}p_1'(yp_2' + zp_3')] - \tfrac{1}{2}[\Theta_{300}p_1'(yp_2' + zp_3')^2] + R_1,$$

where P_1, Q_1, R_1, are magnitudes of the second order of small quantities. Similarly the direction-variables, of a regional geodesic $Y'O'$ at Y' in the direction $Y'O'$, are taken to be P_2', Q_2', R_2', defined by

$$P_2' = p_2' - [\Gamma_{11}p_2'(zp_3' + xp_1')] - \tfrac{1}{2}[\Gamma_{300}p_2'(zp_3' + xp_1')^2] + P_2,$$
$$Q_2' = q_2' - [\Delta_{11}p_2'(zp_3' + xp_1')] - \tfrac{1}{2}[\Delta_{300}p_2'(zp_3' + xp_1')^2] + Q_2,$$
$$R_2' = r_2' - [\Theta_{11}p_2'(zp_3' + xp_1')] - \tfrac{1}{2}[\Theta_{300}p_2'(zp_3' + xp_1')^2] + R_2,$$

where P_2, Q_2, R_2, are of the second order of small quantities. Similarly also the direction-variables, of a regional geodesic $Z'O'$ at Z' in the direction $Z'O'$, are taken to be P_3', Q_3', R_3', defined by

$$P_3' = p_3' - [\Gamma_{11}p_3'(xp_1' + yq_2')] - \tfrac{1}{2}[\Gamma_{300}p_3'(xp_1' + yq_2')^2] + P_3,$$
$$Q_3' = q_3' - [\Delta_{11}p_3'(xp_1' + yq_2')] - \tfrac{1}{2}[\Delta_{300}p_3'(xp_1' + yq_2')^2] + Q_3,$$
$$R_3' = r_3' - [\Theta_{11}p_3'(xp_1' + yq_2')] - \tfrac{1}{2}[\Theta_{300}p_3'(xp_1' + yq_2')^2] + R_3,$$

where P_3, Q_3, R_3, are of the second order of small quantities.

241. To verify that these values, which evidently satisfy the primary conditions of parallelism under all the definitions adopted, satisfy also the intrinsic requirements of the region, and simultaneously to determine the nine quantities P_i, Q_i, R_i (for $i = 1, 2, 3$), there are two kinds of equations.

The first kind consists of a set of three relations, which survive from the permanent arc-relation of the region, at X' for $X'O'$, at Y' for $Y'O'$, at Z' for $Z'O'$. It will appear, in each of the three instances, that the arc-relation is satisfied by the foregoing values of direction-variables, unconditionally for the finite terms and for terms of the first order of small quantities, and by a residuary second-order condition involving magnitudes P_i, Q_i, R_i.

The second kind consists of three sets of two relations, each set arising out of the values of the regional parameters at O'. The value of the p-parameter at O', in relation to its value at O, is the same in the broken geodesic path from O to O', whether this path † ends in $X'O'$, $Y'O'$, or $Z'O'$: the three paths provide three

* Here, and elsewhere, $[\Gamma_{11}p_m'p_n']$ is used as the equivalent of $\sum\limits_i \sum\limits_j \Gamma_{ij}x_i'z_j'$, where the variables x' and z' represent p_m', q_m', r_m', and p_n', q_n', r_n', respectively.

† The equivalence of the paths OYZ' and OXZ', in providing the values of the regional parameters at Z', has already been used (§ 233) in the construction of the regional parallelogram $OXZ'YO$; similarly as regards the paths OXY' and OZY' in the parallelogram $OXY'ZO$, and as regards the paths OYX' and OZX' in the parallelogram $OYX'ZO$.

formally different values which must be equal and therefore provide two relations. Similarly, there are two relations from the q-parameter, and there are two relations from the r-parameter. In all these relations, it will appear that the finite terms, the terms of the first order of small quantities, and the terms of the second order, are exactly the same in each set of three values; in order that they may be the same for terms of the third order, there are two residuary conditions involving the magnitudes P_i, Q_i, R_i.

Consequently, there are nine equations in all, potentially sufficient for the determination of the nine magnitudes P, Q, R. In the arc-relations, terms up to the second order of small quantities inclusive are retained; in the parameter-equalities, terms up to the third order are retained.

We begin with the arc-relations at X', Y', Z', respectively. At X', the relation is

$$\sum A_{X'} P_1'^2 = 1.$$

As we require only terms up to the second order of small quantities in this equation, we take (as in § 234)

$$A_{X'} = A + y\frac{dA}{ds_2} + z\frac{dA}{ds_3} + \tfrac{1}{2}\left(y^2\frac{d^2A}{ds_2^2} + 2yz\frac{d^2A}{ds_2\,ds_3} + z^2\frac{d^2A}{ds_3^2}\right)$$

with the earlier convention (§ 213) as to differentiation along different geodesics. All the various geodesics in question satisfy the convention up to the order retained; for, up to the second order of small quantities, we have

$$p_{X'} = p + yp_2' + zp_3' - \tfrac{1}{2}[\Gamma_{11}(yp_2' + zp_3')^2],$$

with corresponding values of $q_{X'}$ and $r_{X'}$, and

$$A_{X'} = A(p_{X'}, q_{X'}, r_{X'}).$$

There are similar values for the other primary magnitudes at X'. Also, up to the second order,

$$P_1' = p_1' - (y\bar{\gamma}_{12} + z\bar{\gamma}_{13}) + \{P_1 - \tfrac{1}{2}[\Gamma_{300}p_1'(yp_2' + zp_3')^2]\}.$$

Let these values be inserted in the arc-relation at X', which must be satisfied, the small quantities x, y, z, being independent of one another.

From the finite terms, we have

$$\sum Ap_1'^2 = 1,$$

a relation which is satisfied.

From the terms of the first order, we are to have

$$\sum p_1'^2\left(y\frac{dA}{ds_2} + z\frac{dA}{ds_3}\right) - 2\sum Ap_1'(y\bar{\gamma}_{12} + z\bar{\gamma}_{13}) = 0;$$

but (§ 212)

$$\sum \frac{dA}{ds_2} p_1'^2 = 2\{u_1^{(1)}\bar{\gamma}_{12} + u_2^{(1)}\bar{\delta}_{12} + u_3^{(1)}\bar{\theta}_{12}\},$$

$$\sum \frac{dA}{ds_3} p_1'^2 = 2\{u_1^{(1)}\bar{\gamma}_{13} + u_2^{(1)}\bar{\delta}_{13} + u_3^{(1)}\bar{\theta}_{13}\},$$

and therefore the terms of the first order vanish. The required relation is thus far satisfied unconditionally.

From the terms of the second order, we are to have

$$\begin{aligned} 2\sum Ap_1'P_1 - \sum Ap_1'[\Gamma_{300}p_1'(yp_2'+zp_3')^2] + \sum\{A(y\bar{\gamma}_{12}+z\bar{\gamma}_{13})^2\} \\ -2\sum\left\{(y\bar{\gamma}_{12}+z\bar{\gamma}_{13})p_1'\left(y\frac{dA}{ds_2}+z\frac{dA}{ds_3}\right)\right\} \\ +\tfrac{1}{2}y^2\sum\frac{d^2A}{ds_2^2}p_1'^2 + yz\sum\frac{d^2A}{ds_2\,ds_3}p_1'^2 + \tfrac{1}{2}z^2\sum\frac{d^2A}{ds_3^2}p_1'^2 = 0. \end{aligned}$$

When the combinations of the first derivatives of the magnitudes A, and those of their second derivatives, as given in § 212, are inserted, and the condition is reduced, it becomes

$$u_1^{(1)}P_1 + u_2^{(1)}Q_1 + u_3^{(1)}R_1 = \tfrac{1}{6}\sum\{k_{11}(y\xi_{21}+z\xi_{31})^2\},$$

a residuary second-order condition from the arc-relation at X'.

Similarly we have

$$u_1^{(2)}P_2 + u_2^{(2)}Q_2 + u_3^{(2)}R_2 = \tfrac{1}{6}\sum\{k_{11}(z\xi_{32}+x\xi_{12})^2\},$$
$$u_1^{(3)}P_3 + u_2^{(3)}Q_3 + u_3^{(3)}R_3 = \tfrac{1}{6}\sum\{k_{11}(x\xi_{13}+y\xi_{23})^2\},$$

as residuary second-order conditions from the arc-relation, at Y', and at Z', respectively.

Parametric positions of the angular points of the cell in the region.

242. The value of the p-parameter at Z' is, by § 233,

$$p_{Z'} = p + xp_1' + yp_2' - \tfrac{1}{2}[\Gamma_{11}(xp_1'+yp_2')^2] - \tfrac{1}{6}[\Gamma_{300}(xp_1'+yp_2')^3] + \tfrac{1}{6}xyK_{12}T_3,$$

where

$$K_{12}\sin^2\widehat{12} = \sum k_{11}\xi_{12}^2, \quad T_3 = x(p_2' - p_1'\cos\widehat{12}) + y(p_1' - p_2'\cos\widehat{12}),$$

and $\widehat{12}$ in T_3 denotes the angle XOY at O. The value of the p-parameter at O', when O' is attained from O by a broken geodesic path ending with $Z'O'$, is

$$p_{Z'} + zP_3' + \tfrac{1}{2}z^2P_3'' + \tfrac{1}{6}z^3P_3''',$$

up to the third order of small quantities, where P_3'' and P_3''' must be taken at Z'. Now, as in § 229, it is proved that, at Z' for the direction P_3',

$$P_3'' = p_3'' - [\Gamma_{300}p_3'^2(xp_1'+yp_2')] - \frac{2}{3\Omega}\{xK_1(3, 13) + yK_2(3, 23)\},$$

up to the first order of small quantities, higher powers not being necessary because P_3'' has a factor $\frac{1}{2}z^2$. Also, because of the factor $\frac{1}{6}z^3$, we can take

$$P_3''' = p_3'''.$$

Hence substituting for $p_{Z'}$, P_3', P_3'', P_3''', the value of the p-parameter at O', as given by the path ending in $Z'O'$, is

$$\begin{aligned} p + xp_1' + yp_2' - \tfrac{1}{2}[\Gamma_{11}(xp_1' + yp_2')^2] - \tfrac{1}{6}[\Gamma_{300}(xp_1' + yp_2')^3] + \tfrac{1}{6}xyK_{12}T_3 \\ + zp_3' - z[\Gamma_{11}p_3'(xp_1' + yp_2')] - \tfrac{1}{2}z[\Gamma_{300}p_3'(xp_1' + yp_2')^2] + zP_3 \\ + \tfrac{1}{2}z^2p_3'' - \tfrac{1}{2}z^2[\Gamma_{300}p_3'^2(xp_1' + yp_2')] - \frac{z^2}{3\Omega}\{xK_1(3, 13) + yK_2(3, 23)\} \\ - \tfrac{1}{6}z^3[\Gamma_{300}p_3'^3]. \end{aligned}$$

In this expression, the aggregate contributed by the finite term, the terms of the first order in small quantities, the terms of the second order, and those terms of the third order involving the quantities Γ_{ijk}, for $i+j+k=3$, is

$$p + xp_1' + yp_2' + zp_3' - \tfrac{1}{2}[\Gamma_{11}(xp_1' + yp_2' + zp_3')^2] - \tfrac{1}{6}[\Gamma_{300}(xp_1' + yp_2' + zp_3')^3],$$

a total which is symmetrical as regards the three edges OX, OY, OZ, of the cell. The aggregate of the remaining terms, all of the third order of small quantities,

$$= zP_3 + \tfrac{1}{6}xyK_{12}T_3 - \frac{1}{3\Omega}\{xz^2K_1(3, 13) + yz^2K_1(3, 23)\}:$$

or, if

$$P_3 - \frac{1}{3\Omega}\{xzK_1(3, 13) + yzK_1(3, 23)\} = \bar{P}_3,$$

this third-order aggregate

$$= z\bar{P}_3 + \tfrac{1}{6}xyK_{12}T_3.$$

Similarly, if we take

$$P_1 - \frac{1}{3\Omega}\{yxK_1(1, 21) + zxK_1(1, 31)\} = \bar{P}_1,$$

$$P_2 - \frac{1}{3\Omega}\{zyK_1(2, 32) + xyK_1(2, 12)\} = \bar{P}_2,$$

$$y(p_3' - p_2' \cos \widehat{23}) + z(p_2' - p_3' \cos \widehat{23}) = T_1,$$

$$z(p_1' - p_3' \cos \widehat{31}) + x(p_3' - p_1' \cos \widehat{31}) = T_2,$$

the symmetrical aggregate in the expression for p at O' by a broken geodesic path ending in $X'O'$ is the same as the former symmetrical aggregate, and the third-order aggregate

$$= x\bar{P}_1 + \tfrac{1}{6}yzK_{23}T_1;$$

while, in the expression for p at O' by a broken geodesic path ending in $Y'O'$, the symmetrical aggregate is again the same as the former symmetrical aggregate, and the third-order aggregate is

$$= y\bar{P}_2 + \tfrac{1}{6}zxK_{31}T_2.$$

Hence, when the three formally different values of p thus obtained are equated to one another, we have the two relations

$$x\bar{P}_1+\tfrac{1}{6}yzK_{23}T_1=y\bar{P}_2+\tfrac{1}{6}zxK_{31}T_2=z\bar{P}_3+\tfrac{1}{6}xyK_{12}T_3,$$

the three expressions constituting, at this stage, a single unknown quantity. Let the common value be represented by

$$xyz\bar{P}+\tfrac{1}{6}(yzK_{23}T_1+zxK_{31}T_2+xyK_{12}T_3),$$

where $\bar{P}$ is this single unknown quantity; then we have

$$\left.\begin{array}{l}\bar{P}_1=yz\bar{P}+\tfrac{1}{6}(zK_{31}T_2+yK_{12}T_3)\\ \bar{P}_2=zx\bar{P}+\tfrac{1}{6}(xK_{12}T_3+zK_{23}T_1)\\ \bar{P}_3=xy\bar{P}+\tfrac{1}{6}(yK_{23}T_1+xK_{31}T_2)\end{array}\right\},$$

as analytical inferences from the equal values of the p-parameter at O'.

Similarly, if we take

$$Q_1-\frac{1}{3\Omega}\{yxK_2(1,21)+zxK_2(1,31)\}=\bar{Q}_1,$$

$$Q_2-\frac{1}{3\Omega}\{zyK_2(2,32)+xyK_2(2,12)\}=\bar{Q}_2,$$

$$Q_3-\frac{1}{3\Omega}\{xzK_2(3,13)+yzK_2(3,23)\}=\bar{Q}_3,$$

the equal values of the q-parameter at O', attained by the three paths, lead to the analytical inferences

$$\left.\begin{array}{l}\bar{Q}_1=yz\bar{Q}+\tfrac{1}{6}(zK_{31}U_2+yK_{12}U_3)\\ \bar{Q}_2=zx\bar{Q}+\tfrac{1}{6}(xK_{12}U_3+zK_{23}U_1)\\ \bar{Q}_3=xy\bar{Q}+\tfrac{1}{6}(yK_{23}U_1+xK_{31}U_2)\end{array}\right\},$$

where $\bar{Q}$ is an unknown quantity at this stage, and U_1, U_2, U_3, are the same as T_1, T_2, T_3, with the variables q' in place of the variables p'. And, if we take

$$R_1-\frac{1}{3\Omega}\{yxK_3(1,21)+zxK_3(1,31)\}=\bar{R}_1,$$

$$R_2-\frac{1}{3\Omega}\{zyK_3(2,32)+xyK_3(2,12)\}=\bar{R}_2,$$

$$R_3-\frac{1}{3\Omega}\{xzK_3(3,13)+yzK_3(3,23)\}=\bar{R}_3,$$

the equal values of the r-parameter at O', attained by the three paths, lead to the analytical inferences

$$\left.\begin{array}{l}\bar{R}_1=yz\bar{R}+\tfrac{1}{6}(zK_{31}V_2+yK_{12}V_3)\\ \bar{R}_2=zx\bar{R}+\tfrac{1}{6}(xK_{12}V_3+zK_{23}V_1)\\ \bar{R}_3=xy\bar{R}+\tfrac{1}{6}(yK_{23}V_1+xK_{31}V_2)\end{array}\right\},$$

where $\bar{R}$ is an unknown quantity at this stage, and V_1, V_2, V_3, are the same as T_1, T_2, T_3, with the variables r' instead of the variables p'.

We now return to the residual conditions from the arc-relation at X', at Y', and at Z', respectively. When account is taken of the values of the quantities $K_\iota(\lambda, \mu\nu)$ for the various values of i, and of λ, μ, ν, we have

$$u_1^{(1)}K_1(1, 21)+u_2^{(1)}K_2(1, 21)+u_3^{(1)}K_3(1, 21)$$
$$=\begin{vmatrix} au_1^{(1)}+hu_2^{(1)}+gu_3^{(1)}, & p_1', & k_{11}\xi_{21}+k_{12}\eta_{21}+k_{13}\zeta_{21} \\ hu_1^{(1)}+gu_2^{(1)}+fu_3^{(1)}, & q_1', & k_{12}\xi_{21}+k_{22}\eta_{21}+k_{23}\zeta_{21} \\ gu_1^{(1)}+fu_2^{(1)}+cu_3^{(1)}, & r_1', & k_{13}\xi_{21}+k_{23}\eta_{21}+k_{33}\zeta_{21} \end{vmatrix}.$$

The constituents of the first column are $\Omega p_1'$, $\Omega q_1'$, $\Omega r_1'$, and therefore the determinant on the right-hand side vanishes. In the same way, we find

$$u_1^{(1)}K_1(1, 31)+u_2^{(1)}K_2(1, 31)+u_3^{(1)}K_3(1, 31)=0.$$

Accordingly, we have

$$u_1^{(1)}P_1+u_2^{(1)}Q_1+u_3^{(1)}R_1=u_1^{(1)}\bar{P}_1+u_2^{(1)}\bar{Q}_1+u_3^{(1)}\bar{R}_1\,;$$

and therefore the residual condition from the arc-relation at X' becomes

$$u_1^{(1)}\bar{P}_1+u_2^{(1)}\bar{Q}_1+u_3^{(1)}\bar{R}_1=\tfrac{1}{6}\sum\{k_{11}(y\xi_{12}+z\xi_{13})^2\}.$$

In this form of the residual condition, let the inferred expressions for $\bar{P}_1$, $\bar{Q}_1$, $\bar{R}_1$, in terms of $\bar{P}$, $\bar{Q}$, $\bar{R}$, be substituted. Because

$$u_1^{(1)}T_2+u_2^{(1)}U_2+u_3^{(1)}V_2=z\sin^2\widehat{31},$$
$$u_1^{(1)}T_3+u_2^{(1)}U_3+u_3^{(1)}V_3=y\sin^2\widehat{12},$$

the left-hand side becomes

$$yz\{u_1^{(1)}\bar{P}+u_2^{(1)}\bar{Q}+u_3^{(1)}\bar{R}\}+\tfrac{1}{6}z^2K_{31}\sin^2\widehat{31}+\tfrac{1}{6}y^2K_{12}\sin_2\widehat{12}$$
$$=yz\{u_1^{(1)}\bar{P}+u_2^{(1)}\bar{Q}+u_3^{(1)}\bar{R}\}+\tfrac{1}{6}z^2\sum k_{11}\xi_{13}^2+\tfrac{1}{6}y^2\sum k_{11}\xi_{12}^2,$$

and therefore the residual condition from the arc-relation at X' now has the form

$$u_1^{(1)}\bar{P}+u_2^{(1)}\bar{Q}+u_3^{(1)}\bar{R}=\tfrac{1}{3}\sum k_{11}\xi_{12}\xi_{13}.$$

Similarly, the residual conditions from the arc-relation at Y' and the arc-relation at Z' have the respective forms

$$u_1^{(2)}\bar{P}+u_2^{(2)}\bar{Q}+u_3^{(2)}\bar{R}=\tfrac{1}{3}\sum k_{11}\xi_{23}\xi_{21},$$
$$u_1^{(3)}\bar{P}+u_2^{(3)}\bar{Q}+u_3^{(3)}\bar{R}=\tfrac{1}{3}\sum k_{11}\xi_{31}\xi_{32}.$$

These are the only surviving conditions: and they suffice to determine $\bar{P}$, $\bar{Q}$, $\bar{R}$, and therefore (by implication) the quantities P_ι', Q_ι', R_ι'. All the intrinsic requirements of the region have been satisfied by the values postulated initially for the nine quantities P', Q', R'; and these postulated values are such as to constitute the opposite edges of the cell to be directions geodesically parallel to one another in sets.

In the resolution of the three equations for $\bar{P}$, $\bar{Q}$, $\bar{R}$, the determinant of the coefficients

$$= \begin{vmatrix} u_1^{(1)}, & u_2^{(1)}, & u_3^{(1)} \\ u_1^{(2)}, & u_2^{(2)}, & u_3^{(2)} \\ u_1^{(3)}, & u_2^{(3)}, & u_3^{(3)} \end{vmatrix}$$

$$= \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} \begin{vmatrix} p_1', & q_1', & r_1' \\ p_2', & q_2', & r_2' \\ p_3', & q_3', & r_3' \end{vmatrix}$$

$$= (\Omega \Xi)^{\frac{1}{2}},$$

where

$$\Xi = 1 - \cos^2 \widehat{23} - \cos^2 \widehat{31} - \cos^2 \widehat{12} + 2 \cos \widehat{23} \cos \widehat{31} \cos \widehat{12},$$

the usual magnitude connected with the solid angle at O; also

$$u_2^{(2)} u_3^{(3)} - u_2^{(3)} u_3^{(2)} = a\xi_{23} + h\eta_{23} + g\zeta_{23},$$

with corresponding values for the other minors in the determinant. We write

$$\left.\begin{array}{l} \xi_{23}(\sum k_{11}\xi_{21}\xi_{31}) + \xi_{31}(\sum k_{11}\xi_{32}\xi_{12}) + \xi_{12}(\sum k_{11}\xi_{13}\xi_{23}) = K_\xi \\ \eta_{23}(\sum k_{11}\xi_{21}\xi_{31}) + \eta_{31}(\sum k_{11}\xi_{32}\xi_{12}) + \eta_{12}(\sum k_{11}\xi_{13}\xi_{23}) = K_\eta \\ \zeta_{23}(\sum k_{11}\xi_{21}\xi_{31}) + \zeta_{31}(\sum k_{11}\xi_{32}\xi_{12}) + \zeta_{12}(\sum k_{11}\xi_{13}\xi_{23}) = K_\zeta \end{array}\right\};$$

then we have

$$\left.\begin{array}{l} 3(\Omega\Xi)^{\frac{1}{2}}\bar{P} = aK_\xi + hK_\eta + gK_\zeta \\ 3(\Omega\Xi)^{\frac{1}{2}}\bar{Q} = hK_\xi + bK_\eta + fK_\zeta \\ 3(\Omega\Xi)^{\frac{1}{2}}\bar{R} = gK_\xi + fK_\eta + cK_\zeta \end{array}\right\}.$$

Direction-variables of the edges of the regional cell.

243. When these results are gathered together, the direction-variables of $X'O'$ at X' are P_1', Q_1', R_1', where

$$\begin{aligned} P_1' = p_1' &- [\Gamma_{11} p_1'(yp_2' + zp_3')] - \tfrac{1}{2}[\Gamma_{300} p_1'(yp_2' + zp_3')^2] \\ &+ \frac{1}{3\Omega}\{xyK_1(1, 21) + xzK_1(1, 31)\} + \tfrac{1}{6}(zK_{31}T_2 + yK_{12}T_3) \\ &+ \tfrac{1}{3}yz(aK_\xi + hK_\eta + gK_\zeta), \\ Q_1' = q_1' &- [\Delta_{11} p_1'(yp_2' + zp_3')] - \tfrac{1}{2}[\Delta_{300} p_1'(yp_2' + zp_3')^2] \\ &+ \frac{1}{3\Omega}\{xyK_2(1, 21) + xzK_2(1, 31)\} + \tfrac{1}{6}(zK_{31}U_2 + yK_{12}U_3) \\ &+ \tfrac{1}{3}yz(hK_\xi + bK_\eta + fK_\zeta), \\ R_1' = r_1' &- [\Theta_{11} p_1'(yp_2' + zp_3')] - \tfrac{1}{2}[\Theta_{300} p_1'(yp_2' + zp_3')^2] \\ &+ \frac{1}{3\Omega}\{xyK_3(1, 21) + xzK_3(1, 31)\} + \tfrac{1}{6}(zK_{31}V_2 + yK_{12}V_3) \\ &+ \tfrac{1}{3}yz(gK_\xi + fK_\eta + cK_\zeta), \end{aligned}$$

where

$$\left.\begin{aligned} T_2 &= z(p_1' - p_3' \cos \widehat{31}) + x(p_3' - p_1' \cos \widehat{31}) \\ U_2 &= z(q_1' - q_3' \cos \widehat{31}) + x(q_3' - q_1' \cos \widehat{31}) \\ V_2 &= z(r_1' - r_3' \cos \widehat{31}) + x(r_3' - r_1' \cos \widehat{31}) \end{aligned}\right\},$$

$$\left.\begin{aligned} T_3 &= x(p_2' - p_1' \cos \widehat{12}) + y(p_1' - p_2' \cos \widehat{12}) \\ U_3 &= x(q_2' - q_1' \cos \widehat{12}) + y(q_1' - q_2' \cos \widehat{12}) \\ V_3 &= x(r_2' - r_1' \cos \widehat{12}) + y(r_1' - r_2' \cos \widehat{12}) \end{aligned}\right\}.$$

The direction-variables P_2', Q_2', R_2', of $Y'O'$ at Y', and the direction-variables P_3', Q_3', R_3', of $Z'O'$ at Z', are of forms similar to P_1', Q_1', R_1', and can be derived by the cyclical interchanges of x, p_1', q_1', r_1'; y, p_2', q_2', r_2'; z, p_3', q_3', r_3'; with one another, simultaneously with the appropriate changes of the quantities $K_i(\lambda, \mu\nu)$; $K_{\alpha\beta}$; and of T_i, U_i, V_i. The magnitudes K_ξ, K_η, K_ζ, being symmetrical, remain unaltered by all the sets of interchanges.

The values of the regional parameters at O' are

$$\begin{aligned} &p + tp_0' - \tfrac{1}{2}t^2[\Gamma_{11}p_0'^2] - \tfrac{1}{6}t^3[\Gamma_{300}p_0'^3] \\ &\qquad + \tfrac{1}{6}(yzK_{23}T_1 + zxK_{31}T_2 + xyK_{12}T_3) + \frac{xyz}{3(\Omega\Xi)^{\frac{1}{2}}}(aK_\xi + hK_\eta + gK_\zeta), \\ &q + tq_0' - \tfrac{1}{2}t^2[\Delta_{11}p_0'^2] - \tfrac{1}{6}t^3[\Delta_{300}p_0'^3] \\ &\qquad + \tfrac{1}{6}(yzK_{23}U_1 + zxK_{31}U_2 + xyK_{12}U_3) + \frac{xyz}{3(\Omega\Xi)^{\frac{1}{2}}}(hK_\xi + bK_\eta + fK_\zeta), \\ &r + tr_0' - \tfrac{1}{2}t^2[\Theta_{11}p_0'^2] - \tfrac{1}{6}t^3[\Theta_{300}p_0'^3] \\ &\qquad + \tfrac{1}{6}(yzK_{23}V_1 + zxK_{31}V_2 + xyK_{12}V_3) + \frac{xyz}{3(\Omega\Xi)^{\frac{1}{2}}}(gK_\xi + fK_\eta + cK_\zeta), \end{aligned}$$

where

$$tp_0' = xp_1' + yp_2' + zp_3', \quad tq_0' = xq_1' + yq_2' + zq_3', \quad tr_0' = xr_1' + yr_2' + zr_3',$$

$$t^2 = x^2 + y^2 + z^2 + 2yz \cos \widehat{23} + 2zx \cos \widehat{31} + 2xy \cos \widehat{12}.$$

Ex. 1. Prove that the sphericity of the region in an orientation at O, parallel to the orientation of the small triangle XYZ as x, y, z, diminish towards zero, is equal to

$$\frac{\sum k_{11}(yz\xi_{23} + zx\xi_{31} + xy\xi_{12})^2}{\sum a(yz\xi_{23} + zx\xi_{31} + xy\xi_{12})^2}.$$

Ex. 2. The length of the geodesic diagonal OO' of the cell, and the direction-variables at O of that diagonal, can be derived by means of the values of the regional parameters at O' which have just been obtained.

Let the geodesic length of OO' be denoted by $t + T$, where T will prove to be of the third order of small quantities; and let the direction-variables p', q', r', at O of the geodesic OO' be such that

$$p' = p_0' + P, \quad q' = q_0' + Q, \quad r' = r_0' + R,$$

where P, Q, R, will prove to be of the second order of small quantities, within the foregoing approximations.

In the first place, there is the arc-relation $\sum Ap'^2=1$ at O, so that we have

$$\sum A(p_0'^2+zp_0'P)=1,$$

neglecting P^2 and quantities of like order: that is,

$$u_1^{(0)}P+u_2^{(0)}Q+u_3^{(0)}R=0.$$

The p-parameter at O', estimated at an arc-length $t+T$ from O along the geodesic OO' with p', q', r', as its direction-variables at O is

$$=p+(t+T)p'+\tfrac{1}{2}(t+T)^2p''+\tfrac{1}{6}(t+T)^3p'''.$$

Now
$$\begin{aligned} p'' &= -\sum \Gamma_{11}p'^2 \\ &= -\sum \Gamma_{11}(p_0'^2+zp_0'P+P^2), \end{aligned}$$

and p'' is already multiplied by the magnitude $(t+T)^2$; hence the parts in P and P^2 arising out of p'' can be neglected in comparison with the part tP arising out of tp'. Similarly for the term in p'''; and therefore the foregoing value of the parameter p at O'

$$=p+tp_0'-\tfrac{1}{2}t^2[\Gamma_{11}p_0'^2]-\tfrac{1}{6}t^3[\Gamma_{300}p_0'^3]+tP+Tp_0',$$

neglecting magnitudes of the fourth and higher orders. When this value is equated to the value obtained in the text, we have an equation

$$tP+Tp_0'=\tfrac{1}{6}(yzK_{23}T_1+zxK_{31}T_2+xyK_{12}T_3)+\frac{xyz}{3(\Omega\Xi)^{\frac{1}{2}}}(aK_\xi+hK_\eta+gK_\zeta).$$

The corresponding equations arising from the values at O' of the other two parameters are

$$tQ+Tq_0'=\tfrac{1}{6}(yzK_{23}U_1+zxK_{31}U_2+xyK_{12}U_3)+\frac{xyz}{3(\Omega\Xi)^{\frac{1}{2}}}(hK_\xi+bK_\eta+fK_\zeta),$$

$$tR+Tr_0'=\tfrac{1}{6}(yzK_{23}V_1+zxK_{31}V_2+xyK_{12}V_3)+\frac{xyz}{3(\Omega\Xi)^{\frac{1}{2}}}(gK_\xi+fK_\eta+cK_\zeta).$$

Thus there are four equations for the determination of T; P, Q, R; they verify the statements that P, Q, R, are of the second order of small quantities and that T is of the third order.

We multiply the last three equations by $u_1^{(0)}$, $u_2^{(0)}$ $u_3^{(0)}$, and add the products: as

$$\sum u_1^{(0)}p_0'=1,\quad \sum u_1^{(0)}P=0,$$

the result becomes

$$\begin{aligned} T=\tfrac{1}{6}&\sum\{u_1^{(0)}(yzK_{23}T_1+zxK_{31}T_2+xyK_{12}T_3)\} \\ &+\frac{xyz}{3(\Theta\Xi)^{\frac{1}{2}}}\sum\{u_1^{(0)}(aK_\xi+hK_\eta+gK_\zeta)\}. \end{aligned}$$

For the aggregate of terms in the first line of this expression for T, as

$$tu_1^{(0)}=xu_1^{(1)}+yu_1^{(2)}+zu_1^{(3)},$$

with similar values of $u_2^{(0)}$ and $u_3^{(0)}$, we have

$$\sum tu_1^{(0)}T_1 = \sum \{xu_1^{(1)} + yu_1^{(2)} + zu_1^{(3)}\}\{y(p_3' - p_2' \cos \widehat{23}) + z(p_2' - p_3' \cos \widehat{23})\}$$

$$= 2yz \sin^2 \widehat{23} + xy(\cos \widehat{13} - \cos \widehat{12} \cos \widehat{23}) + xz(\cos \widehat{12} - \cos \widehat{13} \cos \widehat{23}) = W_1,$$

$$\sum tu_1^{(0)}T_2 = 2zx \sin^2 \widehat{31} + yz(\cos \widehat{12} - \cos \widehat{13} \cos \widehat{23}) + xy(\cos \widehat{23} - \cos \widehat{12} \cos \widehat{13}) = W_2,$$

$$\sum tu_1^{(0)}T_3 = 2xy \sin^2 \widehat{12} + zx(\cos \widehat{23} - \cos \widehat{12} \cos \widehat{13}) + yz(\cos \widehat{13} - \cos \widehat{12} \cos \widehat{23}) = W_3;$$

and therefore the first line in the expression for T

$$= \tfrac{1}{6}t(yzK_{23}W_1 + zxK_{31}W_2 + xyK_{12}W_3).$$

For the aggregate of terms in the second line, we have

$$\sum u_1^{(0)}(aK_\xi + hK_\eta + gK_\zeta) = \Omega(p_0'K_\xi + q_0'K_\eta + r_0'K_\zeta);$$

also

$$t(p_0'K_\xi + q_0'K_\eta + r_0'K_\zeta) = \begin{vmatrix} p_1', & p_2', & p_3' \\ q_1', & q_2', & q_3' \\ r_1', & r_2', & r_3' \end{vmatrix} \sum K_{11}(x\xi_{12}\xi_{13} + y\xi_{23}\xi_{21} + z\xi_{31}\xi_{32})$$

$$= \left(\frac{\Xi}{\Omega}\right)^{\frac{1}{2}} \sum K_{11}(x\xi_{12}\xi_{13} + y\xi_{23}\xi_{21} + z\xi_{31}\xi_{32}).$$

Thus the second line in the expression for T

$$= \frac{xyz}{3t} \sum K_{11}(x\xi_{12}\xi_{13} + y\xi_{23}\xi_{21} + z\xi_{31}\xi_{32}).$$

Accordingly,

$$tT = \tfrac{1}{6}(W_1yzK_{23} + W_2zxK_{31} + W_3xyK_{12})$$
$$+ \tfrac{1}{3}xyz\sum K_{11}(x\xi_{12}\xi_{13} + y\xi_{23}\xi_{21} + z\xi_{31}\xi_{32});$$

and the length of the geodesic diagonal OO'

$$= t + T,$$

where

$$t^2 = x^2 + y^2 + z^2 + 2yz \cos \widehat{23} + 2zx \cos \widehat{31} + 2xy \cos \widehat{12}.$$

The values of P, Q, R, are inferred by substituting this value of T in the three equations involving those magnitudes.

Ex. 3. In the cell thus constructed, each face is a parallelogram of the type considered in § 233. Thus in the face $OXZ'YO$, the angles OXZ' and OYZ' are equal, up to the second order of small quantities, the common value being (§ 225)

$$\pi - \widehat{12} + \tfrac{1}{2}K_{12}xy \sin \widehat{12};$$

and in the face $OXY'ZO$, the angles OXY' and OZY' are equal, to the same order, their common value being $\pi - \widehat{13} + \frac{1}{2}K_{13}xz \sin \widehat{13}$, where $\widehat{12}$ denotes the angle YOX and $\widehat{13}$ denotes the angle ZOX.

Required the magnitude of the difference between the angles $Y'XZ' - \widehat{23}$, where $\widehat{23}$ denotes the angle YOZ, so as to complete the determination of the three plane angles which are the constituents of the solid angle at X made by the three regional

diagonals, one being the continuation of OX and the other two parallels to OY and to OZ respectively.

Ex. 4. Let the continuations of the geodesics $X'O'$, $Y'O'$, $Z'O'$, through O' be $O'\overline{X}$, $O'\overline{Y}$, $O'\overline{Z}$. The direction-variables at O' of these three continuations are known, as are the parameters at O': required the magnitudes

$$\overline{Y}O'\overline{Z} - \widehat{23},\quad \overline{Z}O'\overline{Y} - \widehat{31},\quad \overline{X}O'\overline{Y} + \widehat{12},$$

connected with the plane angles of the solid angle at O'.

CHAPTER XX

Parametric Curves in a Region

244. A curve in a region can be most simply represented by two relations

$$e(p, q, r)=0, \quad \epsilon(p, q, r)=0,$$

among the parameters of the region. Each relation, by itself, represents a parametric surface wholly contained within the region; and the two relations, combined, represent the curve of intersection of the surfaces. At any point of the curve, each of the associated surfaces has its own tangent plane; and the intersection of the two tangent planes is the tangent line of the curve. In the same direction as that tangent line, and therefore touching the curve, there are three geodesics to which the curve can be specially related: the superficial geodesic on the e-surface, the superficial geodesic on the ϵ-surface, and the regional geodesic. The orthogonal frames of the curve, in relation to flexure in the separate surfaces, to flexure in the region, and to its curvatures in the plenary space, are interlocked with the orthogonal frames of the two superficial geodesics in the region and in space, and with the orthogonal frame of the regional geodesic.

Consider the parts of the orthogonal frames of the two superficial geodesics which lie within the tangent flat of the region. The special lines are, (i), the common tangent in the direction p', q', r', such that

$$e_1p'+e_2q'+e_3r'=0, \quad \epsilon_1p'+\epsilon_2q'+\epsilon_3r'=0;$$

(ii), for each surface, the line lying in its own tangent plane which is perpendicular to this common tangent line, the two lines thus selected being the two binormals of the respective superficial geodesics; and (iii), the regional normals to the two surfaces. The element of regional arc normal to the e-surface will be denoted by dn, and the element of regional arc normal to the ϵ-surface will be denoted by $d\nu$; and the respective normal dilatations (§ 207) of the surfaces will be denoted by e_n, ϵ_ν. Then we have

$$\left.\begin{aligned} \Omega e_n\frac{dp}{dn}&=ae_1+he_2+ge_3 \\ \Omega e_n\frac{dq}{dn}&=he_1+be_2+fe_3 \\ \Omega e_n\frac{dr}{dn}&=ge_1+fe_2+ce_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} \Omega \epsilon_\nu\frac{dp}{d\nu}&=a\epsilon_1+h\epsilon_2+g\epsilon_3 \\ \Omega \epsilon_\nu\frac{dq}{d\nu}&=h\epsilon_1+b\epsilon_2+f\epsilon_3 \\ \Omega \epsilon_\nu\frac{dr}{d\nu}&=g\epsilon_1+f\epsilon_2+c\epsilon_3 \end{aligned}\right\},$$

$$\Omega e_n{}^2=\sum ae_1{}^2, \quad \Omega \epsilon_\nu{}^2=\sum a\epsilon_1{}^2.$$

Let ω denote the angle of intersection of the two surfaces at the place p, q, r, taken to be the angle between the positively-drawn regional normals to the two surfaces, and let it be measured positively from the normal of the e-surface towards the normal of the ϵ-surface, as in Figure 28; then

$$\begin{aligned}\cos\omega &= \sum A\frac{dp}{dn}\frac{dp}{d\nu}\\ &= \frac{1}{\Omega^2 e_n\epsilon_\nu}\sum\{A(ae_1+he_2+ge_3)(a\epsilon_1+h\epsilon_2+g\epsilon_3)\}\\ &= \frac{1}{\Omega e_n\epsilon_\nu}\sum ae_1\epsilon_1,\end{aligned}$$

so that

$$\sum ae_1\epsilon_1 = \Omega e_n\epsilon_\nu\cos\omega.$$

It follows that

$$\begin{aligned}(\Omega e_n\epsilon_\nu)^2\sin^2\omega &= (\sum ae_1{}^2)(\sum a\epsilon_1{}^2) - (\sum ae_1\epsilon_1)^2\\ &= \Omega\sum\{A(e_2\epsilon_3-e_3\epsilon_2)^2\},\end{aligned}$$

so that

$$\sum\{A(e_2\epsilon_3-e_3\epsilon_2)^2\} = \Omega\epsilon_n{}^2\epsilon_\nu{}^2\sin^2\omega.$$

The variables p', q', r', of direction of the curve are given by

$$e_1p'+e_2q'+e_3r'=0,\quad \epsilon_1p'+\epsilon_2q'+\epsilon_3r'=0,$$

so that

$$\frac{p'}{e_2\epsilon_3-e_3\epsilon_2} = \frac{q'}{e_3\epsilon_1-e_1\epsilon_3} = \frac{r'}{e_1\epsilon_2-e_2\epsilon_1} = \frac{1}{\Lambda},$$

where

$$\begin{aligned}\Lambda^2 &= \sum\{A(p'\Lambda)^2\}\\ &= \sum\{A(e_2\epsilon_3-e_3\epsilon_2)^2\} = \Omega e_n{}^2\epsilon_\nu{}^2\sin^2\omega:\end{aligned}$$

that is,

$$\Lambda = \Omega^{\frac{1}{2}}e_n\epsilon_\nu\sin\omega.$$

Let λ_e and λ_ϵ denote the typical direction-cosines of the respective superficial binormals, being the lines at right angles to the tangent in the respective tangent-planes TOB_e and TOB_ϵ: then

$$\Omega^{\frac{1}{2}}e_n\lambda_e = \begin{vmatrix} y_1, & y_2, & y_3\\ u_1, & u_2, & u_3\\ e_1, & e_2, & e_3\end{vmatrix},\qquad \Omega^{\frac{1}{2}}\epsilon_\nu\lambda_\epsilon = \begin{vmatrix} y_1, & y_2, & y_3\\ u_1, & u_2, & u_3\\ \epsilon_1, & \epsilon_2, & \epsilon_3\end{vmatrix},$$

the direction-variables in the magnitudes u_1, u_2, u_3, being those of the curve. Also

$$\sum\lambda_e\lambda_\epsilon = \cos\omega,$$

as is easily verified.

Again, let ON_e and ON_ϵ be the regional normals to the two surfaces, OT being the tangent to the curve of intersection; also let OB_e and OB_ϵ be the respective superficial binormals. Then the four lines ON_e, ON_ϵ, OB_e, OB_ϵ, in the tangent flat of the region, are at right angles to the line OT in the flat; and they therefore lie in a regional plane at right angles to OT. The direction-cosines of sets of three of these four lines must therefore be linearly connected; and the geometry of the diagram leads to these linear relations in the form

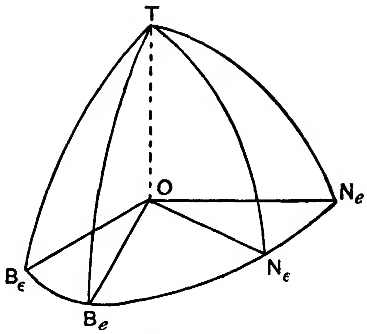

FIG. 28.

$$\left.\begin{aligned}\frac{dy}{dn}\sin\omega &= \lambda_e\cos\omega-\lambda_\epsilon\\ \frac{dy}{d\nu}\sin\omega &= \lambda_e-\lambda_\epsilon\cos\omega\end{aligned}\right\},\qquad \left.\begin{aligned}\lambda_e\sin\omega &= \frac{dy}{d\nu}-\frac{dy}{dn}\cos\omega\\ \lambda_\epsilon\sin\omega &= \frac{dy}{d\nu}\cos\omega-\frac{dy}{dn}\end{aligned}\right\},$$

where

$$N_eON_\epsilon=\omega=B_eOB_\epsilon,$$

while

$$\frac{dy}{dn}=y_1\frac{dp}{dn}+y_2\frac{dq}{dn}+y_3\frac{dr}{dn},\quad \frac{dy}{d\nu}=y_1\frac{dp}{d\nu}+y_2\frac{dq}{d\nu}+y_3\frac{dr}{d\nu},$$

the direction-variables of the respective regional normals ON_e and ON_ϵ being as already stated.

Ex. 1. Obtain the following relations, which will be found useful in a later investigation:

$$\left.\begin{aligned}(u_2e_3-u_3e_2)\sin\omega &= \Omega^{\frac{1}{2}}e_n\left(\frac{dp}{d\nu}-\frac{dp}{dn}\cos\omega\right)\\ (u_3e_1-u_1e_3)\sin\omega &= \Omega^{\frac{1}{2}}e_n\left(\frac{dq}{d\nu}-\frac{dq}{dn}\cos\omega\right)\\ (u_1e_2-u_2e_1)\sin\omega &= \Omega^{\frac{1}{2}}e_n\left(\frac{dr}{d\nu}-\frac{dr}{dn}\cos\omega\right)\end{aligned}\right\},$$

$$\left.\begin{aligned}(u_2\epsilon_3-u_3\epsilon_2)\sin\omega &= \Omega^{\frac{1}{2}}\epsilon_\nu\left(-\frac{dp}{dn}+\frac{dp}{d\nu}\cos\omega\right)\\ (u_3\epsilon_1-u_1\epsilon_3)\sin\omega &= \Omega^{\frac{1}{2}}\epsilon_\nu\left(-\frac{dq}{dn}+\frac{dq}{d\nu}\cos\omega\right)\\ (u_1\epsilon_2-u_2\epsilon_1)\sin\omega &= \Omega^{\frac{1}{2}}\epsilon_\nu\left(-\frac{dr}{dn}+\frac{dr}{d\nu}\cos\omega\right)\end{aligned}\right\}.$$

An analytical interchange of the surfaces e and ϵ, with the corresponding interchange of the sets of the direction-variables of their regional normals, requires a reversal of the direction in which the inclination ω is measured and therefore compels a change in the sign of ω: hence the skew symmetry between the two groups of results.

In connection with these results, it is convenient to re-state the results

$$\left.\begin{aligned} e_2\epsilon_3 - e_3\epsilon_2 &= p'\,\Omega^{\frac{1}{2}} e_n\epsilon_\nu \sin\omega \\ e_3\epsilon_1 - e_1\epsilon_3 &= q'\,\Omega^{\frac{1}{2}} e_n\epsilon_\nu \sin\omega \\ e_1\epsilon_2 - e_2\epsilon_1 &= r'\,\Omega^{\frac{1}{2}} e_n\epsilon_\nu \sin\omega \end{aligned}\right\},$$

which evince the same skew symmetry as regards the two regional surfaces.

Ex. 2. To obtain the value of $\dfrac{d\omega}{ds}$, we proceed from

$$e_n\epsilon_\nu \cos\omega = \sum \frac{a}{\Omega} e_1\epsilon_1,$$

so that

$$\left(\frac{de_n}{ds}\epsilon_\nu + \frac{d\epsilon_\nu}{ds}e_n\right)\cos\omega - \frac{d\omega}{ds}e_n\epsilon_\nu \sin\omega$$

$$= \frac{1}{\Omega}\sum\left[\left(\sum ae_1\right)(\epsilon_{11}p' + \epsilon_{12}q' + \epsilon_{13}r') + \left(\sum a\epsilon_1\right)(e_{11}p' + e_{12}q' + e_{13}r')\right] + \sum\left\{e_1\epsilon_1\frac{d}{ds}\left(\frac{a}{\Omega}\right)\right\}$$

$$= \frac{1}{\Omega}\left[\left(\sum ae_1\right)\bar\epsilon_1 + \left(\sum he_1\right)\bar\epsilon_2 + \left(\sum ge_1\right)\bar\epsilon_3 + \left(\sum a\epsilon_1\right)\bar e_1 + \left(\sum h\epsilon_1\right)\bar e_2 + \left(\sum g\epsilon_1\right)\bar e_3\right],$$

on reduction, where

$$\bar e_i = \sum_k (e_{ik} - e_1\Gamma_{ik} - e_2\Delta_{ik} - e_3\Theta_{ik})x_k',$$

$$\bar\epsilon_i = \sum_k (\epsilon_{ik} - \epsilon_1\Gamma_{ik} - \epsilon_2\Delta_{ik} - \epsilon_3\Theta_{ik})x_k',$$

with the usual convention $x_1, x_2, x_3, = p, q, r$. Also, as in § 207,

$$\frac{de_n}{ds} = \bar e_1\frac{dp}{dn} + \bar e_2\frac{dq}{dn} + \bar e_3\frac{dr}{dn},$$

$$\frac{d\epsilon_\nu}{ds} = \bar\epsilon_1\frac{dp}{d\nu} + \bar\epsilon_2\frac{dq}{d\nu} + \bar\epsilon_3\frac{dr}{d\nu};$$

and therefore

$$\begin{aligned}\frac{d\omega}{ds}e_n\epsilon_\nu\sin\omega = {} & e_n\left\{\bar\epsilon_1\left(\frac{dp}{d\nu}\cos\omega - \frac{dp}{dn}\right) + \bar\epsilon_2\left(\frac{dq}{d\nu}\cos\omega - \frac{dq}{dn}\right) + \bar\epsilon_3\left(\frac{dr}{d\nu}\cos\omega - \frac{dr}{dn}\right)\right\} \\ & + \epsilon_\nu\left\{\bar e_1\left(\frac{dp}{dn}\cos\omega - \frac{dp}{d\nu}\right) + \bar e_2\left(\frac{dp}{dn}\cos\omega - \frac{dp}{d\nu}\right) + \bar e_3\left(\frac{dr}{dn}\cos\omega - \frac{dr}{d\nu}\right)\right\}.\end{aligned}$$

When the values of the coefficients of $\bar\epsilon_1, \bar\epsilon_2, \bar\epsilon_3, \bar e_1, \bar e_2, \bar e_3$, as given in the preceding example are inserted, and after a slight reduction, we find

$$\Omega^{\frac{1}{2}}\frac{d\omega}{ds} = \frac{1}{e_n^2}\begin{vmatrix} u_1, & u_2, & u_3 \\ \bar e_1, & \bar e_2, & \bar e_3 \\ e_1, & e_2, & e_3 \end{vmatrix} - \frac{1}{\epsilon_\nu^2}\begin{vmatrix} u_1, & u_2, & u_3 \\ \bar\epsilon_1, & \bar\epsilon_2, & \bar\epsilon_3 \\ \epsilon_1, & \epsilon_2, & \epsilon_3 \end{vmatrix};$$

and therefore, by the results in § 200,

$$\frac{d\omega}{ds} = \frac{1}{\sigma_e} - \frac{1}{\sigma_\epsilon},$$

where σ_e and σ_ϵ are the radii of regional torsion of the superficial geodesics on the e-surface and the ϵ-surface respectively.

Ex. 3. The quantities y', $\dfrac{dy}{dn}$, $\dfrac{dy}{d\nu}$, are typical direction-cosines of three non-complanar lines in the tangent flat which can be taken as its leading lines. They are equivalent, as leading lines, to the directions of the parametric curves at O; and the typical direction-cosines of each set are linearly expressible in terms of those of the other set. We therefore have relations of the form

$$y_1 = Iy' + P\frac{dy}{dn} + Q\frac{dy}{d\nu}.$$

To determine I, P, Q, we have

$$I = \sum y_1 y' = u_1;$$

$$P + Q\cos\omega = \sum y_1 \frac{dy}{dn} = \frac{e_1}{e_n};$$

$$P\cos\omega + Q = \sum y_1 \frac{dy}{d\nu} = \frac{\epsilon_1}{\epsilon_\nu};$$

and therefore

$$y_1 = y'u_1 + \frac{1}{\sin^2\omega}\left[\frac{dy}{dn}\left(\frac{e_1}{e_n} - \frac{\epsilon_1}{\epsilon_\nu}\cos\omega\right) + \frac{dy}{d\nu}\left(\frac{\epsilon_1}{\epsilon_\nu} - \frac{e_1}{e_n}\cos\omega\right)\right].$$

Similarly

$$y_2 = y'u_2 + \frac{1}{\sin^2\omega}\left[\frac{dy}{dn}\left(\frac{e_2}{e_n} - \frac{\epsilon_2}{\epsilon_\nu}\cos\omega\right) + \frac{dy}{d\nu}\left(\frac{\epsilon_2}{\epsilon_\nu} - \frac{e_2}{e_n}\cos\omega\right)\right],$$

$$y_3 = y'u_3 + \frac{1}{\sin^2\omega}\left[\frac{dy}{dn}\left(\frac{e_3}{e_n} - \frac{\epsilon_3}{\epsilon_\nu}\cos\omega\right) + \frac{dy}{d\nu}\left(\frac{\epsilon_3}{\epsilon_\nu} - \frac{e_3}{e_n}\cos\omega\right)\right].$$

It follows that any expression which is linear and homogeneous in y_1, y_2, y_3, can be transformed into a modified expression which is linear and homogeneous in y', $\dfrac{dy}{dn}$, $\dfrac{dy}{d\nu}$.

Values of p'', q'', r'', along the curve.

245. These relations are useful in evaluating the second arc-derivatives of p, q, r, (to be denoted by p_c'', q_c'', r_c'') along the curve of intersection of the e-surface and the ϵ-surface.

Because the curve lies on the e-surface, we have

$$e_1 p_c'' + e_2 q_c'' + e_3 r_c'' = -\sum e_{11} p'^2,$$

and therefore

$$e_1(p_c'' - p'') + e_2(q_c'' - q'') + e_3(r_c'' - r'') = -\sum (e_{11} - e_1\Gamma_{11} - e_2\Delta_{11} - e_3\Theta_{11})p'^2 = \frac{e_n}{\gamma_e},$$

where $1/\gamma_e$ is the regional flexure of the superficial geodesic on the e-surface touching the curve. Similarly,

$$\epsilon_1(p_c'' - p'') + \epsilon_2(q_c'' - q'') + \epsilon_3(r_c'' - r'') = \frac{\epsilon_\nu}{\gamma_\epsilon},$$

where $1/\gamma_\epsilon$ is the regional flexure of the superficial geodesic on the ϵ-surface touching the curve. Also, because the arc lies in the region so that the relation $\sum Ap'^2=1$ is satisfied, we have

$$u_1(p_c''-p'')+u_2(q_c''-q'')+u_3(r_c''-r'')=0,$$

with the customary significance for u_1, $=Ap'+Hq'+Gr'$, and likewise for u_2 and u_3.

Thus there are three equations for $p_c''-p''$, $q_c''-q''$, $r_c''-r''$. The determinant of the coefficients of these quantities

$$\begin{aligned}
&= \begin{vmatrix} e_1, & e_2, & e_3 \\ \epsilon_1, & \epsilon_2, & \epsilon_3 \\ u_1, & u_2, & u_3 \end{vmatrix} \\
&= (u_1p'+u_2q'+u_3r')\,\Omega^{\frac{1}{2}}e_n\epsilon_\nu \sin\omega \\
&= \Omega^{\frac{1}{2}}e_n\epsilon_\nu \sin\omega.
\end{aligned}$$

Consequently,

$$(p_c''-p'')\,\Omega^{\frac{1}{2}}e_n\epsilon_\nu \sin\omega = \begin{vmatrix} \dfrac{e_n}{\gamma_e}, & e_2, & e_3 \\ \dfrac{\epsilon_\nu}{\gamma_\epsilon}, & \epsilon_2, & \epsilon_3 \\ 0\,, & u_2, & u_3 \end{vmatrix}$$

$$=\frac{e_n}{\gamma_e}\frac{\Omega^{\frac{1}{2}}\epsilon_\nu}{\sin\omega}\left(\frac{dp}{dn}-\frac{dp}{d\nu}\cos\omega\right)+\frac{\epsilon_\nu}{\gamma_\epsilon}\frac{\Omega^{\frac{1}{2}}e_n}{\sin\omega}\left(\frac{dp}{d\nu}-\frac{dp}{dn}\cos\omega\right);$$

and therefore

$$p_c''-p''=\frac{1}{\sin^2\omega}\left\{\frac{dp}{dn}\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)+\frac{dp}{d\nu}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\right\}.$$

Similarly we find

$$q_c''-q''=\frac{1}{\sin^2\omega}\left\{\frac{dq}{dn}\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)+\frac{dq}{d\nu}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\right\},$$

$$r_c''-r''=\frac{1}{\sin^2\omega}\left\{\frac{dr}{dn}\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)+\frac{dr}{d\nu}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\right\}.$$

These are the values for p'', q'', r'', along the curve which is the intersection of the e-surface and the ϵ-surface. For the superficial geodesic on the e-surface touching this curve, we have

$$\left.\begin{aligned}
p_e''-p''&=\frac{1}{\gamma_e}\frac{dp}{dn}\\
q_e''-q''&=\frac{1}{\gamma_e}\frac{dq}{dn}\\
r_e''-r''&=\frac{1}{\gamma_e}\frac{dr}{dn}
\end{aligned}\right\};$$

and for the superficial geodesic on the ϵ-surface touching the curve, we have

$$\left.\begin{aligned} p_\epsilon''-p''&=\frac{1}{\gamma_\epsilon}\frac{dp}{dv} \\ q_\epsilon''-q''&=\frac{1}{\gamma_\epsilon}\frac{dq}{dv} \\ r_\epsilon''-r''&=\frac{1}{\gamma_\epsilon}\frac{dr}{dv} \end{aligned}\right\}.$$

The quantities p_e'', q_e'', r_e'', and p_ϵ'', q_ϵ'', r_ϵ'', belong to the geodesics on the e-surface and the ϵ-surface respectively, while the quantities p'', q'', r'', belong to the regional geodesic, all these geodesics touching the curve and touching one another. (The curve of intersection can be regarded as its own geodesic.)

Circular curvature of the curve.

246. We now can obtain the circular curvature and the direction-cosines of the prime normal of the curve of intersection of the surfaces.

Along the curve, we have, for a typical variation,

$$\begin{aligned} y_c''&=y_1p_c''+y_2q_c''+y_3r_c''+\sum y_{11}p'^2 \\ &=y_1(p_c''-p'')+y_2(q_c''-q'')+y_3(r_c''-r'')+\sum \eta_{11}p'^2. \end{aligned}$$

Let $1/\rho_c$ denote the circular curvature of the curve, and let Y_c denote the typical direction-cosine associated with the typical space-variable y, so that

$$Y_c=\rho_c y_c''.$$

Now, from the values obtained for p_c'', q_c'', r_c'',

$$\begin{aligned} &y_1(p_c''-p'')+y_2(q_c''-q'')+y_3(r_c''-r'') \\ &\quad=\frac{1}{\sin^2\omega}\left\{\frac{dy}{dn}\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)+\frac{dy}{dv}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\right\} \\ &\quad=\frac{1}{\sin^2\omega}\left\{\frac{1}{\gamma_e}\left(\frac{dy}{dn}-\frac{dy}{dv}\cos\omega\right)+\frac{1}{\gamma_\epsilon}\left(\frac{dy}{dv}-\frac{dy}{dn}\cos\omega\right)\right\} \\ &\quad=\left(\frac{\lambda_e}{\gamma_\epsilon}-\frac{\lambda_\epsilon}{\gamma_e}\right)\frac{1}{\sin\omega}, \end{aligned}$$

by the expressions (§ 244) for λ_e and λ_ϵ. The foregoing equation for y_c'' thus becomes

$$\frac{Y_c}{\rho_c}=\frac{Y}{\rho}+\left(\frac{\lambda_e}{\gamma_\epsilon}-\frac{\lambda_\epsilon}{\gamma_e}\right)\frac{1}{\sin\omega},$$

which accordingly is the typical equation for the circular curvature and the direction-cosines of the prime normal of the curve.

For the magnitude of the circular curvature, we have

$$\sum Y\lambda_e=0,\quad \sum Y\lambda_\epsilon=0,\quad \sum \lambda_e\lambda_\epsilon=\cos\omega\,;$$

and therefore

$$\frac{1}{\rho_c^2}=\frac{1}{\rho^2}+\left(\frac{1}{\gamma_\epsilon^2}-\frac{2\cos\omega}{\gamma_\epsilon\gamma_e}+\frac{1}{\gamma_e^2}\right)\frac{1}{\sin^2\omega}.$$

The equation for the typical direction-cosine can also be written in the equivalent forms

$$\begin{vmatrix} \dfrac{Y_c}{\rho_c}-\dfrac{Y}{\rho}, & \dfrac{dy}{dn}, & \dfrac{dy}{dv} \\ \dfrac{1}{\gamma_e}, & 1, & \cos\omega \\ \dfrac{1}{\gamma_\epsilon}, & \cos\omega, & 1 \end{vmatrix}=0, \quad \begin{vmatrix} \dfrac{Y_c}{\rho_c}-\dfrac{Y}{\rho}, & \dfrac{Y_e}{\rho_e}-\dfrac{Y}{\rho}, & \dfrac{Y_\epsilon}{\rho_\epsilon}-\dfrac{Y}{\rho} \\ \dfrac{1}{\gamma_e}, & \dfrac{1}{\gamma_e}, & \dfrac{\cos\omega}{\gamma_\epsilon} \\ \dfrac{1}{\gamma_\epsilon}, & \dfrac{\cos\omega}{\gamma_e}, & \dfrac{1}{\gamma_\epsilon} \end{vmatrix}=0,$$

with the significance for Y_e, Y_ϵ, ρ_e, ρ_ϵ, indicated below.

Ex. The tangent to the curve can be taken in the form

$$\bar{y}-y=\kappa y_1+\lambda y_2+\mu y_3,$$

where the parameters κ, λ, μ, satisfy the two conditions

$$\kappa e_1+\lambda e_2+\mu e_3=0, \quad \kappa\epsilon_1+\lambda\epsilon_2+\mu\epsilon_3=0.$$

Let Π be the perpendicular drawn to the tangent from a consecutive point of the curve at a small arc-distance δ from O; and let $\bar{Y}$ denote the typical direction-cosine of this perpendicular. Then, using the method of §§ 166, 194 to determine this perpendicular and its direction, obtain the equation

$$\bar{Y}\Pi=\tfrac{1}{2}\delta^2\left\{\frac{Y}{\rho}+\left(\frac{\lambda_e}{\gamma_\epsilon}-\frac{\lambda_\epsilon}{\gamma_e}\right)\frac{1}{\sin\omega}\right\},$$

accurate up to δ^2 inclusive, leading to the result stated in the text.

There is a geometrical construction for the direction of the prime normal of the curve of intersection of two parametric surfaces in a region.

Let the circular curvature and the typical direction-cosine of the superficial geodesic on $e=0$, touching the curve of intersection, be denoted by $1/\rho_e$ and Y_e; and let the corresponding magnitudes for the superficial geodesic on $\epsilon=0$, in the same direction, be denoted by $1/\rho_\epsilon$ and Y_ϵ; then we have (§ 195) the respective relations

$$\frac{Y_e}{\rho_e}=\frac{Y}{\rho}+\frac{1}{\gamma_e}\frac{dy}{dn}, \quad \frac{Y_\epsilon}{\rho_\epsilon}=\frac{Y}{\rho}+\frac{1}{\gamma_\epsilon}\frac{dy}{dv}.$$

We take the regional normals ON_e and ON_ϵ at O as in § 244, and the two superficial binormals OB_e and OB_ϵ, all these four lines lying in one plane—the plane which, in the tangent flat of the region, is at right angles to the tangent line of the curve. With this plane, we associate the prime normal OY of the regional geodesic; and

we thus have another flat. In this new flat, let a sphere, centre O, be drawn; and consider its intersection by the various lines. The points N_e, N_ϵ, B_e, B_ϵ, lie on a great circle, of which Y is the pole.

The direction of the radius of circular curvature of the superficial geodesic on the e-surface lies in a plane through OY and ON_e, by § 150; accordingly, it intersects the sphere in a point on the quadrantal arc YN_e, and we take OC_e to be the direction. Similarly, the direction of the radius of circular curvature of the superficial geodesic on the ϵ-surface lies in the plane through OY and ON_ϵ; we take OC_ϵ to be the direction.

Again, the curve lies on the e-surface; OC_e is the prime normal of the geodesic on this surface in the same direction as the curve; and the line, perpendicular to this direction and in the tangent plane of the surface, is OB_e. Consequently, the direction of the prime normal of the curve lies in the plane determined by OC_e and OB_e, and it therefore meets the sphere in some point on the great circle B_eC_e.

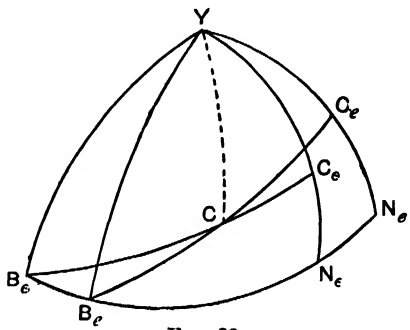

FIG. 29.

Similarly, because it lies on the ϵ-surface, the prime normal of the curve meets the sphere in some point on the great circle $B_\epsilon C_\epsilon$. Let those two great circles intersect in C; then OC is the direction of the prime normal of the curve, and the prime normal itself is the intersection of the planes B_eOC_e and $B_\epsilon OC_\epsilon$.

As regards the spherical trigonometry of Fig. 29, we have

$$N_eN_\epsilon=\omega=B_eB_\epsilon;$$

the point B_e is the pole of the great circle N_eC_eY, and B_ϵ is the pole of $N_\epsilon C_\epsilon Y$. Let $YC_e=\alpha$, $YC_\epsilon=\beta$, so that

$$\left.\begin{aligned}\frac{\cos\alpha}{\rho_e}&=\frac{1}{\rho}\\ \frac{\sin\alpha}{\rho_e}&=\frac{1}{\gamma_e}\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{\cos\beta}{\rho_\epsilon}&=\frac{1}{\rho}\\ \frac{\sin\beta}{\rho_\epsilon}&=\frac{1}{\gamma_\epsilon}\end{aligned}\right\}.$$

Let $CY=\phi$, $CYC_e=u$, so that $CYC_\epsilon=u-\omega$; then as the angles at C_e and C_ϵ are right angles, we have

$$\frac{\tan\alpha}{\tan\phi}=\cos u,\quad \frac{\tan\beta}{\tan\phi}=\cos(u-\omega).$$

Eliminating u, we find

$$\sin^2\omega\tan^2\phi=\tan^2\alpha-2\tan\alpha\tan\beta\cos\omega+\tan^2\beta,$$

and therefore

$$\frac{\tan^2\phi}{\rho^2}=\left(\frac{1}{\gamma_e^2}-\frac{2\cos\omega}{\gamma_e\gamma_\epsilon}+\frac{1}{\gamma_\epsilon^2}\right)\frac{1}{\sin^2\omega}.$$

But

$$\frac{1}{\rho_c^2}=\frac{1}{\rho^2}+\left(\frac{1}{\gamma_e^2}-\frac{2\cos\omega}{\gamma_e\gamma_\epsilon}+\frac{1}{\gamma_\epsilon^2}\right)\frac{1}{\sin^2\omega},$$

and therefore

$$\frac{\cos\phi}{\rho_c}=\frac{1}{\rho}.$$

Again, in the right-angled triangle YC_eC, we have

$$\cos CC_e \cos YC_e=\cos CY=\cos\phi,$$

that is,

$$\frac{\rho_e}{\rho}\cos CC_e=\cos\phi=\frac{\rho_c}{\rho},$$

and therefore

$$\frac{\cos CC_e}{\rho_c}=\frac{1}{\rho_e}.$$

Similarly, we find

$$\frac{\cos CC_\epsilon}{\rho_c}=\frac{1}{\rho_\epsilon}.$$

Thus we have

$$\rho\cos CY=\rho_e\cos CC_e=\rho_\epsilon\cos CC_\epsilon=\rho_c,$$

three equations which assign the angular distances of C, the intersection of the two great circles B_eC_e and $B_\epsilon C_\epsilon$, from the centres of circular curvature of the three several geodesics touching the curve.

Next, let C_0 be the point on the sphere where the sphere is met by the prime normal of the curve, with a typical direction-cosine given by the equation

$$\frac{Y_c}{\rho_c}=\frac{Y}{\rho}+\left(\frac{\lambda_e}{\gamma_\epsilon}-\frac{\lambda_\epsilon}{\gamma_e}\right)\frac{1}{\sin\omega}.$$

The angular distance YC_0 is given by

$$\cos YC_0=\sum YY_c.$$

Now, with the relations expressing λ_e and λ_ϵ in terms of $\frac{dy}{dn}$ and $\frac{dy}{d\nu}$ in § 244, we have

$$\sum Y\lambda_e=0,\quad \sum Y\lambda_\epsilon=0;$$

and therefore

$$\cos YC_0=\frac{\rho_c}{\rho}=\cos YC.$$

Again, the angular distance C_0C_e is given by

$$\cos C_0C_e=\sum Y_cY_e.$$

Also

$$\frac{Y_e}{\rho_e}=\frac{Y}{\rho}+\frac{1}{\gamma_e}\frac{dy}{dn},$$

and therefore

$$\frac{1}{\rho_c\rho_e}\cos C_0C_e=\sum\left\{\frac{Y}{\rho}+\left(\frac{\lambda_e}{\gamma_\epsilon}-\frac{\lambda_\epsilon}{\gamma_e}\right)\frac{1}{\sin\omega}\right\}\left\{\frac{Y}{\rho}+\frac{1}{\gamma_e}\frac{dy}{dn}\right\};$$

but

$$\sum\lambda_\epsilon\frac{dy}{dn}=-\sin\omega,\quad \sum\lambda_e\frac{dy}{dn}=0,$$

so that

$$\frac{1}{\rho_c\rho_e}\cos C_0C_e=\frac{1}{\rho^2}+\frac{1}{\gamma_e^{\,2}}=\frac{1}{\rho_e^{\,2}};$$

hence

$$\cos C_0C_e=\frac{\rho_c}{\rho_e}=\cos CC_e.$$

Similarly, we find

$$\cos C_0C_\epsilon=\frac{\rho_c}{\rho_\epsilon}=\cos CC_\epsilon.$$

Consequently the angular distances of C_0 from Y, C_e, C_ϵ, are equal to the angular distances, from the same points respectively, of the intersection of the circles B_eC_e and $B_\epsilon C_\epsilon$: that is, C_0 coincides with that intersection—which is the geometrical construction indicated.

Ex. In Fig. 29, the great circle C_eC_ϵ and the great circle YC meet $N_eN_\epsilon B_eB_\epsilon$ in X and Z respectively; shew that $B_eX=N_\epsilon Z$. Shew also that

$$\gamma_\epsilon\sin XN_e=\gamma_e\sin XN_\epsilon=\gamma.$$

Superficial flexures, regional flexure, of the curve.

247. Account also must be taken of the relation of the curve to the two surfaces, the intersection of which constitutes the curve, as well as of its relation to the region, which contains the intersecting surfaces. We therefore require the measure of its respective rates of deviation from the geodesics on the two surfaces, and the measure of its rate of deviation from the regional geodesic, all these geodesics being tangential to the curve.

We proceed as before (§ 149). At a point on the curve, at a small arc-distance δ from O, the value of the typical space-coordinate is

$$y_c=y+y'\delta+\tfrac{1}{2}y_c''\delta^2+\dots,$$

while at a point on the regional geodesic in the same direction, at an equal arc-distance δ from O, the value of the same typical space-coordinate is

$$y_r=y+y'\delta+\tfrac{1}{2}y''\delta^2+\dots,$$

with the customary notation. Hence, up to the second power of δ inclusive, the

projection of the deviation of the point on the curve from the point on the regional geodesic

$$=y_c-y_r=\tfrac{1}{2}(y_c''-y'')\delta^2.$$

This deviation, manifestly within the region, is at right angles to the prime normal of the regional geodesic, because the prime normal is orthogonal to the region. Let l denote the typical direction-cosine corresponding to the variable y, and let D_r denote the magnitude of the deviation; also, let $1/\gamma$ denote the regional flexure, as a measure of the rate of deviation; then we have

$$lD_r=\tfrac{1}{2}(y_c''-y'')\delta^2, \quad 2\gamma D_r=\delta^2,$$

and therefore

$$\begin{aligned}\frac{l}{\gamma}&=y_c''-y''\\ &=\frac{Y_c}{\rho_c}-\frac{Y}{\rho}=\left(\frac{\lambda_e}{\gamma_\epsilon}-\frac{\lambda_\epsilon}{\gamma_e}\right)\frac{1}{\sin\omega},\end{aligned}$$

while $\sum lY=0$.

Let ϕ denote, as before (§ 246), the inclination of the prime normal of the curve to the prime normal to the geodesic, so that

$$\sum YY_c=\cos\phi=\frac{\rho_c}{\rho};$$

then, because of the relations

$$\| \; l, \quad Y_c, \quad Y \; \|=0,$$

the direction of the regional deviation D_r lies in the plane through these two prime normals, and

$$\sum lY_c=\sin\phi=\frac{\rho_c}{\gamma}.$$

Similarly for the separate superficial flexures. Let $1/\Gamma_e$ denote the measure of flexure of the curve by the e-surface, and let l_e denote the typical direction-cosine of the deviation of a neighbouring point on the curve from the corresponding neighbouring point on the superficial geodesic; then, in the same way as for the regional deviation, we find

$$\begin{aligned}\frac{l_e}{\Gamma_e}&=y_c''-y_e''\\ &=\frac{Y_c}{\rho_c}-\frac{Y_e}{\rho_e}\\ &=\left\{\frac{Y}{\rho}+\left(\frac{\lambda_e}{\gamma_\epsilon}-\frac{\lambda_\epsilon}{\gamma_e}\right)\frac{1}{\sin\omega}\right\}-\left\{\frac{Y}{\rho}+\frac{1}{\gamma_e}\frac{dy}{dn}\right\}\\ &=\left(\frac{\lambda_e}{\gamma_\epsilon}-\frac{\lambda_\epsilon}{\gamma_e}\right)\frac{1}{\sin\omega}-\frac{1}{\gamma_e\sin\omega}(\lambda_e\cos\omega-\lambda_\epsilon)\\ &=\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\frac{\lambda_e}{\sin\omega}.\end{aligned}$$

This relation is typical of the direction-cosines of the radius Γ_e of superficial flexure of the curve by the e-surface; hence

$$\left.\begin{aligned} l_e&=\lambda_e \\ \frac{\sin\omega}{\Gamma_e}&=\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e} \end{aligned}\right\}.$$

The former typical relation verifies the known property (§ 150) that the radius of superficial flexure of the curve lies in the tangent plane of the surface, at right angles to the tangent to the curve: that is, in the line OB_e.

So, also, for the ϵ-surface: let $1/\Gamma_\epsilon$ denote the superficial flexure of the curve by the ϵ-surface; we find

$$\frac{\sin\omega}{\Gamma_\epsilon}=\frac{\cos\omega}{\gamma_\epsilon}-\frac{1}{\gamma_e};$$

and the radius of this superficial flexure lies in the tangent plane of the ϵ-surface at right angles to the tangent to the curve: that is, in the line OB_ϵ.

The inclinations of the respective radii of superficial flexure of the curve by the surfaces, to the radius of regional flexure of the curve, are given by the equations

$$\sum l\lambda_e=\frac{\gamma}{\Gamma_e},\quad \sum l\lambda_\epsilon=\frac{\gamma}{\Gamma_\epsilon};$$

also we have

$$\begin{aligned}\frac{\sin^2\omega}{\gamma^2}&=\frac{1}{\gamma_e^2}-\frac{2\cos\omega}{\gamma_e\gamma_\epsilon}+\frac{1}{\gamma_\epsilon^2}\\ &=\frac{1}{\Gamma_e^2}-\frac{2\cos\omega}{\Gamma_e\Gamma_\epsilon}+\frac{1}{\Gamma_\epsilon^2}.\end{aligned}$$

248. The relations between the various circular curvatures of the curve and the three geodesics, including also the various flexures of all but the regional geodesic, can be illustrated geometrically. Let the dotted line OZ denote the tangent at O; and let K denote a point on the curve adjacent * to O. We take the block, the leading lines of which are the three lines of the tangent flat of the region and the prime normal of the regional geodesic in the direction OZ; in that block, we select the flat which is perpendicular to OZ; and in that selected flat, we construct a sphere on ZK as diameter. Let the regional geodesic, the superficial geodesic on the e-surface, and the superficial geodesic on the ϵ-surface, meet this sphere in the points R, S_e, S_ϵ, respectively, so that the angles KRZ, KS_eZ,

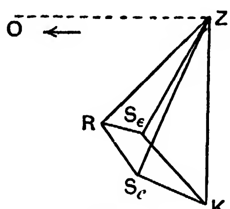

FIG. 30.

* In the diagram, ZO represents a small magnitude of the first order of small quantities, while ZK and all the lines connected with the sphere on ZK as diameter represent small magnitudes of the second order of small quantities.

$KS_\epsilon Z$, are right angles; then as OZ tends towards zero, the limiting positions of the lines ZR, ZS_e, ZS_ϵ, ZK, are the prime normals of the regional geodesic, the superficial geodesic on the e-surface, the superficial geodesic on the ϵ-surface, and the curve respectively. In the selected flat, the lines RK, RS_e, RS_ϵ, are at right angles to RZ, and consequently they lie in one plane $RS_eS_\epsilon K$. Moreover, the angles ZS_eK and $ZS_\epsilon K$ are right angles, SK being the diameter of the sphere. Then as ZO and (up to the second order of small quantities inclusive) the lines RO, S_eO, $S_\epsilon O$, KO, are equal to one another, these lines all being orthogonal to the selected flat (up to the same order of small quantities), we have

$$OZ^2 = 2KZ \,.\, \rho_c,$$

from the curve; also

$$OZ^2 = 2RZ \,.\, \rho = 2RK \,.\, \gamma,$$

from the circular curvature of the regional geodesic and the regional flexure of the curve; also

$$OZ^2 = 2S_eZ \,.\, \rho_e = 2S_eR \,.\, \gamma_e = 2S_eK \,.\, \Gamma_e,$$

from the circular curvature and the regional flexure of the superficial geodesic on the e-surface, and from the superficial flexure of the curve by that e-surface; and

$$OZ^2 = 2S_\epsilon Z \,.\, \rho_\epsilon = 2S_\epsilon R \,.\, \gamma_\epsilon = 2S_\epsilon K \,.\, \Gamma_\epsilon,$$

from the circular curvature and the regional flexure of the superficial geodesic on the ϵ-surface, and from the superficial flexure of the curve by that ϵ-surface.

A comparison of this diagram with the diagram in § 246, shews that

RZS_e = the angle between the prime normal ZR of the regional geodesic and the prime normal ZS_e of the superficial geodesic on the e-surface

$= YC_e = a$,

with the former notation; also, similarly

$$RZS_\epsilon = YC_\epsilon = \beta;$$

and, as connected with corresponding lines associated with the respective combinations of curvatures,

$$RZK = YC = \phi, \quad S_eZK = CC_e, \quad S_\epsilon ZK = CC_\epsilon.$$

The foregoing sets of equal values for OZ^2 now give

$$\left.\begin{array}{lcl} \rho\,\dfrac{RZ}{KZ} = \rho_c, & \text{so that} & \rho\cos\phi = \rho_c \\ \gamma\,\dfrac{RK}{KZ} = \rho_c, & \ldots\ldots & \gamma\sin\phi = \rho_c \end{array}\right\};$$

$$\left.\begin{aligned} \rho\frac{RZ}{S_eZ}&=\rho_e, &\text{so that}\quad \rho\cos a&=\rho_e\\ \gamma_e\frac{RZ_e}{S_eZ}&=\rho_e, &\ldots\ldots\quad \gamma_e\sin a&=\rho_e\end{aligned}\right\};$$

$$\left.\begin{aligned} \rho\frac{RZ}{S_\epsilon Z}&=\rho_\epsilon, &\ldots\ldots\quad \rho\cos\beta&=\rho_c\\ \gamma_\epsilon\frac{RS_\epsilon}{S_\epsilon Z}&=\rho_\epsilon, &\ldots\ldots\quad \gamma_\epsilon\sin\beta&=\rho_\epsilon\end{aligned}\right\};$$

$$\left.\begin{aligned} \Gamma_e\frac{S_eK}{KZ}&=\rho_c, &\ldots\ldots\quad \Gamma_e\sin CC_e&=\rho_c\\ \Gamma_\epsilon\frac{S_\epsilon K}{KZ}&=\rho_c, &\ldots\ldots\quad \Gamma_\epsilon\sin CC_\epsilon&=\rho_c\end{aligned}\right\}.$$

But we had (§ 246)

$$\frac{\tan a}{\tan\phi}=\cos u,\quad \frac{\tan\beta}{\tan\phi}=\cos(u-\omega);$$

and therefore

$$\begin{aligned}\sin u\sin\omega&=\frac{1}{\tan\phi}(\tan\beta-\tan a\cos\omega)\\ &=\frac{\rho}{\tan\phi}\left(\frac{1}{\gamma_c}-\frac{\cos\omega}{\gamma_e}\right)=\frac{\Gamma_e\sin CC_e}{\sin\phi}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\end{aligned}$$

The former diagram (p. 179) gives

$$\sin CC_e=\sin CYC_e\sin CY=\sin u\sin\phi;$$

and therefore

$$\frac{\sin\omega}{\Gamma_e}=\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}.$$

Similarly we find

$$\frac{\sin\omega}{\Gamma_\epsilon}=\frac{\cos\omega}{\gamma_\epsilon}-\frac{1}{\gamma_e}.$$

All these results are in accordance with the results already (§ 247) established.

249. To obtain the torsion and the direction-cosines of the binormal of the parametric curve in the region, we could proceed by regarding the curve as belonging to the e-surface and as arising from an intersection by the ϵ-surface: also by regarding it as belonging to the ϵ-surface and as arising from an intersection by the e-surface. In each instance, we should combine the results of § 152 with those of § 195; and for a diagram, we should combine the constructions of §§ 152, 202, 246.

We can proceed, more directly to the analytical formulæ, in a different manner;

and the actual result admits of a simple geometrical construction. In the investigation, the symbol [L] will be used (when convenient) to denote generically any quantity which is linear and homogeneous in $\frac{dy}{dn}$, $\frac{dy}{dv}$, y', whether y' does or does not occur in a completed expression.

The equation for the circular curvature and the direction of the prime normal of the curve is

$$\frac{Y_c}{\rho_c}=\frac{Y}{\rho}+\frac{l}{\gamma},$$

where

$$\frac{l}{\gamma}=\left[\frac{dy}{dn}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)+\frac{dy}{dv}\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)\right]\frac{1}{\sin^2\omega}\,.$$

We denote differentiation along the curve by ds_c as usual. Also we have (§ 245)

$$p_c''-p''=\left[\frac{dp}{dn}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)+\frac{dp}{dv}\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)\right]\frac{1}{\sin^2\omega}=P\,;$$

and, with similar expressions, we shall write

$$q_c''-q''=Q,\quad r_c''-r''=R.$$

In the first place, because

$$\frac{Y}{\rho}=\sum\eta_{11}p'^2,$$

we have

$$\frac{d}{ds_c}\left(\frac{Y}{\rho}\right)-\frac{d}{ds}\left(\frac{Y}{\rho}\right)=2(\eta_1P+\eta_2Q+\eta_3R),$$

while

$$\frac{d}{ds}\left(\frac{Y}{\rho}\right)=\frac{1}{\rho}\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)+Y\frac{d}{ds}\left(\frac{1}{\rho}\right).$$

The quantity l_3 (which belongs to the binormal of the regional geodesic tangent) is linear and homogeneous in y_1, y_2, y_3, and therefore (*Ex.* 1, § 244) can be made linear and homogeneous in y', $\frac{dy}{dn}$, $\frac{dy}{dv}$, so that we can write

$$\frac{1}{\rho}\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)=[L],$$

where, for the immediate purpose, a knowledge of the actual coefficients of y', $\frac{dy}{dn}$, $\frac{dy}{dv}$, in [L] is not specially relevant. Thus we have

$$\frac{d}{ds_c}\left(\frac{Y}{\rho}\right)=2(\eta_1P+\eta_2Q+\eta_3R)+Y\frac{d}{ds}\left(\frac{1}{\rho}\right)+[L].$$

In the next place, when we differentiate the expression for l/γ along the curve, we have

$$\frac{d}{ds_c}\left(\frac{l}{\gamma}\right)=[L]+\frac{1}{\sin^2\omega}\left\{\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\frac{d}{ds}\left(\frac{dy}{dn}\right)+\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)\frac{d}{ds}\left(\frac{dy}{d\nu}\right)\right\},$$

where $[L]$ again is used generically and not specifically. But by the result in § 208, we have (for any direction of differentiation)

$$\frac{d}{ds}\left(\frac{dy}{dn}\right)+\frac{e_n'}{e_n}\frac{dy}{dn}=\eta_1\frac{dp}{dn}+\eta_2\frac{dq}{dn}+\eta_3\frac{dr}{dn}+\frac{1}{\Omega e_n}(a \between \bar{e}_1, \bar{e}_2, \bar{e}_3 \between y_1, y_2, y_3):$$

that is, we can write

$$\frac{d}{ds_c}\left(\frac{dy}{dn}\right)=\eta_1\frac{dp}{dn}+\eta_2\frac{dq}{dn}+\eta_3\frac{dr}{dn}+[L];$$

and similarly

$$\frac{d}{ds_c}\left(\frac{dy}{d\nu}\right)=\eta_1\frac{dp}{d\nu}+\eta_2\frac{dq}{d\nu}+\eta_3\frac{dr}{d\nu}+[L].$$

Consequently, the coefficient of η_1 on the right-hand side

$$=\frac{1}{\sin^2\omega}\left\{\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\frac{dp}{dn}+\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)\frac{dp}{d\nu}\right\}=P,$$

and similarly for the coefficients of η_2 and η_3; hence

$$\frac{d}{ds_c}\left(\frac{l}{\gamma}\right)=\eta_1 P+\eta_2 Q+\eta_3 R+[L].$$

Finally, we have

$$\frac{d}{ds_c}\left(\frac{Y_c}{\rho_c}\right)=\frac{1}{\rho_c}\left(\frac{\lambda_3}{\sigma_c}-\frac{y'}{\rho_c}\right)+Y_c\frac{d}{ds_c}\left(\frac{1}{\rho_c}\right),$$

where λ_3 denotes the typical direction-cosine of the binormal of the curve and $1/\sigma_c$ denotes the torsion of the curve. Also

$$Y_c=\frac{\rho_c}{\rho}Y+\frac{\rho_c}{\gamma}l=\frac{\rho_c}{\rho}Y+[L],$$

so that

$$\frac{d}{ds_c}\left(\frac{Y_c}{\rho_c}\right)=\frac{\lambda_3}{\rho_c\sigma_c}+Y\frac{\rho_c}{\rho}\frac{d}{ds}\left(\frac{1}{\rho_c}\right)+[L].$$

When these values are substituted in the arc-derivative of the equation on p. 186 for the direction of the prime normal, we have

$$\frac{\lambda_3}{\rho_c\sigma_c}=3(\eta_1 P+\eta_2 Q+\eta_3 R)+Y\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{\rho_c}{\rho}\frac{d}{ds}\left(\frac{1}{\rho_c}\right)\right\}+[L],$$

where now $[L]$ denotes the combination of all the preceding generic terms $[L]$.

But (§ 188) it was proved that

$$\eta_1=Yv_1+l_5u_5,\quad \eta_2=Yv_2+l_5v_5,\quad \eta_3=Yv_3+l_5w_5;$$

and therefore we have

$$\frac{\lambda_3}{\rho_c\sigma_c} = YS + l_5 T + [L]$$

$$= YS + l_5 T + Jy' + D\frac{dy}{dn} + E\frac{dy}{d\nu} = \square,$$

where the coefficients S and T can be regarded as known quantities, where J, D, E, are the complete coefficients in the aggregate expression for $[L]$, and where $\square$ denotes the whole of the right-hand side in the expression for λ_3. The values of S and T are

$$S = \frac{d}{ds}\left(\frac{1}{\rho}\right) - \frac{\rho_c}{\rho}\frac{d}{ds}\left(\frac{1}{\rho_c}\right) + 3(v_1 P + v_2 Q + v_3 R),$$

$$T = 3(u_5 P + v_5 Q + w_5 R),$$

while the values of u_5, v_5, w_5, have been given in § 189. It remains to obtain relations for the determination of J, D, E.

Ex. 1. Prove that

$$\frac{d}{ds_c}\left(\frac{1}{\rho}\right) - \frac{d}{ds}\left(\frac{1}{\rho}\right) = \frac{2}{\sin\omega}\left(\frac{1}{\gamma_e \tau_\epsilon} - \frac{1}{\gamma_\epsilon \tau_e}\right),$$

where $1/\tau_e$ and $1/\tau_\epsilon$ are the respective regional tilts of the geodesic on the e-surface and the geodesic on the ϵ-surface touching the curve.

Ex. 2. Prove that

$$\frac{1}{\rho^2}\left(\frac{dY}{ds_c} - \frac{dY}{ds}\right) = 2l_5 \begin{vmatrix} p_c'' - p'', & q_c'' - q'', & r_c'' - r'' \\ p', & q', & r' \\ \bar{\mathfrak{a}}^{\frac{1}{2}}, & -\bar{\mathfrak{b}}^{\frac{1}{2}}, & \bar{\mathfrak{c}}^{\frac{1}{2}} \end{vmatrix},$$

where the quantities denoted by $\bar{\mathfrak{a}}^{\frac{1}{2}}$, $\bar{\mathfrak{b}}^{\frac{1}{2}}$, $\bar{\mathfrak{c}}^{\frac{1}{2}}$, are given in §§ 188, 189.

250. Two inferences can be made at once from the geometrical property that the binormal of a curve is at right angles to the tangent and to the prime normal.

Because the binormal is at right angles to the tangent, we must have

$$\sum y'\lambda_3 = 0.$$

But

$$\sum y'Y = 0, \quad \sum y'l_5 = 0, \quad \sum y'\frac{dy}{dn} = 0, \quad \sum y'\frac{dy}{d\nu} = 0,$$

the two last holding, because $\frac{dy}{dn}$ and $\frac{dy}{d\nu}$ are typical direction-cosines of the regional normals to the two surfaces which intersect in the parametric curve; hence

$$J = 0.$$

Because the binormal is at right angles to the prime normal, we must have

$$\sum Y_c \lambda_3 = 0:$$

that is,

$$\frac{1}{\rho}\sum Y\lambda_3 + \sum \lambda_3 \frac{l}{\gamma} = 0.$$

Now

$$\sum Yl = 0, \quad \sum l_5 l = 0, \quad \sum y' l = 0,$$

$$\sum \frac{l}{\gamma}\frac{dy}{dn} = \frac{1}{\gamma_\epsilon}, \quad \sum \frac{l}{\gamma}\frac{dy}{d\nu} = \frac{1}{\gamma_e};$$

and therefore

$$\frac{1}{\rho_c \sigma_c}\sum \lambda_3 \frac{l}{\gamma} = \frac{D}{\gamma_c} + \frac{E}{\gamma_\epsilon}.$$

Also

$$\frac{1}{\rho_c \sigma_c}\sum \lambda_3 Y = S:$$

and therefore we have the equation

$$\frac{S}{\rho} + \frac{D}{\gamma_e} + \frac{E}{\gamma_\epsilon} = 0.$$

But no further relations can be derived merely from the geometrical properties of the curve, taken without special consideration to the source of the curve as the intersection of the two surfaces.

Because

$$\frac{Y_c}{\rho_c} = \frac{Y}{\rho} + \left[\frac{dy}{dn}\left(\frac{1}{\gamma_\epsilon} - \frac{\cos\omega}{\gamma_e}\right) + \frac{dy}{d\nu}\left(\frac{1}{\gamma_c} - \frac{\cos\omega}{\gamma_\epsilon}\right)\right]\frac{1}{\sin^2\omega},$$

it follows that

$$\frac{1}{\rho_c}\sum Y_c \frac{dy}{dn} = \frac{1}{\gamma_\epsilon}, \quad \frac{1}{\rho_c}\sum Y_c \frac{dy}{d\nu} = \frac{1}{\gamma_e}.$$

Now

$$\frac{d}{ds}\left(\sum Y_c \frac{dy}{dn}\right) = \sum\left\{Y_c \frac{d}{ds_c}\left(\frac{dy}{dn}\right)\right\} + \sum \frac{dy}{dn}\left(\frac{\lambda_3}{\sigma_c} - \frac{y'}{\rho_c}\right).$$

But

$$\sum y' \frac{dy}{dn} = 0;$$

and, on substituting the formal value of λ_3,

$$\frac{1}{\rho_c \sigma_c}\sum \lambda_3 \frac{dy}{dn} = D + E\cos\omega:$$

consequently

$$\frac{1}{\rho_c}\frac{d}{ds_c}\left(\frac{\rho_c}{\gamma_\epsilon}\right) - (D + E\cos\omega) = \sum \frac{Y_c}{\rho_c}\frac{d}{ds}\left(\frac{dy}{dn}\right).$$

The value of $\dfrac{d}{ds}\left(\dfrac{dy}{dn}\right)$, already cited, is

$$\frac{d}{ds}\left(\frac{dy}{dn}\right)=-\frac{e_n'}{e_n}\frac{dy}{dn}+\left(\eta_1\frac{dp}{dn}+\eta_2\frac{dq}{dn}+\eta_3\frac{dr}{dn}\right)+\frac{1}{\Omega e_n}(a\between\bar{e}_1,\,\bar{e}_2,\,\bar{e}_3\between y_1,\,y_2,\,y_3)\,;$$

and thus the right-hand side of the preceding equation, when evaluated, will consist of three terms.

The contribution from the first term

$$=-\frac{e_n'}{e_n}\sum\left(\frac{Y_c}{\rho_c}\frac{dy}{dn}\right)=-\frac{e_n'}{e_n}\frac{1}{\gamma_\epsilon}.$$

Again, because

$$\sum Y\eta_\lambda=v_\lambda,\quad \sum l\eta_\lambda=0,$$

we have

$$\sum Y_c\eta_\lambda=v_\lambda\,;$$

and therefore the contribution from the second term

$$=\frac{1}{\rho_c}\left(v_1\frac{dp}{dn}+v_2\frac{dq}{dn}+v_3\frac{dr}{dn}\right).$$

Further, writing

$$C_\epsilon=\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\omega}{\gamma_e}\right)\frac{1}{\sin^2\omega},\quad C_e=\left(\frac{1}{\gamma_e}-\frac{\cos\omega}{\gamma_\epsilon}\right)\frac{1}{\sin^2\omega},$$

we have

$$\frac{1}{\rho_c}\left(\sum Y_c y_i\right)=\frac{e_1}{e_n}C_\epsilon+\frac{\epsilon_1}{\epsilon_\nu}C_e,$$

for $i=1, 2, 3$; and therefore the contribution from the third term

$$=\frac{C_\epsilon}{\Omega e_n{}^2}(a\between\bar{e}_1,\,\bar{e}_2,\,\bar{e}_3\between e_1,\,e_2,\,e_3)+\frac{C_e}{\Omega e_n\epsilon_\nu}(a\between\bar{e}_1,\,\bar{e}_2,\,\bar{e}_3\between\epsilon_1,\,\epsilon_2,\,\epsilon_3).$$

By the result in § 207, we have

$$e_n'=\frac{1}{\Omega e_n}(a\between\bar{e}_1,\,\bar{e}_2,\,\bar{e}_3\between e_1,\,e_2,\,e_3),$$

so that the contribution from the first term combines with a part of the contribution from the third term.

When the results are gathered together, the equation can be arranged in the form

$$\begin{aligned}D+E\cos\omega= &\ \frac{1}{\rho_c}\left[\left\{\frac{d}{ds_c}\left(\frac{\rho_c}{\gamma_\epsilon}\right)\right\}-\left(v_1\frac{dp}{dn}+v_2\frac{dq}{dn}+v_3\frac{dr}{dn}\right)\right]\\ &+\frac{C_e}{\Omega e_n}\left(a\between\bar{e}_1,\,\bar{e}_2,\,\bar{e}_3\between\frac{\epsilon_1}{\epsilon_\nu}-\frac{e_1}{e_n}\cos\omega,\ \frac{\epsilon_2}{\epsilon_\nu}-\frac{e_2}{e_n}\cos\omega,\ \frac{\epsilon_3}{\epsilon_\nu}-\frac{e_3}{e_n}\cos\omega\right).\end{aligned}$$

Proceeding in like manner from the relation

$$\sum Y_c \frac{dy}{dv} = \frac{\rho_c}{\gamma_e},$$

we obtain an equation

$$D \cos \omega + E = \frac{1}{\rho_c}\left[\left\{\frac{d}{ds_c}\left(\frac{\rho_c}{\gamma_e}\right)\right\} - \left(v_1 \frac{dp}{dv} + v_2 \frac{dq}{dv} + v_3 \frac{dr}{dv}\right)\right]$$
$$+ \frac{C_\epsilon}{\Omega \epsilon_\nu}\left(a \between \bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3 \between \frac{e_1}{e_n} - \frac{\epsilon_1}{\epsilon_\nu}\cos \omega,\ \frac{e_2}{e_n} - \frac{\epsilon_2}{\epsilon_\nu}\cos \omega,\ \frac{e_3}{e_n} - \frac{\epsilon_3}{\epsilon_\nu}\cos \omega\right).$$

These two equations suffice to determine values of D and E, independently of the condition

$$\frac{S}{\rho} + \frac{D}{\gamma_\epsilon} + \frac{E}{\gamma_e} = 0.$$

With these values, the typical equation for the direction-cosine of the binormal and the torsion of the curve is

$$\frac{\lambda_3}{\rho_c \sigma_c} = YS + l_5 T + D \frac{dy}{dn} + E \frac{dy}{dv};$$

and the magnitude of the torsion is given by the equation

$$\frac{1}{\rho_c^2 \sigma_c^2} = S^2 + T^2 + D^2 + 2DE \cos \omega + E^2.$$

CHAPTER XXI

REGIONS: SOME SPECIAL PROBLEMS

Dimensionality of the plenary space, as affecting a region.

251. Hitherto no specific account has been taken of the precise dimensionality of the plenary homaloidal space of a region. When that space is quadruple, any special characteristics of the region have had a partial discussion elsewhere *; we shall therefore assume that the plenary homaloidal space of the region is of more than four dimensions.

The most immediate issue, demanding a consideration of the dimensionality of the plenary space, would seem to be provided by the tale of the successive grades of curvature appertaining to geodesics. For a quadruple space, a regional geodesic possesses circular curvature, torsion, tilt; and no curve in such a space, however it arises, possesses curvature of more advanced grade than these three. For a quintuple space, a regional geodesic possesses coil, in addition to the three curvatures just specified: and no curve in a quintuple homaloidal space can possess more than the four curvatures of these successive grades. Generally, if the plenary space is of N dimensions, the number of grades of curvature that can be possessed is $N-1$.

The dimensionality demands immediate consideration when we investigate such a matter as the spatial range of the centre of circular curvature of geodesics drawn through a point in the configuration. Thus, in a region, let y_0 be the typical space-coordinate of the centre of circular curvature of a regional geodesic in a direction p', q', r'; because $y_0 - y = Y\rho$, the typical initial equation for the locus of centres is

$$\frac{y_0 - y}{\rho^2} = \eta_{11}p'^2 + 2\eta_{12}p'q' + \eta_{22}q'^2 + 2\eta_{13}p'r' + 2\eta_{23}q'r' + \eta_{33}r'^2 = \sum \eta_{11}p'^2.$$

Now for all the values $i, j, k, = 1, 2, 3$, in any combination, we have

$$\sum y_k y_{ij} = 0;$$

and therefore

$$\sum y_1(y_0 - y) = 0, \quad \sum y_2(y_0 - y) = 0, \quad \sum y_3(y_0 - y) = 0,$$

so that the centre of curvature certainly lies in a homaloid of $N-3$ dimensions orthogonal to the tangent flat of the region. The locus of the centre will lie in a

* *G.F.D.*, vol. ii, *passim.*

more restricted plenary space, when the plenary space of the whole configuration is of dimensionality greater than nine.

When the plenary space of the region is quadruple, this orthogonal homaloid is a line, being the unique normal to the region at the point; and the portions of this line, lying between the centres of greatest curvature and the centres of smallest curvature, constitute the locus in question. It is, however, hardly a locus in the customary sense of the term: every point on a due portion of the line being a centre of curvature for an infinitude of geodesics through O.

When the plenary space of the region is quintuple, the orthogonal homaloid is a plane. The locus of centres of circular curvature of regional geodesics, which have their initial direction at O contained within an assigned orientation, is a curve in that plane; and the curve varies from one orientation to another.

When the plenary space is sextuple, the orthogonal homaloid of a region is a flat; the locus of the centres of circular curvature of regional geodesics through O is a surface in that flat.

When the plenary space of the region is of dimensions higher than six, the locus of the centre of circular curvature of regional geodesics through O still is a surface, though not necessarily (nor generally) contained in a flat. The typical equation for a point on the locus of centres is

$$y_0 - y = \rho^2 \sum \eta_{11} p'^2$$
$$= \frac{(\sum A p'^2)(\sum \eta_{11} p'^2)}{\sum \kappa_{400} p'^4};$$

thus the coordinates of any point on the locus involve the two independent parameters $p' : q' : r'$; consequently, the locus is two-dimensional and therefore is a surface. This surface lies in the homaloid orthogonal to the region. But if the plenary homaloidal space is of more than nine dimensions, the equations

$$\| \; y_0 - y, \quad \eta_{11}, \quad \eta_{12}, \quad \eta_{22}, \quad \eta_{13}, \quad \eta_{23}, \quad \eta_{33} \; \| = 0$$

are satisfied by the coordinates of the centre: and thus, in the least restricted instance, the surface certainly lies in a sextuple homaloid, itself manifestly orthogonal to the region.

We shall begin with the consideration of the centre-locus for regional geodesics, when the region exists freely in a quintuple plenary space: that is, when the region is not given as a sub-amplitude of a domain in such a space.

Orthogonal plane of a region in a quintuple plenary space.

252. The tangent flat of the region can be represented in two ways, according to the selection of its leading lines. When these are taken to be the tangents to the parametric curves, the (two) equations of the tangent flat are

$$\| \; \bar{y} - y, \quad y_1, \quad y_2, \quad y_3 \; \| = 0.$$

When they are taken to be the tangent, the binormal, and the trinormal, of a regional geodesic in the direction p', q', r', the (two) equations of the tangent flat are

$$\| \bar{y}-y, \; y', \; l_3, \; l_4 \| = 0.$$

The remaining principal lines of the geodesic are its prime normal (with Y as its typical direction-cosine) and its quartinormal (with l_5 as its typical direction-cosine), each of them being normal to the tangent flat; and therefore the homaloid orthogonal to the region in the quintuple plenary space is a plane, the (three) equations of which may be taken either in the form

$$\| \bar{y}-y, \; Y, \; l_5 \| = 0,$$

or in the form

$$\sum (\bar{y}-y) y' = 0, \quad \sum (\bar{y}-y) l_3 = 0, \quad \sum (\bar{y}-y) l_4 = 0,$$

or in the form

$$\sum (\bar{y}-y) y_1 = 0, \quad \sum (\bar{y}-y) y_2 = 0, \quad \sum (\bar{y}-y) y_3 = 0.$$

The plane will be called the orthogonal plane of the region in the quintuple plenary space.

The five quantities η_{ij}, arising out of the five space-coordinates, determine a direction with direction-cosines represented by

$$\frac{\eta_{ij}}{(\sum \eta_{ij}^2)^{\frac{1}{2}}};$$

and as

$$\sum y_k \eta_{ij} = 0,$$

for $k=1, 2, 3$, this direction lies in the orthogonal plane. The property is true for all the six combinations $i, j, =1, 2, 3$; and thus there are six such directions in the orthogonal plane. In a plane, two lines (being guiding lines) suffice for a parametric expression of the direction of any line. In the orthogonal plane, we shall take the prime normal and the quartinormal as the guiding lines; and therefore we have relations

$$\eta_{ij} = YP_{ij} + l_5 Q_{ij},$$

where P_{ij} and Q_{ij} are the same for all the magnitudes in the set η_{ij} belonging to the different coordinates. In particular, we have

$$P_{ij} = \sum Y \eta_{ij}, \quad Q_{ij} = \sum l_5 \eta_{ij},$$

these relations holding for all the values $i, j, =1, 2, 3$. Thus, with the notation of § 182,

$$\left.\begin{aligned} \eta_{11} &= Y\bar{A} + l_5 A_5, & \eta_{23} &= Y\bar{F} + l_5 F_5 \\ \eta_{22} &= Y\bar{B} + l_5 B_5, & \eta_{31} &= Y\bar{G} + l_5 G_5 \\ \eta_{33} &= Y\bar{C} + l_5 C_5, & \eta_{12} &= Y\bar{H} + l_5 H_5 \end{aligned}\right\}.$$

Further, we have the relations in the array

$$\left\| \begin{matrix} \eta_{ij}^{(\mu)} \\ \eta_{kl}^{(\mu)} \\ \eta_{mn}^{(\mu)} \end{matrix} \right\| = 0,$$

where there are five columns, for $\mu = 1, 2, 3, 4, 5$, corresponding to the five space-coordinates, and where i, j; k, l; m, n; can have the values 1, 2, 3, independently of one another. Consequently, we have relations

$$\begin{vmatrix} \sum \eta_{ij}^2, & \sum \eta_{ij}\eta_{kl}, & \sum \eta_{ij}\eta_{mn} \\ \sum \eta_{ij}\eta_{kl}, & \sum \eta_{kl}^2, & \sum \eta_{kl}\eta_{mn} \\ \sum \eta_{ij}\eta_{mn}, & \sum \eta_{kl}\eta_{mn}, & \sum \eta_{mn}^2 \end{vmatrix} = 0,$$

for all the non-evanescent combinations, the summations being taken over the space-combinations. The constituents in each such relation are expressible in terms of the magnitudes $\kappa_{\lambda\mu\nu}$ and $k_{\alpha\beta}$ of § 168; and therefore there exist fundamental relations among those quantities. A comprehensive statement of the result is that every minor of three rows and columns (every third minor) in the determinant

$$\begin{vmatrix} \sum \eta_{11}^2, & \sum \eta_{11}\eta_{12}, & \sum \eta_{11}\eta_{22}, & \sum \eta_{11}\eta_{13}, & \sum \eta_{11}\eta_{23}, & \sum \eta_{11}\eta_{33} \\ \sum \eta_{11}\eta_{12}, & \sum \eta_{12}^2, & \sum \eta_{12}\eta_{22}, & \sum \eta_{12}\eta_{13}, & \sum \eta_{12}\eta_{23}, & \sum \eta_{12}\eta_{33} \\ \sum \eta_{11}\eta_{22}, & \sum \eta_{12}\eta_{22}, & \sum \eta_{22}^2, & \sum \eta_{22}\eta_{13}, & \sum \eta_{22}\eta_{23}, & \sum \eta_{22}\eta_{33} \\ \sum \eta_{11}\eta_{13}, & \sum \eta_{12}\eta_{13}, & \sum \eta_{22}\eta_{13}, & \sum \eta_{13}^2, & \sum \eta_{13}\eta_{23}, & \sum \eta_{13}\eta_{33} \\ \sum \eta_{11}\eta_{23}, & \sum \eta_{12}\eta_{23}, & \sum \eta_{22}\eta_{23}, & \sum \eta_{13}\eta_{23}, & \sum \eta_{23}^2, & \sum \eta_{23}\eta_{33} \\ \sum \eta_{11}\eta_{33}, & \sum \eta_{12}\eta_{33}, & \sum \eta_{22}\eta_{33}, & \sum \eta_{13}\eta_{33}, & \sum \eta_{23}\eta_{33}, & \sum \eta_{33}^2 \end{vmatrix}$$

vanishes when the plenary space is quintuple.

Another mode of using the relations in the array arises when a set of magnitudes $\eta_{ij}^{(\mu)}$ is expressed in terms of any two other sets which can be made to become leading lines in the orthogonal plane. Thus there are relations of the form

$$\eta_{12} = \gamma\eta_{11} + \delta\eta_{22},$$

where γ and δ are the same throughout the set of five equations corresponding to the space-coordinates; and their values are given by

$$\kappa_{310} = \sum \eta_{11}\eta_{12} = \gamma \sum \eta_{11}^2 + \delta \sum \eta_{11}\eta_{22} = \gamma\kappa_{400} + \delta(\kappa_{220} + \tfrac{2}{3}k_{33})$$
$$\kappa_{130} = \sum \eta_{12}\eta_{22} = \gamma \sum \eta_{11}\eta_{22} + \delta \sum \eta_{22}^2 = \gamma(\kappa_{220} + \tfrac{2}{3}k_{33}) + \delta\kappa_{040}.$$

And so for other instances.

253. As one main purpose at this stage is the expression of the six magnitudes A_5, B_5, C_5, F_5, G_5, H_5, in terms of the regional magnitudes which have already been defined and of the direction-variables p', q', r', these relations will be developed only so far as to serve this purpose.

In particular, where the relations between the magnitudes $\kappa_{\lambda\mu\nu}$ and $k_{\alpha\beta}$ are concerned, a few instances may suffice. The relation

$$\begin{vmatrix} \sum\eta_{11}^2, & \sum\eta_{11}\eta_{12}, & \sum\eta_{11}\eta_{22} \\ \sum\eta_{11}\eta_{12}, & \sum\eta_{12}^2, & \sum\eta_{12}\eta_{22} \\ \sum\eta_{11}\eta_{22}, & \sum\eta_{12}\eta_{22}, & \sum\eta_{22}^2 \end{vmatrix} = 0,$$

when the values (§ 168) of the constituents are substituted, becomes

$$\begin{vmatrix} \kappa_{400} & , & \kappa_{310} & , & \kappa_{220}+\frac{2}{3}k_{33} \\ \kappa_{310} & , & \kappa_{220}-\frac{1}{3}k_{33}, & & \kappa_{130} \\ \kappa_{220}+\frac{2}{3}k_{33}, & & \kappa_{130} & , & \kappa_{040} \end{vmatrix} = 0,$$

an equation expressing the magnitude k_{33} in terms of the magnitudes $\kappa_{\lambda\mu\nu}$. The equation can be stated in the form

$$\tfrac{4}{27}k_{33}^3 - \tfrac{1}{3}I_3 k_{33} + J_3 = 0,$$

where I_3 and J_3 are the quadrinvariant and the cubinvariant of the binary quartic

$$(\kappa_{400}, \kappa_{310}, \kappa_{220}, \kappa_{130}, \kappa_{040} \between u, v)^4.$$

Similarly we have

$$\tfrac{4}{27}k_{11}^3 - \tfrac{1}{3}I_1 k_{11} + J_1 = 0,$$

where I_1 and J_1 are the quadrinvariant and the cubinvariant of the binary quartic

$$(\kappa_{040}, \kappa_{031}, \kappa_{022}, \kappa_{013}, \kappa_{004} \between u, v)^4\,;$$

and

$$\tfrac{4}{27}k_{22}^3 - \tfrac{1}{3}I k_{222} + J_2 = 0,$$

where I_2 and J_2 are the quadrinvariant and the cubinvariant of the binary quartic

$$(\kappa_{400}, \kappa_{301}, \kappa_{202}, \kappa_{103}, \kappa_{004} \between u, v)^4.$$

Again, the relation

$$\begin{vmatrix} \sum\eta_{11}^2, & \sum\eta_{11}\eta_{12}, & \sum\eta_{11}\eta_{13} \\ \sum\eta_{11}\eta_{12}, & \sum\eta_{12}^2, & \sum\eta_{12}\eta_{13} \\ \sum\eta_{11}\eta_{13}, & \sum\eta_{12}\eta_{13}, & \sum\eta_{13}^2 \end{vmatrix} = 0,$$

when similar substitution is made for the constituents, can be expressed in the form

$$\begin{aligned} &\{\tfrac{1}{3}k_{23}\kappa_{400} - (\kappa_{310}\kappa_{301} - \kappa_{400}\kappa_{211})\}^2 \\ &\qquad = (\tfrac{1}{3}k_{22}\kappa_{400} + \kappa_{301}^2 - \kappa_{400}\kappa_{202})(\tfrac{1}{3}k_{33}\kappa_{400} + \kappa_{310}^2 - \kappa_{400}\kappa_{220}), \end{aligned}$$

so that, as k_{22} and k_{33} are expressible in terms of the magnitudes $\kappa_{\lambda\mu\nu}$, so also is k_{23}. Similarly, there are equations

$$\begin{aligned} &\{\tfrac{1}{3}k_{13}\kappa_{040} - (\kappa_{130}\kappa_{031} - \kappa_{121}\kappa_{040})\}^2 \\ &\qquad = (\tfrac{1}{3}k_{33}\kappa_{040} + \kappa_{130}^2 - \kappa_{220}\kappa_{040})(\tfrac{1}{3}k_{11}\kappa_{040} + \kappa_{031}^2 - \kappa_{040}\kappa_{022}), \\ &\{\tfrac{1}{3}k_{12}\kappa_{004} - (\kappa_{103}\kappa_{013} - \kappa_{112}\kappa_{004})\}^2 \\ &\qquad = (\tfrac{1}{3}k_{11}\kappa_{004} + \kappa_{013}^2 - \kappa_{022}\kappa_{004})(\tfrac{1}{3}k_{22}\kappa_{004} + \kappa_{103}^2 - \kappa_{202}\kappa_{004}). \end{aligned}$$

As a last instance, the relation

$$\begin{vmatrix} \sum \eta_{23}^2, & \sum \eta_{23}\eta_{13}, & \sum \eta_{23}\eta_{12} \\ \sum \eta_{23}\eta_{13}, & \sum \eta_{13}^2, & \sum \eta_{13}\eta_{12} \\ \sum \eta_{23}\eta_{12}, & \sum \eta_{13}\eta_{12}, & \sum \eta_{12}^2 \end{vmatrix} = 0$$

takes the form

$$\begin{vmatrix} \kappa_{022} - \tfrac{1}{3}k_{11}, & \kappa_{112} + \tfrac{1}{3}k_{12}, & \kappa_{122} + \tfrac{1}{3}k_{13} \\ \kappa_{112} + \tfrac{1}{3}k_{12}, & \kappa_{202} - \tfrac{1}{3}k_{22}, & \kappa_{211} + \tfrac{1}{3}k_{23} \\ \kappa_{122} + \tfrac{1}{3}k_{13}, & \kappa_{211} + \tfrac{1}{3}k_{23}, & \kappa_{220} - \tfrac{1}{3}k_{33} \end{vmatrix} = 0.$$

Magnitudes connected with the quartinormal of a region in quintuple space.

254. At this stage, and in connection with the second type of relations, we interpolate the construction of the magnitudes A_5, B_5, C_5, F_5, G_5, H_5, for a region in quintuple space. The formal expressions for the quantities $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, of a region are already known (§ 168).

As the third-order minors of the six-row determinant in § 252 vanish, the second-order minors of that determinant are connected by relations; and therefore it is desirable to select a set which may be used to frame canonical expressions. We associate the integers 1, 2, 3, 4, 5, 6, with the quantities η_{11}, η_{12}, η_{22}, η_{13}, η_{23}, η_{33}, respectively; and (as in § 168, footnote) we write

$$\sum \eta_{11}^2 = s_{11}, \quad \sum \eta_{11}\eta_{12} = s_{12}, \quad \sum \eta_{12}^2 = s_{22},$$
$$\sum \eta_{11}\eta_{22} = s_{13}, \quad \sum \eta_{12}\eta_{22} = s_{23}, \quad \sum \eta_{22}^2 = s_{33},$$

and so on; and thus, inserting the values of the quantities $\eta_{,,}$ in terms of Y and l_5, as given in § 252, we have

$$\left.\begin{aligned} \bar{A}^2 + A_5^2 = s_{11}, \quad \bar{B}^2 + B_5^2 = s_{33}, \quad \bar{C}^2 + C_5^2 = s_{66} \\ \bar{F}^2 + F_5^2 = s_{55}, \quad \bar{G}^2 + G_5^2 = s_{44}, \quad \bar{H}^2 + H_5^2 = s_{22} \end{aligned}\right\}.$$

Accordingly, let

$$\left.\begin{aligned} \bar{A} = s_{11}^{\frac{1}{2}} \cos\theta_1 \\ A_5 = s_{11}^{\frac{1}{2}} \sin\theta_1 \end{aligned}\right\}, \quad \left.\begin{aligned} \bar{H} = s_{22}^{\frac{1}{2}} \cos\theta_2 \\ H_5 = s_{22}^{\frac{1}{2}} \sin\theta_2 \end{aligned}\right\}, \quad \left.\begin{aligned} \bar{B} = s_{33}^{\frac{1}{2}} \cos\theta_3 \\ B_5 = s_{33}^{\frac{1}{2}} \sin\theta_3 \end{aligned}\right\},$$

$$\left.\begin{aligned} \bar{G} = s_{44}^{\frac{1}{2}} \cos\theta_4 \\ G_5 = s_{44}^{\frac{1}{2}} \sin\theta_4 \end{aligned}\right\}, \quad \left.\begin{aligned} \bar{F} = s_{55}^{\frac{1}{2}} \cos\theta_5 \\ F_5 = s_{55}^{\frac{1}{2}} \sin\theta_5 \end{aligned}\right\}, \quad \left.\begin{aligned} \bar{C} = s_{66}^{\frac{1}{2}} \cos\theta_6 \\ C_5 = s_{66}^{\frac{1}{2}} \sin\theta_6 \end{aligned}\right\}.$$

It will appear at once that, for a general region, no two of the angular quantities θ_1, θ_2, θ_3, θ_4, θ_5, θ_6, are equal; various possibilities can exist as regards their relative magnitudes; as a standard of reference, we shall assume

$$\theta_1 > \theta_2 > \theta_3 > \theta_4 > \theta_5 > \theta_6.$$

With these values of the quantities $\bar{A}$, A_5, and their like, we have

$$s_{12} = \sum \eta_{11}\eta_{12} = \bar{A}\bar{H} + A_5H_5 = (s_{11}s_{22})^{\frac{1}{2}} \cos(\theta_1 - \theta_2);$$

and similarly, for all six values of i and of j,

$$s_{ij}=(s_{ii}s_{jj})^{\frac{1}{2}}\cos(\theta_i-\theta_j),$$

the positive sign being implicitly associated with each of the radicals $(s_{ii})^{\frac{1}{2}}$. We shall require quantities

$$(s_{ii}s_{jj}-s_{ij}^2)^{\frac{1}{2}}=m_{ij}\,;$$

as m_{ii} is zero, and as the quantity under the radical is symmetrical in i and j, we shall take $j>i$ for all the non-vanishing quantities m_{ij}. We also take

$$m_{ij}=(s_{ii}s_{jj})^{\frac{1}{2}}\sin(\theta_i-\theta_j),$$

so that m_{ij} is a positive quantity in the standard case, a positive sign being associated with the definition of m_{ij} as a radical.

Then we have

$$\begin{aligned} s_{11}s_{23}-s_{12}s_{13} &= s_{11}(s_{22}-s_{33})^{\frac{1}{2}}\{\cos(\theta_2-\theta_3)-\cos(\theta_1-\theta_2)\cos(\theta_1-\theta_3)\}\\ &= s_{11}(s_{22}s_{33})^{\frac{1}{2}}\sin(\theta_1-\theta_2)\sin(\theta_1-\theta_3)\\ &= m_{12}m_{13}. \end{aligned}$$

More generally, choose four integers i, j, k, l, from the set 1, 2, 3, 4, 5, 6, such that

$$j>i,\quad l>k,\quad i\leqq k,\quad j\neq l,$$

the first two selections being made in connection with the significance of the quantities $m_{\lambda\mu}$, and the last two in order to avoid repetition of instances; then

$$\begin{aligned} \begin{vmatrix} s_{ik}, & s_{il} \\ s_{jk}, & s_{jl} \end{vmatrix} &= (s_{ii}s_{jj}s_{kk}s_{ll})^{\frac{1}{2}}\{\cos(\theta_i-\theta_k)\cos(\theta_j-\theta_l)-\cos(\theta_i-\theta_l)\cos(\theta_j-\theta_k)\}\\ &= (s_{ii}s_{jj}s_{kk}s_{ll})^{\frac{1}{2}}\sin(\theta_i-\theta_j)\sin(\theta_k-\theta_l)\\ &= m_{ij}m_{kl}. \end{aligned}$$

Now each of the quantities m_{ij}^2 is a diagonal minor of the second order; hence all minors of the second order, framed from the six-row determinant, are expressible in terms of diagonal minors of that order.

The last equation leads to a further identity, to be satisfied by the square roots of those diagonal minors of the second order. If the four integers i, j, k, l, satisfy the more restricted inequalities

$$l>k>j>i,$$

then the relation

$$m_{ij}m_{kl}-m_{ik}m_{jl}+m_{il}m_{jk}=0$$

is satisfied identically. There are fifteen such relations, not, however, algebraically independent.

We now can verify the values of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, expressed (§ 168) in terms

of the circular curvature and of the quantities s_{ij}, there expressed in terms of the magnitudes $\kappa_{\lambda\mu\nu}$ and $k_{\alpha\beta}$. The relation

$$A_5p'^2+2H_5p'q'+B_5q'^2+2G_5p'r'+2F_5q'r'+C_5r'^2=0$$

has been established (§ 182). When the immediately preceding values of the six coefficients of the type A_5 are inserted, the relation becomes

$$s_{11}^{\frac{1}{2}}p'^2\sin\theta_1+2s_{22}^{\frac{1}{2}}p'q'\sin\theta_2+s_{33}^{\frac{1}{2}}q'^2\sin\theta_3$$
$$+2s_{44}^{\frac{1}{2}}p'r'\sin\theta_4+2s_{55}^{\frac{1}{2}}q'r'\sin\theta_5+s_{66}^{\frac{1}{2}}r'^2\sin\theta_6=0.$$

For all the values $i=2, 3, 4, 5, 6$, we have

$$\sin\theta_i=\sin\theta_1\cos(\theta_1-\theta_i)-\cos\theta_1\sin(\theta_1-\theta_i),$$

that is,

$$(s_{11}s_{ii})^{\frac{1}{2}}\sin\theta_i=s_{1i}\sin\theta_1-m_{1i}\cos\theta_1;$$

and so the relation, on multiplication throughout by $s_{11}^{\frac{1}{2}}$, can be written

$$P\sin\theta_1-Q\cos\theta_1=0,$$

where

$$P=s_{11}p'^2+2s_{12}p'q'+s_{13}q'^2+2s_{14}p'r'+2s_{15}q'r'+s_{16}r'^2,$$
$$Q=\qquad 2m_{12}p'q'+m_{13}q'^2+2m_{14}p'r'+2m_{15}q'r'+m_{16}r'^2.$$

By repeated use of the formula

$$s_{ik}s_{jl}-s_{il}s_{jk}=m_{ij}m_{kl},$$

when we take $i=1$, $k=1$, we have

$$P^2+Q^2=s_{11}\{s_{11}p'^4+4s_{12}p'^3q'+2(s_{13}+2s_{22})p'^2q'^2+4s_{23}p'q'^3+s_{33}q'^4$$
$$+4s_{14}p'^3r'+4(s_{15}+2s_{24})p'^2q'r'+4(s_{34}+2s_{25})p'q'^2r'+4s_{35}q'^3r'$$
$$+2(s_{16}+2s_{44})p'^2r'^2+4(s_{26}+2s_{46})p'q'r'^2+2(s_{36}+2s_{55})q'^2r'^2$$
$$+4s_{46}p'r'^3+4s_{56}q'r'^3+s_{66}r'^4\}$$
$$=\frac{s_{11}}{\rho^2},$$

by the expression for the curvature in § 168. Hence

$$\frac{\cos\theta_1}{P}=\frac{\sin\theta_1}{Q}=\frac{\rho}{s_{11}^{\frac{1}{2}}},$$

and therefore

$$\frac{\bar{A}}{\rho}=\frac{s_{11}^{\frac{1}{2}}}{\rho}\cos\theta_1=P$$
$$=s_{11}p'^2+2s_{12}p'q'+s_{13}q'^2+2s_{14}p'r'+2s_{15}q'r'+s_{16}r'^2,$$

in agreement with the earlier value. Also, we have

$$\begin{aligned}\frac{A_5}{\rho}=\frac{s_{11}^{\frac{1}{2}}}{\rho}\sin\theta_1&=Q\\ &=2m_{12}p'q'+m_{13}q'^2+2m_{14}p'r'+2m_{15}q'r'+m_{16}r'^2,\end{aligned}$$

thus giving an expression for A_5.

Proceeding similarly to make θ_i (for $i=2, 3, 4, 5, 6$, in turn) the central quantity instead of θ_1 in the modified expression of the relation

$$\sum A_5 p'^2=0,$$

we verify the earlier values of $\overline{B}, \overline{C}, \overline{F}, \overline{G}, \overline{H}$, in all, and we obtain expressions for B_5, C_5, F_5, G_5, H_5. The full tale of these expressions is

$$\left.\begin{aligned}
\frac{A_5}{\rho}&= \qquad\qquad +2m_{12}p'q'+m_{13}q'^2+2m_{14}p'r'+2m_{15}q'r'+m_{16}r'^2\\
\frac{H_5}{\rho}&=-m_{12}p'^2 \qquad\qquad +m_{23}q'^2+2m_{24}p'r'+2m_{25}q'r'+m_{26}r'^2\\
\frac{B_5}{\rho}&=-m_{13}p'^2-2m_{23}p'q' \qquad\qquad +2m_{34}p'r'+2m_{35}q'r'+m_{36}r'^2\\
\frac{G_5}{\rho}&=-m_{14}p'^2-2m_{24}p'q'-m_{34}q'^2 \qquad\qquad +2m_{45}q'r'+m_{46}r'^2\\
\frac{F_5}{\rho}&=-m_{15}p'^2-2m_{25}p'q'-m_{35}q'^2-2m_{45}p'r' \qquad\qquad +m_{56}r'^2\\
\frac{C_5}{\rho}&=-m_{16}p'^2-2m_{26}p'q'-m_{36}q'^2-2m_{46}p'r'-2m_{56}q'r'
\end{aligned}\right\},$$

it being remembered, in connection with this distribution of signs with the magnitudes m_{ij}, that a conventional standard of reference for unequal quantities of the region in quintuple space has been adopted.

These quantities m_{ij} are useful in expressing the relations between the variables of the complanar directions η_{ij}. It will be convenient later (p. 202) to select the two directions η_{11} and η_{22} as the directions of reference, so that, as before (§ 252), there are quantities γ and δ such that

$$\eta_{12}=\gamma\eta_{11}+\delta\eta_{22}.$$

Then

$$s_{12}=\gamma s_{11}+\delta s_{13},\quad s_{23}=\gamma s_{13}+\delta s_{33},$$

and so

$$\begin{aligned}\gamma m_{13}^2&=s_{12}s_{33}-s_{13}s_{23}=m_{13}m_{23},\\ \delta m_{13}^2&=s_{11}s_{23}-s_{12}s_{13}=m_{12}m_{13}:\end{aligned}$$

that is,

$$m_{13}\eta_{12}=m_{23}\eta_{11}+m_{12}\eta_{22}.$$

Similarly, we find

$$m_{13}\eta_{13} = -m_{34}\eta_{11} + m_{14}\eta_{22},$$
$$m_{13}\eta_{23} = -m_{35}\eta_{11} + m_{15}\eta_{22},$$
$$m_{13}\eta_{33} = -m_{36}\eta_{11} + m_{16}\eta_{22}.$$

There are cognate relations for every pair of directions chosen in the orthogonal plane of the region in quintuple space.

The six secondary magnitudes $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, are the values of

$$\sum Y\eta_{ij},$$

for the various combinations i, j; and therefore, multiplying these various relations by Y and, for each of them, adding the products through the range of the quintuple space, we have

$$\left.\begin{aligned} m_{13}\bar{H} &= m_{23}\bar{A} + m_{12}\bar{B} \\ m_{13}\bar{G} &= -m_{34}\bar{A} + m_{14}\bar{B} \\ m_{13}\bar{F} &= -m_{35}\bar{A} + m_{15}\bar{B} \\ m_{13}\bar{C} &= -m_{36}\bar{A} + m_{16}\bar{B} \end{aligned}\right\},$$

which are relations connecting these secondary magnitudes with the quantities $\kappa_{\lambda\mu\nu}$ and $k_{\alpha\beta}$. Each such relation contains three radicals, with signs that are definite under the explicitly postulated convention on p. 197; but by means of the formula

$$s_{ik}s_{jl} - s_{il}s_{jk} = m_{ij}m_{kl}, \quad (i < j, \ k < l, \ i \leqslant k, \ j \leqslant l),$$

each of them can be made rational after multiplication by m_{13}. Thus the first of the four relations becomes

$$(s_{11}s_{33} - s_{13}^2)\bar{H} = (s_{33}s_{12} - s_{13}s_{23})\bar{A} + (s_{11}s_{23} - s_{13}s_{12})\bar{B},$$

and the last of them becomes

$$(s_{11}s_{33} - s_{13}^2)\bar{C} = (s_{33}s_{16} - s_{13}s_{36})\bar{A} + (s_{11}s_{36} - s_{13}s_{16})\bar{B}.$$

Ex. Let i, j, k, such that $i < j < k$, be any three integers from the set 1, 2, 3, 4, 5, 6, with which η_{11}, η_{12}, η_{22}, η_{13}, η_{23}, η_{33}, are respectively associated; and let $\bar{\eta}_i$, $\bar{\eta}_j$, $\bar{\eta}_k$, denote the quantities η thus selected. Shew that

$$m_{jk}\bar{\eta}_i + m_{ik}\bar{\eta}_j - m_{ij}\bar{\eta}_k = 0.$$

Locus of centre of circular curvature of concurrent geodesics of a region.

255. Now consider in detail the locus of the centre of circular curvature of concurrent geodesics belonging to a region in a quintuple plenary homaloidal space.

We have seen that the centre of curvature of any such geodesic lies in the orthogonal plane of the region. Let the spatial coordinates of any point be referred

to five new axes, the origin being at the point through which the geodesics are drawn; let three of these axes be the directions of the parametric curves in the region, and let the other two be selected directions in the orthogonal plane. For the former set of three axes, we take new quantities $\bar{U}$, $\bar{V}$, $\bar{W}$, such that

$$\sum y_1(\bar{y}-y)=A^{\frac{1}{2}}\bar{U},\quad \sum y_2(\bar{y}-y)=B^{\frac{1}{2}}\bar{V},\quad \sum y_3(\bar{y}-y)=C^{\frac{1}{2}}\bar{W};$$

and we denote by ω_{23}, ω_{31}, ω_{12}, the angles between the parametric curves, so that

$$(BC)^{\frac{1}{2}}\cos\omega_{23}=F,\quad (CA)^{\frac{1}{2}}\cos\omega_{31}=G,\quad (AB)^{\frac{1}{2}}\cos\omega_{12}=H.$$

For the latter set of two axes, we take two new quantities $\bar{X}$, $\bar{Y}$, such that

$$\sum \eta_{11}(\bar{y}-y)=s_{11}{}^{\frac{1}{2}}\bar{X},\quad \sum \eta_{22}(\bar{y}-y)=s_{33}{}^{\frac{1}{2}}\bar{Y};$$

and we denote by $\bar{\omega}$ the angle between these axes, which are orthogonal to the first three, so that

$$(s_{11}s_{33})^{\frac{1}{2}}\cos\bar{\omega}=s_{13}.$$

Of the five coordinates $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$, $\bar{x}_0$, $\bar{y}_0$, of the centre of circular curvature referred to these new axes, the first three are zero because the centre of curvature of any geodesic lies in the orthogonal plane; and the other two are given by

$$\bar{X}_0=\bar{x}_0+\bar{y}_0\cos\bar{\omega}=\frac{1}{s_{11}^{\frac{1}{2}}}\sum \eta_{11}(y_0-y),$$

$$\bar{Y}_0=\bar{x}_0\cos\bar{\omega}+\bar{y}_0=\frac{1}{s_{33}^{\frac{1}{2}}}\sum \eta_{22}(y_0-y).$$

Manifestly

$$\bar{x}_0{}^2+2\bar{x}_0\bar{y}_0\cos\bar{\omega}+\bar{y}_0{}^2=\rho^2,$$

where ρ is the length of the radius of curvature.

But, owing to the values of $\bar{X}_0$, $\bar{Y}_0$, we have

$$\frac{s_{11}^{\frac{1}{2}}}{\rho^2}(\bar{x}_0+\bar{y}_0\cos\bar{\omega})=\sum \eta_{11}\frac{y_0-y}{\rho^2}$$

$$=\sum \eta_{11}\frac{Y}{\rho}=s_{11}p'^2+2s_{12}p'q'+s_{13}q'^2+2s_{14}p'r'+2s_{15}q'r'+s_{16}r'^2,$$

and

$$\frac{s_{33}^{\frac{1}{2}}}{\rho^2}(\bar{x}_0\cos\bar{\omega}+\bar{y}_0)=\sum \eta_{22}\frac{Y}{\rho}=s_{13}p'^2+2s_{23}p'q'+s_{33}q'^2+2s_{34}p'r'+2s_{35}q'r'+s_{36}r'^2.$$

Moreover, there is the permanent relation

$$Ap'^2+2Hp'q'+Bq'^2+2Gp'r'+2Fq'r'+Cr'^2=1.$$

Manifestly these three equations do not determine a locus (in the form of a curve) for $\bar{x}_0$, $\bar{y}_0$, in the orthogonal plane; they assign definite unique values to those coordinates for each set of values of p', q', r', satisfying the permanent relation. In fact, the full aggregate of values for $\bar{x}_0$, $\bar{y}_0$, is two-fold in range, instead of the one-fold range proper to a curve.

Let a single additional limitation be imposed upon p', q', r'; and then, instead of the two-fold range, there results the one-fold range proper to a curve. In particular, let the initial direction p', q', r', of the geodesic be contained in a regional orientation ξ, η, ζ, so that

$$\xi p'+\eta q'+\zeta r'=0.$$

When this is associated with the preceding three relations, there are four equations in all: the three variables p', q', r', can be eliminated. The eliminant involves $\bar{x}_0$ and $\bar{y}_0$, as well as the magnitudes of the equation; and therefore it represents a curve in the orthogonal plane, which in fact is the locus of the centres of circular curvature of regional geodesics originating in the orientation ξ, η, ζ. To frame the eliminant, let

$$t_1=\frac{s_{11}^{\frac{1}{2}}}{\rho^2}(\bar{x}_0+\bar{y}_0\cos\bar{\omega}),\quad t_2=\frac{s_{33}^{\frac{1}{2}}}{\rho^2}(\bar{x}_0\cos\bar{\omega}+\bar{y}_0),$$

$$\begin{aligned}U_1&=(a_1, b_1, c_1, f_1, g_1, h_1 \between p', q', r')^2\\&=(s_{11}-t_1A,\ s_{13}-t_1B,\ s_{16}-t_1C,\ s_{15}-t_1F,\ s_{14}-t_1G,\ s_{12}-t_1H \between p', q', r')^2,\\U_2&=(a_2, b_2, c_2, f_2, g_2, h_2 \between p', q', r')^2\\&=(s_{13}-t_2A,\ s_{33}-t_2B,\ s_{36}-t_2C,\ s_{35}-t_2F,\ s_{34}-t_2G,\ s_{23}-t_2H \between p', q', r')^2;\end{aligned}$$

then we have to eliminate p', q', r', between the three homogeneous equations

$$U_1=0,\quad U_2=0,\quad \xi p'+\eta q'+\zeta r'=0.$$

Let Θ_1, Θ_2, Θ_3, denote the three contravariants

$$\begin{aligned}\Theta_1&=\sum\{(b_1c_1-f_1^2)\xi^2\},\\\Theta_2&=\sum\{(b_2c_2-f_2^2)\xi^2\},\\\Theta_{12}&=\sum\{(b_1c_2-2f_1f_2+c_1b_2)\xi^2\},\end{aligned}$$

in the system of ternariants; the required eliminant is

$$\Theta_{12}^2=4\Theta_1\Theta_2,$$

and it is necessary to develop this equation, so far as to shew its character in the variables $\bar{x}_0$, $\bar{y}_0$.

Let

$$\Phi_1 = \sum\{(s_{13}s_{16} - s_{15}^2)\xi^2\}, \quad \Phi_2 = \sum\{(s_{33}s_{36} - s_{35}^2)\xi^2\},$$
$$2\Psi_1 = \sum\{(s_{13}C - 2s_{15}F + s_{16}B)\xi^2\},$$
$$2\Psi_2 = \sum\{(s_{33}C - 2s_{35}F + s_{36}B)\xi^2\},$$
$$2\Phi_{12} = \sum\{(s_{13}s_{36} - 2s_{15}s_{35} + s_{16}s_{33})\xi^2\},$$

these quantities being contravariants in the system of the three simultaneous ternary forms

$$\sum s_{11}p'^2, \quad \sum s_{13}p'^2, \quad \sum Ap'^2.$$

Moreover, we have

$$\sum(BC - F^2)\xi^2 = \sum a\xi^2 = 1,$$

also a contravariant of the system. By actual substitution,

$$\Theta_1 = \Phi_1 - 2t_1\Psi_1 + t_1^2,$$
$$\Theta_2 = \Phi_2 - 2t_2\Psi_2 + t_2^2,$$
$$\tfrac{1}{2}\Theta_{12} = \Phi_{12} - t_1\Psi_2 - t_2\Psi_1 + t_1t_2\,;$$

and the developed eliminant-equation is found to be

$$t_1^2(\Phi_2 - \Psi_2^2) - 2t_1t_2(\Phi_{12} - \Psi_1\Psi_2) + t_2^2(\Phi_1 - \Psi_1^2)$$
$$+ 2t_1(\Phi_{12}\Psi_2 - \Phi_2\Psi_1) + 2t_2(\Phi_{12}\Psi_1 - \Phi_1\Psi_2) + \Phi_1\Phi_2 - \Phi_{12}^2 = 0.$$

Various forms can be given to this equation. For the present purpose, it is sufficient to notice that, because

$$t_1 = \frac{s_{11}^{\frac{1}{2}}}{\rho^2}(\bar{x}_0 + \bar{y}_0 \cos\bar{\omega}), \quad t_2 = \frac{s_{33}^{\frac{1}{2}}}{\rho^2}(\bar{x}_0 \cos\bar{\omega} + \bar{y}_0),$$

the equation is of the form

$$\rho^4 + \rho^2 u_1 + u_2 = 0,$$

where $\rho^2 = \bar{x}_0^2 + 2\bar{x}_0\bar{y}_0 \cos\bar{\omega} + \bar{y}_0^2$, the inclination of the axes in the orthogonal plane being $\bar{\omega}$, while u_1 is homogeneous of the first degree in $\bar{x}_0$ and $\bar{y}_0$, and u_2 is homogeneous of the second degree in $\bar{x}_0$ and $\bar{y}_0$.

It thus appears that, for regional geodesics through a point in the region which originate in an assigned orientation at the point, the locus of the centres of circular curvature is a lemniscate curve in the orthogonal plane of the region at the point, the parameters in the equation of the curve depending on the variables of the assigned orientation. Further, for all regional geodesics through a point, the aggregate configuration formed by their centres of circular curvature is the area in the orthogonal plane bounded by the complete envelope of all the lemniscate curves.

We note that the foregoing equation in t_1 and t_2 can be expressed in the form

$$\begin{vmatrix} \Phi_1, & \Phi_{12}, & \Psi_1, & t_1 \\ \Phi_{12}, & \Phi_2, & \Psi_2, & t_2 \\ \Psi_1, & \Psi_2, & 1, & -1 \\ t_1, & t_2, & -1, & 0 \end{vmatrix} = 0;$$

and it thus suggests the known property that the pedal of a conic with respect to any point in its plane is a lemniscate curve *.

It may be pointed out that, by taking the orientation ξ, η, ζ, to be that of a parametric surface through O, we effectively have the locus of the centre of circular curvature of regional geodesics touching this surface. The foregoing analysis is applicable throughout, by use of the relations

$$\frac{\xi}{\theta_1} = \frac{\eta}{\theta_2} = \frac{\zeta}{\theta_3},$$

and transforming the equation in t_1 and t_2 which is homogeneous in ξ, η, ζ. The locus in question is a lemniscate curve in the orthogonal plane of the region.

256. It is natural to enquire whether there is any locus in the orthogonal plane of a region in quintuple space, corresponding to Kommerell's conic in the orthogonal plane of a surface existing freely in quadruple space. For the purpose, we use both forms of the (three) equations of the orthogonal plane,

$$\sum y_1(\bar{y}-y)=0, \quad \sum y_2(\bar{y}-y)=0, \quad \sum y_3(\bar{y}-y)=0,$$

where no special account of the dimensionality of the space appears, and

$$\| \bar{y}-y, \quad Y, \quad l_5 \| = 0,$$

where there is implicitly an assumption that the plenary space is quintuple.

To obtain the equations for the intersection (if any) of this plane with the orthogonal plane of the region at a consecutive point along the geodesic, we associate, with the three equations, the three additional equations of the type

$$\sum y_1'(\bar{y}-y) - \sum y_1 y' = 0.$$

But $\sum y_1 y' = u_1$; and

$$\begin{aligned} y_1' &= y_{11}p' + y_{12}q' + y_{13}r' \\ &= \eta_1 + y_1\alpha_1 + y_2\xi_1 + y_3\phi_1 \\ &= Yv_1 + l_5u_5 + y_1\alpha_1 + y_2\xi_1 + y_3\phi_1. \end{aligned}$$

When this value is substituted, and when the equations of the initial orthogonal

* The result is similar to the result for the centro-locus of concurrent superficial geodesics, when the surface exists in a quadruple plenary homaloidal space: see *G.F.D.*, vol. i, § 242.

plane are used in connection with the possible intersection, the cited additional equation becomes

$$v_1 \sum \{Y(\bar{y}-y)\} + u_5 \sum \{l_5(\bar{y}-y)\} - u_1 = 0.$$

Similarly, the other two additional equations, to be associated with the initial equation, are

$$v_2 \sum Y(\bar{y}-y) + v_5 \sum l_5(\bar{y}-y) - u_2 = 0,$$
$$v_3 \sum Y(\bar{y}-y) + w_5 \sum l_5(\bar{y}-y) - u_3 = 0.$$

Thus there are six linear equations in all, not all of them homogeneous, to be satisfied by the five magnitudes $\bar{y}-y$; hence they cannot be satisfied unconditionally.

To obtain values (if any) of the magnitudes, we use the second form of the equations of the original orthogonal plane. Any point in it can be represented by

$$\bar{y}-y = \lambda Y + \mu l_5,$$

where λ and μ are parameters; and, for all such points,

$$\sum Y(\bar{y}-y) = \lambda, \quad \sum l_5(\bar{y}-y) = \mu.$$

Thus the possible intersections are given by

$$\lambda v_1 + \mu u_5 - u_1 = 0,$$
$$\lambda v_2 + \mu v_5 - u_2 = 0,$$
$$\lambda v_3 + \mu w_5 - u_3 = 0,$$

three equations with two unknowns.

We first multiply by p', q', r', and add: then we find

$$\lambda = \rho,$$

a result to be expected in connection with the orthogonal plane. Next, the elimination of λ and μ leads to the relation

$$\begin{vmatrix} u_5, & u_1, & v_1 \\ v_5, & u_2, & v_2 \\ w_5, & u_3, & v_3 \end{vmatrix} = 0;$$

and therefore (§ 182)

$$\frac{1}{\sigma\kappa} = 0.$$

If $1/\sigma = 0$, the direction is that of a curve of circular curvature in the region. If $1/\kappa$, the direction is that of a curve of globular curvature in the region. In either case, the direction of the regional geodesic must be special, if consecutive orthogonal planes of the region in quintuple space are to intersect.

The result can be established also as follows. Because the tangent, the

binormal, and the trinormal, of a geodesic lie in the tangent flat of the region, the orthogonal plane of the region can be represented by the equations

$$\sum(\bar{y}-y)y'=0, \quad \sum(\bar{y}-y)l_3=0, \quad \sum(\bar{y}-y)l_4=0.$$

The three additional equations, which must be associated with these three, for the determination of an orthogonal centre (if any) when the plenary space is quintuple, are

$$\sum(\bar{y}-y)y''-\sum y'^2=0,$$
$$\sum(\bar{y}-y)\frac{dl_3}{ds}-\sum y'l_3=0,$$
$$\sum(\bar{y}-y)\frac{dl_4}{ds}-\sum y'l_4=0.$$

The first of these is

$$\sum(\bar{y}-y)Y=\rho.$$

The second of them is

$$\frac{1}{\sigma}\sum(\bar{y}-y)Y=\frac{1}{\tau}\sum(\bar{y}-y)l_4:$$

that is, when all the equations are combined,

$$\frac{\rho}{\sigma}=0.$$

The third of them is

$$\frac{1}{\kappa}\sum(\bar{y}-y)l_5=\frac{1}{\tau}\sum(\bar{y}-y)l_3:$$

that is, when all the equations are combined

$$\frac{1}{\kappa}\sum(\bar{y}-y)l_5=0.$$

Now ρ does not vanish; and therefore

$$\frac{1}{\sigma}=0.$$

Assuming that we have not to deal with an isolated point on a geodesic where the torsion may be zero, this relation requires the geodesic to be a curve of curvature of the region (§ 190).

The fifth equation can be satisfied by

$$\frac{1}{\kappa}=0,$$

a result necessarily holding when the plenary space is quadruple; but the orthogonal homaloid of the region is then merely the prime normal of any geodesic.

If the coil be not zero (we may assume that we are not dealing with an isolated point where the coil of a geodesic happens to vanish), then the fifth equation becomes

$$\sum (\bar{y}-y) l_5 = 0.$$

As the original three equations of the orthogonal homaloid can be taken in the form

$$\| \bar{y}-y, \quad Y, \quad l_5 \| = 0,$$

that is,

$$\bar{y}-y = \lambda Y + \mu l_5,$$

when the plenary space is quintuple, we see at once that the intersection is given by

$$\bar{y}-y = Y\rho:$$

that is, even in the exceptional instance when there is a geodesic line of curvature of a region in a plenary quintuple space, the intersection of the consecutive orthogonal planes is the centre of circular curvature of the geodesic.

257. Next, consider the locus of the centre of circular curvature of regional geodesics when the plenary homaloidal space of the region is sextuple. The orthogonal homaloid of such a region is a flat.

As before (§ 255), we change the spatial axes of coordinates, so that three of them coincide with the directions of the parametric curves at O, by taking

$$\sum y_1(\bar{y}-y) = A^{\frac{1}{2}}\bar{U}, \quad \sum y_2(\bar{y}-y) = B^{\frac{1}{2}}\bar{V}, \quad \sum y_3(\bar{y}-y) = C^{\frac{1}{2}}\bar{W}.$$

Three of the coordinates for the centro-locus then become

$$\bar{U}=0, \quad \bar{V}=0, \quad \bar{W}=0.$$

The six directions indicated by the set of six quantities η_{ij}, from the six combinations i, j, lie in the orthogonal homaloid of the region. They therefore lie in the flat when the plenary space is sextuple, so that three of them can be taken as directions of axial reference provided the chosen three be non-complanar. We shall take the directions indicated by the three quantities $\eta_{11}, \eta_{22}, \eta_{33}$, for these new axes; and we write

$$\bar{X} s_{11}^{\frac{1}{2}} = \sum \eta_{11}(\bar{y}-y), \quad \bar{Y} s_{33}^{\frac{1}{2}} = \sum \eta_{22}(\bar{y}-y), \quad \bar{Z} s_{66}^{\frac{1}{2}} = \sum \eta_{33}(\bar{y}-y).$$

We denote by k_1, k_2, k_3, the cosines of the respective angles between these axes in pairs, so that

$$k_1 (s_{33}s_{66})^{\frac{1}{2}} = \sum \eta_{22}\eta_{33} = s_{36},$$
$$k_2 (s_{66}s_{11})^{\frac{1}{2}} = \sum \eta_{33}\eta_{11} = s_{16},$$
$$k_3 (s_{11}s_{33})^{\frac{1}{2}} = \sum \eta_{11}\eta_{22} = s_{13};$$

and now, if $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$, are the coordinates of the centre of circular curvature, referred to the axes in the flat, we have

$$\left.\begin{aligned} X_0 &= \bar{x}_0 + k_3\bar{y}_0 + k_2\bar{z}_0 = \frac{1}{s_{11}^{\frac{1}{2}}}\sum \eta_{11}(y_0 - y) \\ Y_0 &= k_3\bar{x}_0 + \bar{y}_0 + k_1\bar{z}_0 = \frac{1}{s_{33}^{\frac{1}{2}}}\sum \eta_{22}(y_0 - y) \\ Z_0 &= k_2\bar{x}_0 + k_1\bar{y}_0 + \bar{z}_0 = \frac{1}{s_{66}^{\frac{1}{2}}}\sum \eta_{33}(y_0 - y) \end{aligned}\right\},$$

while

$$\rho^2 = \bar{x}_0^2 + \bar{y}_0^2 + \bar{z}_0^2 + 2k_1\bar{y}_0\bar{z}_0 + 2k_2\bar{z}_0\bar{x}_0 + 2k_3\bar{x}_0\bar{y}_0.$$

We shall write

$$t_1 = \frac{s_{11}^{\frac{1}{2}}X_0}{\rho^2}, \quad t_2 = \frac{s_{33}^{\frac{1}{2}}Y_0}{\rho^2}, \quad t_3 = \frac{s_{66}^{\frac{1}{2}}Z_0}{\rho^2},$$

so that t_1, t_2, t_3, are explicit algebraic functions of $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$. From the relation $y_0 - y = Y\rho$, for the typical space-coordinate of the centre of circular curvature referred to the earlier spatial axes, we have

$$\begin{aligned} t_1 &= s_{11}p'^2 + 2s_{12}p'q' + s_{13}q'^2 + 2s_{14}p'r' + 2s_{15}q'r' + s_{16}r'^2, \\ t_2 &= s_{13}p'^2 + 2s_{23}p'q' + s_{33}q'^2 + 2s_{34}p'r' + 2s_{35}q'r' + s_{36}r'^2, \\ t_3 &= s_{16}p'^2 + 2s_{26}p'q' + s_{36}q'^2 + 2s_{46}p'r' + 2s_{56}q'r' + s_{66}r'^2; \end{aligned}$$

and there is the permanent relation

$$Ap'^2 + 2Hp'q' + Bq'^2 + 2Gp'r' + 2Fq'r' + Cr'^2 = 1.$$

The elimination of the three quantities p', q', r', among these four equations leads to the equation of the locus of the centre of circular curvature, the locus being referred to the selected axes in the orthogonal flat.

To construct the eliminant, we form the homogeneous ternary quantics of the second order

$$\begin{aligned} U_1 &= \sum(s_{11}p'^2) - t_1\sum(Ap'^2) = \sum\{(s_{11} - t_1A)p'^2\} = 0, \\ U_2 &= \sum(s_{13}p'^2) - t_2\sum(Ap'^2) = \sum\{(s_{13} - t_2A)p'^2\} = 0, \\ U_3 &= \sum(s_{16}p'^2) - t_3\sum(Ap'^2) = \sum\{(s_{16} - t_3A)p'^2\} = 0; \end{aligned}$$

and, to frame the eliminant of these three ternary homogeneous quadratic equations, we use the dialytic process due to Sylvester *. Let J denote the

* *Coll. Math. Papers*, vol. i, pp. 61-65, 298-302; see also Salmon's *Higher Algebra*, (2nd. edn., 1866), §§ 85, 86.

Jacobian of U_1, U_2, U_3, so that

$$J=\tfrac{1}{8}\cdot\frac{\partial(U_1, U_2, U_3)}{\partial(p', q', r')}=(e \between p', q', r')^3,$$

where these coefficients e_{ijk} are linear in the coefficients of each of the quantities. Then, for values of p', q', r', that make U_1, U_2, U_3, vanish simultaneously, not merely does J vanish, but also (*l.c.*) we have

$$\frac{\partial J}{\partial p'}=0, \quad \frac{\partial J}{\partial q'}=0, \quad \frac{\partial J}{\partial r'}=0:$$

and as

$$3J=p'\frac{\partial J}{\partial p'}+q'\frac{\partial J}{\partial q'}+r'\frac{\partial J}{\partial r'},$$

the last three of these four relations secure also the relation $J=0$. Thus

$$\begin{aligned}
e_{111}p'^2+2e_{112}p'q'+e_{122}q'^2+2e_{113}p'r'+2e_{123}q'r'+e_{133}r'^2&=0,\\
e_{112}p'^2+2e_{122}p'q'+e_{222}q'^2+2e_{123}p'r'+2e_{223}q'r'+e_{233}r'^2&=0,\\
e_{113}p'^2+2e_{123}p'q'+e_{223}q'^2+2e_{133}p'r'+2e_{233}q'r'+e_{333}r'^2&=0.
\end{aligned}$$

Also we write

$$U_i=a_ip'^2+2h_ip'q'+b_iq'^2+2g_ip'r'+2f_iq'r'+c_ir'^2=0,$$

for $i=1, 2, 3$. There now are six equations, each linear and homogeneous in the six quantities p'^2, $p'q'$, q'^2, $p'r'$, $q'r'$, r'^2; consequently their eliminant is

$$S=\begin{vmatrix}
e_{111}, & e_{112}, & e_{122}, & e_{113}, & e_{123}, & e_{133}\\
e_{112}, & e_{122}, & e_{222}, & e_{123}, & e_{223}, & e_{233}\\
e_{113}, & e_{123}, & e_{223}, & e_{133}, & e_{233}, & e_{333}\\
a_1, & h_1, & b_1, & g_1, & f_1, & c_1\\
a_2, & h_2, & b_2, & g_2, & f_2, & c_2\\
a_3, & h_3, & b_3, & g_3, & f_3, & c_3
\end{vmatrix}=0.$$

Accordingly this equation represents the locus of the centre of circular curvature of regional geodesics when the plenary space is sextuple. It is a single equation involving the three coordinates $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$, referred to the orthogonal flat of the region; and therefore the locus is a surface.

The degree of the surface can be inferred by considering representative groups of terms in the expansion of this determinant: such a group is the aggregate contained in the quantity

$$\begin{vmatrix}
a_1, & h_1, & g_1\\
a_2, & h_2, & g_2\\
a_3, & h_3, & g_3
\end{vmatrix}
\begin{vmatrix}
e_{122}, & e_{123}, & e_{133}\\
e_{222}, & e_{223}, & e_{233}\\
e_{223}, & e_{233}, & e_{333}
\end{vmatrix}.$$

By actual expansion after the respective values have been substituted, we find, for the first factor,

$$\begin{vmatrix} a_1, & h_1, & g_1 \\ a_2, & h_2, & g_2 \\ a_3, & h_3, & g_3 \end{vmatrix} - \begin{vmatrix} s_{11}, & s_{12}, & s_{14} \\ s_{13}, & s_{23}, & s_{34} \\ s_{16}, & s_{26}, & s_{46} \end{vmatrix}$$

$$= -t_1 \begin{vmatrix} A, & H, & G \\ s_{13}, & s_{23}, & s_{34} \\ s_{16}, & s_{26}, & s_{46} \end{vmatrix} - t_2 \begin{vmatrix} A, & H, & G \\ s_{16}, & s_{26}, & s_{46} \\ s_{11}, & s_{12}, & s_{14} \end{vmatrix} - t_3 \begin{vmatrix} A, & H, & G \\ s_{11}, & s_{12}, & s_{14} \\ s_{13}, & s_{23}, & s_{34} \end{vmatrix}.$$

To obtain the character of the second factor, it is necessary to revert to the form of J, in which the quantities e are coefficients. We write

$$\left.\begin{aligned} s_{i1}p' + s_{i2}q' + s_{i4}r' &= \lambda_1, \lambda_2, \lambda_3 \\ s_{i2}p' + s_{i3}q' + s_{i5}r' &= \mu_1, \mu_2, \mu_3 \\ s_{i4}p' + s_{i5}q' + s_{i6}r' &= \nu_1, \nu_2, \nu_3 \end{aligned}\right\}$$

respectively, when $i = 1, 3, 6$; and now, with the usual significance for u_1, u_2, u_3, we have

$$J = \begin{vmatrix} \lambda_1 - t_1u_1, & \mu_1 - t_1u_2, & \nu_1 - t_1u_3 \\ \lambda_2 - t_2u_1, & \mu_2 - t_2u_2, & \nu_2 - t_2u_3 \\ \lambda_3 - t_3u_1, & \mu_3 - t_3u_2, & \nu_3 - t_3u_3 \end{vmatrix};$$

and therefore

$$J = \begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \lambda_3, & \mu_3, & \nu_3 \end{vmatrix} - t_1 \begin{vmatrix} u_1, & \mu_1, & \nu_1 \\ u_2, & \mu_2, & \nu_2 \\ u_3, & \mu_3, & \nu_3 \end{vmatrix} - t_2 \begin{vmatrix} \lambda_1, & u_1, & \nu_1 \\ \lambda_2, & u_2, & \nu_2 \\ \lambda_3, & u_3, & \nu_3 \end{vmatrix} - t_3 \begin{vmatrix} \lambda_1, & \mu_1, & u_1 \\ \lambda_2, & \mu_2, & u_2 \\ \lambda_3, & \mu_3, & u_3 \end{vmatrix}.$$

Thus J is a linear non-homogeneous function of t_1, t_2, t_3. Hence each of the coefficients e in the expanded form of J is a linear non-homogeneous function of t_1, t_2, t_3; the part of each coefficient e, which is independent of t_1, t_2, t_3, is of the third degree in the quantities s_{lm}, while the parts involving t_1, t_2, t_3, linearly are of the second degree in those quantities s_{lm}. It follows that a factor of the type

$$\begin{vmatrix} e_{122}, & e_{123}, & e_{133} \\ e_{222}, & e_{223}, & e_{233} \\ e_{223}, & e_{233}, & e_{333} \end{vmatrix}$$

is an expression which is of degree three in the variables t_1, t_2, t_3, but is not homogeneous in those variables.

Consequently the eliminant S, the sum of the products of factors of these two types, is of degree four in the variables t_1, t_2, t_3, but is not homogeneous in those variables. Having regard to the values of t_1, t_2, t_3, in terms of the coordinates $\bar{x}_0, \bar{y}_0, \bar{z}_0$, referred to the orthogonal flat, we see that the equation of the surface

$S=0$, which is the locus of the centre of circular curvature of concurrent regional geodesics when the plenary space of the region is sextuple, has the form

$$E_0+\frac{E_1}{\rho^2}+\frac{E_2}{\rho^4}+\frac{E_3}{\rho^6}+\frac{E_4}{\rho^8}=0,$$

where

$$\rho^2=\bar{x}_0{}^2+\bar{y}_0{}^2+\bar{z}_0{}^2+2k_1\bar{y}_0\bar{z}_0+2k_2\bar{z}_0\bar{x}_0+2k_3\bar{x}_0\bar{y}_0,$$

and where E_i is homogeneous of degree i in $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$, for $i=0, 1, 2, 3, 4$. Thus the centro-surface is of degree eight, and it has a conical point (real or imaginary) of the fourth order at the origin O in the region.

Magnitudes of a region in sextuple space.

258. There are questions, corresponding to those in § 254, to be answered when the plenary homaloidal space of a region is sextuple. In particular, the orthogonal homaloid of the region then is a flat; and thus there can be only three independent directions in that homaloid. All the directions indicated by the set of six magnitudes η_{ij} (for the six combinations ij) lie in the homaloid; and therefore all these six magnitudes are expressible in terms of any three of them, so that all the determinants

$$\begin{Vmatrix} \eta_{ij}^{(a)} \\ \eta_{kl}^{(a)} \\ \eta_{mn}^{(a)} \\ \eta_{\lambda\mu}^{(a)} \end{Vmatrix}$$

(where there are six columns in the array, for $a=1, 2, 3, 4, 5, 6$, corresponding to the six space-coordinates) vanish. Therefore also the quantities

$$\sum \begin{vmatrix} \eta_{ij}^{(1)}, & \eta_{ij}^{(2)}, & \eta_{ij}^{(3)}, & \eta_{ij}^{(4)} \\ \eta_{kl}^{(1)}, & \eta_{kl}^{(2)}, & \eta_{kl}^{(3)}, & \eta_{kl}^{(4)} \\ \eta_{mn}^{(1)}, & \eta_{mn}^{(2)}, & \eta_{mn}^{(3)}, & \eta_{mn}^{(4)} \\ \eta_{\lambda\mu}^{(1)}, & \eta_{\lambda\mu}^{(2)}, & \eta_{\lambda\mu}^{(3)}, & \eta_{\lambda\mu}^{(4)} \end{vmatrix}^2$$

vanish: that is, every minor of order four in the six-row determinant in § 252 vanishes.

The same result can be obtained thus. As the plenary space is sextuple, the orthogonal flat contains three principal lines of any regional geodesic, the prime normal, the fourth normal, and the fifth normal, with typical direction-cosines Y, l_5, l_6, respectively. The differential equations of the second order satisfied by the point-coordinates thus (§ 184) assume the typical form

$$\eta_{11}=Y\bar{A}+l_5A_5+l_6A_6,\quad \eta_{22}=Y\bar{B}+l_5B_5+l_6B_6,\quad \eta_{33}=Y\bar{C}+l_5C_5+l_6C_6,$$
$$\eta_{23}=Y\bar{F}+l_5F_5+l_6F_6,\quad \eta_{31}=Y\bar{G}+l_5G_5+l_6G_6,\quad \eta_{12}=Y\bar{H}+l_5H_5+l_6H_6.$$

The constituents of the six-row determinant thus have values

$$s_{11}=\bar{A}^2+A_5{}^2+A_6{}^2,\quad s_{12}=\bar{A}\bar{H}+A_5H_5+A_6H_6,$$

and so on. Now any determinant such as

$$\begin{vmatrix} P^2+\ \ P_5{}^2+P_6{}^2\ , & PQ+P_5Q_5+P_6Q_6, & PR+P_5R_5+P_6R_6, & PS+P_5S_5+P_6S_6 \\ PQ+P_5Q_5+P_6Q_6, & Q^2+\ \ Q_5{}^2+Q_6{}^2\ , & QR+Q_5R_5+Q_6R_6, & QS+Q_5S_5+Q_6S_6 \\ PR+P_5R_5+P_6R_6, & QR+Q_5R_5+Q_6R_6, & R^2+\ \ R_5{}^2+R_6{}^2\ , & RS+R_5S_5+R_6S_6 \\ PS+P_5S_5+P_6S_6, & QS+Q_5S_5+Q_6S_6, & RS+R_5S_5+R_6S_6, & R^2+\ \ R_5{}^2+R_6{}^2 \end{vmatrix}$$

vanishes identically: that is, in connection with the six-row determinant in § 252, every fourth-order minor vanishes.

Thus there must exist relations among the third-order minors of the determinant. For example, we have, always,

$$\begin{vmatrix} t_{11}, & t_{12}, & t_{13} \\ t_{12}, & t_{22}, & t_{23} \\ t_{13}, & t_{23}, & t_{33} \end{vmatrix} \begin{vmatrix} t_{11}, & t_{12}, & t_{14} \\ t_{12}, & t_{22}, & t_{24} \\ t_{14}, & t_{24}, & t_{44} \end{vmatrix} - \begin{vmatrix} t_{11}, & t_{12}, & t_{14} \\ t_{12}, & t_{22}, & t_{24} \\ t_{13}, & t_{23}, & t_{34} \end{vmatrix}^2 = \begin{vmatrix} t_{11}, & t_{12} \\ t_{12}, & t_{22} \end{vmatrix} \begin{vmatrix} t_{11}, & t_{12}, & t_{13}, & t_{14} \\ t_{12}, & t_{22}, & t_{23}, & t_{24} \\ t_{13}, & t_{23}, & t_{33}, & t_{34} \\ t_{14}, & t_{24}, & t_{34}, & t_{44} \end{vmatrix}$$

identically; hence, when the four-row determinant on the right-hand side vanishes, the vanishing left-hand side provides a relation between the three-row determinants. For a more concise expression of these third-order minors (as indeed of minors of other orders), the following notation is adopted. With the columns in succession from left to right, we associate the integers 1, 2, 3, 4, 5, 6: and with the rows in succession from top to bottom, we associate the same integers: then minors are represented, to take types,

$$\begin{vmatrix} s_{il}, & s_{im} \\ s_{jl}, & s_{jm} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} i\ j \\ l\ m \end{vmatrix},$$

$$\begin{vmatrix} s_{il}, & s_{im}, & s_{in} \\ s_{jl}, & s_{jm}, & s_{jn} \\ s_{kl}, & s_{km}, & s_{kn} \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} i\ j\ k \\ l\ m\ n \end{vmatrix},$$

and so on, the upper line and the lower line in the symbol being interchangeable. Thus, when we take $t_{\alpha\beta}=s_{\alpha\beta}$ in the preceding identity, we obtain a relation

$$\begin{vmatrix} 1\ 2\ 3 \\ 1\ 2\ 3 \end{vmatrix} \begin{vmatrix} 1\ 2\ 4 \\ 1\ 2\ 4 \end{vmatrix} = \begin{vmatrix} 1\ 2\ 3 \\ 1\ 2\ 4 \end{vmatrix}^2.$$

More generally, we have relations of the forms

$$\begin{vmatrix} i\ j \\ i\ j \end{vmatrix} \begin{vmatrix} i\ k \\ i\ l \end{vmatrix} - \begin{vmatrix} i\ j \\ i\ k \end{vmatrix} \begin{vmatrix} i\ j \\ i\ l \end{vmatrix} = s_{ii} \begin{vmatrix} i\ j\ k \\ i\ j\ l \end{vmatrix},$$

$$\begin{vmatrix} i\ j\ k \\ i\ j\ m \end{vmatrix} \begin{vmatrix} i\ j\ l \\ i\ j\ n \end{vmatrix} = \begin{vmatrix} i\ j\ k \\ i\ j\ n \end{vmatrix} \begin{vmatrix} i\ j\ l \\ i\ j\ m \end{vmatrix},$$

whatever be the integers i, j, k, l, m, n, chosen from 1, 2, 3, 4, 5, 6, the first of which is valid universally, and the second of which is valid because the fourth-order minors vanish.

By means of these relations, used in connection with the equations of the type expressing s_{ij} in terms of the non-gremial magnitudes $\bar{A}$, A_5, H_5, and so on, and having regard to the relations

$$\sum A_5 p'^2 = 0,$$

$$A_6 p' + A_6 q' + G_6 r' = 0, \quad H_6 p' + B_6 q' + F_6 r' = 0, \quad G_6 p' + F_6 q' + C_6 r' = 0,$$

we can obtain the values of the quantities such as A_5, A_6, in terms of the magnitudes s_{ij}, the values of the secondary magnitudes $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, being known (§ 168). We have

$$\begin{aligned}\frac{1}{\rho^2} = {} & s_{11} p'^4 + 4 s_{12} p'^3 q' + 4 s_{14} p'^3 r' \\ & + (2 s_{13} + 4 s_{22}) p'^2 q'^2 + (4 s_{15} + 8 s_{24}) p'^2 q' r' + (2 s_{16} + 4 s_{44}) p'^2 r'^2 \\ & + 4 s_{23} p' q'^3 + (8 s_{25} + 4 s_{34}) p' q'^2 r' + (4 s_{26} + 8 s_{45}) p' q' r'^2 + 4 s_{46} p' r'^3 \\ & + s_{33} q'^4 + 4 s_{35} q'^3 r' + (2 s_{36} + 4 s_{55}) q'^2 r'^2 + 4 s_{56} q' r'^3 + s_{66} r'^4,\end{aligned}$$

and

$$\frac{\bar{A}}{\rho} = s_{11} p'^2 + 2 s_{12} p' q' + s_{13} q'^2 + 2 s_{14} p' r' + 2 s_{15} q' r' + s_{16} r'^2.$$

Then, as

$$s_{11} = \bar{A}^2 + A_5^2 + A_6^2,$$

we have

$$\begin{aligned}\frac{1}{\rho^2}(A_5^2 + A_6^2) &= \frac{s_{11}}{\rho^2} - \left(\frac{\bar{A}}{\rho}\right)^2 \\ &= 4 p'^2 q'^2 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + 8 p'^2 q' r' \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} + 4 p'^2 r'^2 \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix} \\ &\quad + 4 p' q'^3 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + p' q'^2 r' \left\{ 4 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 8 \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \right\} \\ &\quad + p' q' r'^2 \left\{ 4 \begin{vmatrix} 1 & 2 \\ 1 & 6 \end{vmatrix} + 8 \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} \right\} + 4 p' r'^3 \begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix} \\ &\quad + q'^4 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 4 q'^3 r' \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} + q'^2 r'^2 \left\{ 2 \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} + 4 \begin{vmatrix} 1 & 5 \\ 1 & 5 \end{vmatrix} \right\} \\ &\quad + 4 q' r'^3 \begin{vmatrix} 1 & 5 \\ 1 & 6 \end{vmatrix} + r'^4 \begin{vmatrix} 1 & 6 \\ 1 & 6 \end{vmatrix}.\end{aligned}$$

(It may be remarked, in passing, that the right-hand side can be expressed in the form

$$\frac{P_A{}^2}{\begin{vmatrix}1\,2\\1\,2\end{vmatrix}}+\frac{s_{11}\,Q_A{}^2}{\begin{vmatrix}1\,2\\1\,2\end{vmatrix}\begin{vmatrix}1\,2\,3\\1\,2\,3\end{vmatrix}},$$

where

$$P_A=2\begin{vmatrix}1\,2\\1\,2\end{vmatrix}p'q'+\begin{vmatrix}1\,3\\1\,2\end{vmatrix}q'^2+2\begin{vmatrix}1\,4\\1\,2\end{vmatrix}p'r'+2\begin{vmatrix}1\,5\\1\,2\end{vmatrix}q'r'+\begin{vmatrix}1\,6\\1\,2\end{vmatrix}r'^2,$$

$$Q_A=\begin{vmatrix}1\,2\,3\\1\,2\,3\end{vmatrix}q'^2+2\begin{vmatrix}1\,2\,4\\1\,2\,3\end{vmatrix}p'r'+2\begin{vmatrix}1\,2\,5\\1\,2\,3\end{vmatrix}q'r'+\begin{vmatrix}1\,2\,6\\1\,2\,3\end{vmatrix}r'^2;$$

and there are similar expressions admissible for the other magnitudes of the same type. But we cannot infer relations

$$\frac{A_5}{\rho}=\frac{P_A}{\begin{vmatrix}1\,2\\1\,2\end{vmatrix}^{\frac{1}{2}}},\quad \frac{A_6}{\rho}=-\frac{s_{11}{}^{\frac{1}{2}}\,Q_A}{\begin{vmatrix}1\,2\\1\,2\end{vmatrix}^{\frac{1}{2}}\begin{vmatrix}1\,2\,3\\1\,2\,3\end{vmatrix}^{\frac{1}{2}}};$$

these would satisfy some, but not all, of the equations.)

Similarly, we find

$$\begin{aligned}
\frac{1}{\rho^2}(A_5H_5+A_6H_6)&=\frac{s_{12}}{\rho^2}-\frac{\bar{A}\bar{H}}{\rho^2}\\
&=-2p'^3q'\begin{vmatrix}1\,2\\1\,2\end{vmatrix}-2p'^3r'\begin{vmatrix}1\,2\\1\,4\end{vmatrix}\\
&\quad-p'^2q'^2\begin{vmatrix}1\,2\\1\,3\end{vmatrix}+p'^2q'r'\left\{-2\begin{vmatrix}1\,2\\1\,5\end{vmatrix}+4\begin{vmatrix}1\,2\\2\,4\end{vmatrix}\right\}\\
&\quad+p'^2r'^2\left\{-\begin{vmatrix}1\,1\\2\,6\end{vmatrix}+4\begin{vmatrix}1\,4\\2\,4\end{vmatrix}\right\}\\
&\quad+p'r'^3\left\{2\begin{vmatrix}1\,4\\2\,6\end{vmatrix}+2\begin{vmatrix}1\,6\\2\,4\end{vmatrix}\right\}+p'q'^2r'\left\{4\begin{vmatrix}1\,2\\2\,5\end{vmatrix}+2\begin{vmatrix}1\,3\\2\,4\end{vmatrix}+2\begin{vmatrix}1\,4\\2\,3\end{vmatrix}\right\}\\
&\quad+p'q'r'^2\left\{2\begin{vmatrix}1\,2\\5\,6\end{vmatrix}+4\begin{vmatrix}1\,4\\2\,5\end{vmatrix}+4\begin{vmatrix}1\,5\\2\,4\end{vmatrix}\right\}+2p'q'^3\begin{vmatrix}1\,2\\2\,3\end{vmatrix}\\
&\quad+q'^4\begin{vmatrix}1\,3\\2\,3\end{vmatrix}+q'^3r'\left\{2\begin{vmatrix}1\,3\\2\,5\end{vmatrix}+2\begin{vmatrix}1\,5\\2\,3\end{vmatrix}\right\}\\
&\quad+q'^2r'^2\left\{4\begin{vmatrix}1\,5\\2\,5\end{vmatrix}+\begin{vmatrix}1\,3\\2\,6\end{vmatrix}+\begin{vmatrix}1\,6\\2\,3\end{vmatrix}\right\}\\
&\quad+q'r'^3\left\{2\begin{vmatrix}1\,5\\2\,6\end{vmatrix}+2\begin{vmatrix}1\,6\\2\,5\end{vmatrix}\right\}+r'^4\begin{vmatrix}1\,6\\2\,6\end{vmatrix};
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\rho^2}(H_5^2+H_6^2)&=\frac{s_{22}}{\rho^2}-\left(\frac{\bar{H}}{\rho}\right)^2\\
&=p'^4\begin{vmatrix}1&2\\1&2\end{vmatrix}+4p'^3r'\begin{vmatrix}2&1\\2&4\end{vmatrix}\\
&\quad+2p'^2q'^2\begin{vmatrix}2&1\\2&3\end{vmatrix}+4p'^2q'r'\begin{vmatrix}2&1\\2&5\end{vmatrix}+p'^2r'^2\left\{2\begin{vmatrix}2&1\\2&6\end{vmatrix}+4\begin{vmatrix}2&4\\2&4\end{vmatrix}\right\}\\
&\quad+4p'q'^2r'\begin{vmatrix}2&3\\2&4\end{vmatrix}+8p'q'r'^2\begin{vmatrix}2&4\\2&5\end{vmatrix}+4p'r'^3\begin{vmatrix}2&4\\2&6\end{vmatrix}\\
&\quad+q'^4\begin{vmatrix}2&3\\2&3\end{vmatrix}+4q'^3r'\begin{vmatrix}2&3\\2&5\end{vmatrix}+q'^2r'^2\left\{2\begin{vmatrix}2&3\\2&6\end{vmatrix}+4\begin{vmatrix}2&5\\2&5\end{vmatrix}\right\}\\
&\quad+4q'r'^3\begin{vmatrix}2&5\\2&6\end{vmatrix}+r'^4\begin{vmatrix}2&6\\2&6\end{vmatrix};
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\rho^2}(A_5F_5+A_6F_6)&=\frac{s_{15}}{\rho^2}-\frac{\bar{A}\bar{F}}{\rho^2}\\
&=2p'^3q'\begin{vmatrix}1&2\\1&5\end{vmatrix}+2p'^3r'\begin{vmatrix}1&4\\1&5\end{vmatrix}\\
&\quad+p'^2q'^2\left\{4\begin{vmatrix}2&1\\2&5\end{vmatrix}-\begin{vmatrix}1&3\\1&5\end{vmatrix}\right\}+p'^2q'r'\left\{-2\begin{vmatrix}1&5\\1&5\end{vmatrix}+4\begin{vmatrix}1&2\\5&4\end{vmatrix}+4\begin{vmatrix}1&4\\5&2\end{vmatrix}\right\}\\
&\quad+p'^2r'^2\left\{4\begin{vmatrix}4&1\\4&5\end{vmatrix}-\begin{vmatrix}1&5\\1&6\end{vmatrix}\right\}\\
&\quad+p'q'^3\left\{2\begin{vmatrix}2&1\\3&5\end{vmatrix}+2\begin{vmatrix}2&5\\3&1\end{vmatrix}\right\}+p'q'^2r'\left\{4\begin{vmatrix}1&2\\5&5\end{vmatrix}+2\begin{vmatrix}3&1\\4&5\end{vmatrix}+2\begin{vmatrix}3&5\\4&1\end{vmatrix}\right\}\\
&\quad+p'q'r'^2\left\{-4\begin{vmatrix}1&4\\5&5\end{vmatrix}+2\begin{vmatrix}2&1\\6&5\end{vmatrix}+2\begin{vmatrix}2&5\\6&1\end{vmatrix}\right\}\\
&\quad+p'r'^3\left\{2\begin{vmatrix}1&4\\5&6\end{vmatrix}+2\begin{vmatrix}1&6\\5&4\end{vmatrix}\right\}\\
&\quad+q'^4\begin{vmatrix}1&3\\5&3\end{vmatrix}-2q'^3r'\begin{vmatrix}1&5\\3&5\end{vmatrix}+q'^2r'^2\left\{\begin{vmatrix}3&1\\6&5\end{vmatrix}+\begin{vmatrix}3&5\\6&1\end{vmatrix}\right\}\\
&\quad-2q'r'^3\begin{vmatrix}1&5\\6&5\end{vmatrix}+r'^4\begin{vmatrix}1&6\\5&6\end{vmatrix}.
\end{aligned}$$

The values of all the remaining expressions of the same type can be derived from one or other of the three which are thus stated, by due interchanges of the parameters p, q, r, with the appropriate interchange of the row-and-column numbers in the six-row determinant of § 252. Thus, an interchange of q and r while p is

left unaltered, requires 2 and 4, 3 and 6, in these row-and-column numbers to be interchanged, while 1 and 5 are unchanged; the expression given for

$$(A_5H_5+A_6H_6)/\rho^2$$

is unaltered. An interchange of p and q, while r is left unaltered, requires 1 and 3, 4 and 5, to be interchanged, while 2 and 6 are unchanged; the expression given for $(H_5^2+H_6^2)/\rho^2$ is unaltered; while, from the expression for $(A_5^2+A_6^2)\rho^2$, we have

$$\begin{aligned}
\frac{1}{\rho^2}(B_5^2+B_6^2)&=\frac{s_{33}}{\rho^2}-\left(\frac{\overline{B}}{\rho}\right)^2\\
&=p'^4\begin{vmatrix}1&3\\1&3\end{vmatrix}+4p'^3q'\begin{vmatrix}3&1\\3&2\end{vmatrix}+4p'^3r'\begin{vmatrix}3&1\\3&4\end{vmatrix}\\
&+4p'^2q'^2\begin{vmatrix}2&3\\2&3\end{vmatrix}+p'^2q'r'\left\{4\begin{vmatrix}3&1\\3&5\end{vmatrix}+8\begin{vmatrix}3&2\\3&4\end{vmatrix}\right\}\\
&\qquad+p'^2r'^2\left\{2\begin{vmatrix}3&1\\3&6\end{vmatrix}+4\begin{vmatrix}3&4\\3&4\end{vmatrix}\right\}\\
&+8p'q'^2r'\begin{vmatrix}3&2\\3&5\end{vmatrix}+p'q'r'^2\left\{4\begin{vmatrix}3&2\\3&6\end{vmatrix}+8\begin{vmatrix}3&4\\3&5\end{vmatrix}\right\}+4p'r'^3\begin{vmatrix}3&4\\3&6\end{vmatrix}\\
&+4q'^2r'^2\begin{vmatrix}3&5\\3&5\end{vmatrix}+4q'r'^3\begin{vmatrix}3&5\\3&6\end{vmatrix}+r'^4\begin{vmatrix}3&6\\3&6\end{vmatrix}.
\end{aligned}$$

Similarly for the remaining expressions of this type.

259. To resolve these relations for the twelve quantities such as A_5 and A_6, we can proceed as follows.

(i) First of all, let

$$\left.\begin{aligned}
\sigma_{11}&=\frac{1}{\rho^2}(A_5^2+A_6^2), & \sigma_{44}&=\frac{1}{\rho^2}(G_5^2+G_6^2)\\
\sigma_{22}&=\frac{1}{\rho^2}(H_5^2+H_6^2), & \sigma_{55}&=\frac{1}{\rho^2}(F_5^2+F_6^2)\\
\sigma_{33}&=\frac{1}{\rho^2}(B_5^2+B_6^2), & \sigma_{66}&=\frac{1}{\rho^2}(C_5^2+C_6^2)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\sigma_{12}&=\frac{1}{\rho^2}(A_5H_5+A_6H_6), & \sigma_{13}&=\frac{1}{\rho^2}(A_5B_5+A_6B_6), & \sigma_{23}&=\frac{1}{\rho^2}(H_5B_5+H_6B_6)\\
\sigma_{14}&=\frac{1}{\rho^2}(A_5G_5+A_6G_6), & \sigma_{15}&=\frac{1}{\rho^2}(A_5F_5+A_6F_6), & \sigma_{16}&=\frac{1}{\rho^2}(A_5C_5+A_6C_6)\\
\sigma_{24}&=\frac{1}{\rho^2}(H_5G_5+H_6G_6), & \sigma_{25}&=\frac{1}{\rho^2}(H_5F_5+H_6F_6), & \sigma_{26}&=\frac{1}{\rho^2}(H_5C_5+H_6C_6)\\
\sigma_{34}&=\frac{1}{\rho^2}(B_5G_5+B_6G_6), & \sigma_{35}&=\frac{1}{\rho^2}(B_5F_5+B_6F_6), & \sigma_{36}&=\frac{1}{\rho^2}(B_5C_5+B_6C_6)\\
\sigma_{45}&=\frac{1}{\rho^2}(G_5F_5+G_6F_6), & \sigma_{46}&=\frac{1}{\rho^2}(G_5C_5+G_6C_6), & \sigma_{56}&=\frac{1}{\rho^2}(F_5C_5+F_6C_6)
\end{aligned}\right\}.$$

Now (§ 182) we have, as regards the quantities of the type A_5,

$$u_5 = A_5 p' + H_5 q' + G_5 r', \quad v_5 = H_5 p' + B_5 q' + F_5 r', \quad w_5 = G_5 p' + F_5 q' + C_5 r',$$
$$u_5 p' + v_5 q' + w_5 r' = 0.$$

It is easy to verify the equations

$$\left.\begin{aligned}
\sigma_{11} p'^2 + 2\sigma_{12} p'q' + \sigma_{22} q'^2 + 2\sigma_{14} p'r' + 2\sigma_{24} q'r' + \sigma_{44} r'^2 &= \frac{u_5^2}{\rho^2} \\
\sigma_{22} p'^2 + 2\sigma_{23} p'q' + \sigma_{33} q'^2 + 2\sigma_{25} p'r' + 2\sigma_{35} q'r' + \sigma_{55} r'^2 &= \frac{v_5^2}{\rho^2} \\
\sigma_{44} p'^2 + 2\sigma_{45} p'q' + \sigma_{55} q'^2 + 2\sigma_{46} p'r' + 2\sigma_{56} q'r' + \sigma_{66} r'^2 &= \frac{w_5^2}{\rho^2}
\end{aligned}\right\};$$

and these values for u_5, v_5, w_5, may be compared with the apparently more succinct expressions in § 189.

Also we have

$$\left.\begin{aligned}
\frac{A_5}{\rho}\frac{u_5}{\rho} &= \sigma_{11} p' + \sigma_{12} q' + \sigma_{14} r' \\
\frac{A_5}{\rho}\frac{v_5}{\rho} &= \sigma_{12} p' + \sigma_{13} q' + \sigma_{15} r' \\
\frac{A_5}{\rho}\frac{w_5}{\rho} &= \sigma_{14} p' + \sigma_{15} q' + \sigma_{16} r'
\end{aligned}\right\}, \qquad
\left.\begin{aligned}
\frac{F_5}{\rho}\frac{u_5}{\rho} &= \sigma_{15} p_2' + \sigma_{25} q' + \sigma_{45} r' \\
\frac{F_5}{\rho}\frac{v_5}{\rho} &= \sigma_{25} p' + \sigma_{35} q' + \sigma_{55} r' \\
\frac{F_5}{\rho}\frac{w_5}{\rho} &= \sigma_{45} p' + \sigma_{55} q' + \sigma_{56} r'
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\frac{B_5}{\rho}\frac{u_5}{\rho} &= \sigma_{13} p' + \sigma_{23} q' + \sigma_{34} r' \\
\frac{B_5}{\rho}\frac{v_5}{\rho} &= \sigma_{23} p' + \sigma_{33} q' + \sigma_{35} r' \\
\frac{B_5}{\rho}\frac{w_5}{\rho} &= \sigma_{34} p' + \sigma_{35} q' + \sigma_{36} r'
\end{aligned}\right\}, \qquad
\left.\begin{aligned}
\frac{G_5}{\rho}\frac{u_5}{\rho} &= \sigma_{14} p' + \sigma_{24} q' + \sigma_{44} r' \\
\frac{G_5}{\rho}\frac{v_5}{\rho} &= \sigma_{24} p' + \sigma_{34} q' + \sigma_{45} r' \\
\frac{G_5}{\rho}\frac{w_5}{\rho} &= \sigma_{44} p' + \sigma_{45} q' + \sigma_{46} r'
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\frac{C_5}{\rho}\frac{u_5}{\rho} &= \sigma_{16} p' + \sigma_{26} q' + \sigma_{46} r' \\
\frac{C_5}{\rho}\frac{v_5}{\rho} &= \sigma_{26} p' + \sigma_{36} q' + \sigma_{56} r' \\
\frac{C_5}{\rho}\frac{w_5}{\rho} &= \sigma_{46} p' + \sigma_{56} q' + \sigma_{66} r'
\end{aligned}\right\}, \qquad
\left.\begin{aligned}
\frac{H_5}{\rho}\frac{u_5}{\rho} &= \sigma_{12} p' + \sigma_{22} q' + \sigma_{24} r' \\
\frac{H_5}{\rho}\frac{v_5}{\rho} &= \sigma_{22} p' + \sigma_{23} q' + \sigma_{25} r' \\
\frac{H_5}{\rho}\frac{w_5}{\rho} &= \sigma_{24} p' + \sigma_{25} q' + \sigma_{26} r'
\end{aligned}\right\}:$$

relations which give expressions for A_5, B_5, C_5, F_5, G_5, H_5, when the cited values of u_5, v_5, w_5, are inserted. These values may be compared with the values for A_5/κ and the rest, obtained in § 187.

(ii) In the next place, it is to be noted that, in the six-row determinant

$$\| \sigma_{ij} \|, \qquad (i, j, = 1, 2, 3, 4, 5, 6),$$

every third-order minor vanishes, a result that is verified at once by substituting the values of the quantities σ_{ij} in terms of magnitudes such as A_5 and A_6. As every third-order minor vanishes, there are relations among the second-order minors; and these relations facilitate the expressions to be obtained for the quantities A_6, B_6, C_6, F_6, G_6, H_6. We write

$$\left.\begin{aligned}\frac{A_5}{\rho}&=\sigma_{11}^{\frac{1}{2}}\cos\phi_1\\ \frac{A_6}{\rho}&=\sigma_{11}^{\frac{1}{2}}\sin\phi_1\end{aligned}\right\},\quad \left.\begin{aligned}\frac{B_5}{\rho}&=\sigma_{33}^{\frac{1}{2}}\cos\phi_3\\ \frac{B_6}{\rho}&=\sigma_{33}^{\frac{1}{2}}\sin\phi_3\end{aligned}\right\},\quad \left.\begin{aligned}\frac{C_5}{\rho}&=\sigma_{66}^{\frac{1}{2}}\cos\phi_6\\ \frac{C_6}{\rho}&=\sigma_{66}^{\frac{1}{2}}\sin\phi_6\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{F_5}{\rho}&=\sigma_{55}^{\frac{1}{2}}\cos\phi_5\\ \frac{F_6}{\rho}&=\sigma_{55}^{\frac{1}{2}}\sin\phi_5\end{aligned}\right\},\quad \left.\begin{aligned}\frac{G_5}{\rho}&=\sigma_{44}^{\frac{1}{2}}\cos\phi_4\\ \frac{G_6}{\rho}&=\sigma_{44}^{\frac{1}{2}}\sin\phi_4\end{aligned}\right\},\quad \left.\begin{aligned}\frac{H_5}{\rho}&=\sigma_{22}^{\frac{1}{2}}\cos\phi_2\\ \frac{H_6}{\rho}&=\sigma_{22}^{\frac{1}{2}}\sin\phi_2\end{aligned}\right\}.$$

Then we have

$$\frac{\sigma_{12}}{(\sigma_{11}\sigma_{22})^{\frac{1}{2}}}=\cos(\phi_1-\phi_2),\quad \frac{\sigma_{14}}{(\sigma_{11}\sigma_{44})^{\frac{1}{2}}}=\cos(\phi_1-\phi_4),\quad \frac{\sigma_{24}}{(\sigma_{22}\sigma_{24})^{\frac{1}{2}}}=\cos(\phi_2-\phi_4);$$

as $(\phi_1-\phi_2)-(\phi_1-\phi_4)+(\phi_2-\phi_4)=0$, the equivalent relation

$$\begin{vmatrix}\sigma_{11}, & \sigma_{12}, & \sigma_{14}\\ \sigma_{12}, & \sigma_{22}, & \sigma_{24}\\ \sigma_{14}, & \sigma_{24}, & \sigma_{44}\end{vmatrix}=0$$

is satisfied. Let

$$\mu_{ij}=(\sigma_{ii}\sigma_{jj}-\sigma_{ij}^2)^{\frac{1}{2}},$$

and let a positive sign be affixed to the square root, when for the left-hand side we take $i<j$; then we take

$$\mu_{ij}=(\sigma_{ii}\sigma_{jj})^{\frac{1}{2}}\sin(\phi_i-\phi_j),$$

so that the convention adopted implies a relation $\phi_i>\phi_j$. Thus we assume, as a standard of reference,

$$\phi_1>\phi_2>\phi_3>\phi_4>\phi_5>\phi_6.$$

Then for the minors of the foregoing determinant other than μ_{12}, μ_{14}, μ_{24}, we have

$$\begin{aligned}\sigma_{11}\sigma_{24}-\sigma_{12}\sigma_{14}&=\sigma_{11}(\sigma_{22}\sigma_{44})^{\frac{1}{2}}\{\cos(\phi_2-\phi_4)-\cos(\phi_1-\phi_2)\cos(\phi_1-\phi_4)\}\\ &=\sigma_{11}(\sigma_{22}\sigma_{24})^{\frac{1}{2}}\sin(\phi_1-\phi_2)\sin(\phi_1-\phi_4)\\ &=\mu_{12}\mu_{14};\end{aligned}$$

and similarly

$$\begin{aligned}\sigma_{22}\sigma_{14}-\sigma_{12}\sigma_{24}&=-\mu_{12}\mu_{24},\\ \sigma_{44}\sigma_{12}-\sigma_{14}\sigma_{24}&=\mu_{14}\mu_{24}.\end{aligned}$$

Likewise for any second-order minor: thus, for example,

$$\sigma_{14}\sigma_{26}-\sigma_{16}\sigma_{24}= \mu_{12}\mu_{46},$$
$$\sigma_{34}\sigma_{16}-\sigma_{14}\sigma_{36}=-\mu_{13}\mu_{46};$$

every second-order minor is expressible in terms of the square roots of diagonal second-order minors.

(iii) Further, we have three relations

$$A_6p'+H_6q'+G_6r'=0,\quad H_6p'+B_6q'+F_6r'=0,\quad G_6p'+F_6q'+C_6r'=0.$$

When the postulated values of A_6, H_6, G_6, are substituted in the first of those relations, then

$$p'\sigma_{11}^{\frac{1}{2}}\sin\phi_1+q'\sigma_{22}^{\frac{1}{2}}\sin\phi_2+r'\sigma_{44}^{\frac{1}{2}}\sin\phi_4=0,$$

and therefore

$$\{p'\sigma_{11}^{\frac{1}{2}}+q'\sigma_{22}^{\frac{1}{2}}\cos(\phi_1-\phi_2)+r'\sigma_{44}^{\frac{1}{2}}\cos(\phi_1-\phi_4)\}\sin\phi_1$$
$$-\{q'\sigma_{22}^{\frac{1}{2}}\sin(\phi_1-\phi_2)+r'\sigma_{44}^{\frac{1}{2}}\sin(\phi_1-\phi_4)\}\cos\phi_1=0.$$

When this equation is written in the form

$$P_1\sin\phi_1-Q_1\cos\phi_1=0,$$

we have

$$P_1\sigma_{11}^{\frac{1}{2}}=\sigma_{11}p'+\sigma_{12}q'+\sigma_{14}r'=\frac{A_5u_5}{\rho_2},$$
$$Q_1\sigma_{11}^{\frac{1}{2}}=(\sigma_{11}\sigma_{22}-\sigma_{12}^2)^{\frac{1}{2}}q'+(\sigma_{11}\sigma_{44}-\sigma_{14}^2)^{\frac{1}{2}}r'=\mu_{12}q'+\mu_{14}r';$$

also

$$\sigma_{11}(P_1^2+Q_1^2)=(\sigma_{11}p'+\sigma_{12}q'+\sigma_{14}r')^2+(\mu_{12}q'+\mu_{14}r')^2$$
$$=\sigma_{11}(\sigma_{11}p'^2+2\sigma_{12}p'q'+\sigma_{22}q'^2+2\sigma_{14}p'r'+2\sigma_{24}q'r'+\sigma_{44}r'^2)=\sigma_{11}\frac{\mu_5^2}{\rho^2},$$

on inserting the values of μ_{12}^2 and μ_{14}^2, and using the relation $\mu_{12}\mu_{14}=\sigma_{11}\sigma_{24}-\sigma_{12}\sigma_{14}$. Thus

$$\frac{\sin\phi_1}{Q_1}=\frac{\cos\phi_1}{P_1}=\frac{1}{(P_1^2+Q_1^2)^{\frac{1}{2}}}=\frac{\rho}{u_5}.$$

Accordingly

$$\frac{A_5}{\rho}\frac{u_5}{\rho}=\sigma_{11}^{\frac{1}{2}}\cos\phi_1=\sigma_{11}p'+\sigma_{12}q'+\sigma_{14}r',$$
$$\frac{A_6}{\rho}\frac{u_5}{\rho}=\sigma_{11}^{\frac{1}{2}}\sin\phi_1=\mu_{12}q'+\mu_{14}r'.$$

If, in the same relation, the angle ϕ_2 be retained instead of the angle ϕ_1, then similar analysis leads to the results

$$\frac{H_5}{\rho}\frac{u_5}{\rho}=\sigma_{12}p'+\sigma_{22}q'+\sigma_{24}r',\quad \frac{H_6}{\rho}\frac{u_5}{\rho}=-\mu_{12}p'+\mu_{24}r';$$

and there are corresponding results for G_5 and G_6 when the angle ϕ_3 is retained in that relation.

Similarly for the other two relations: the full tale of results is

$$\left.\begin{aligned}\frac{1}{\rho^2}A_5u_5&=\sigma_{11}p'+\sigma_{12}q'+\sigma_{14}r'\\ \frac{1}{\rho^2}H_5u_5&=\sigma_{12}p'+\sigma_{22}q'+\sigma_{24}r'\\ \frac{1}{\rho^2}G_5u_5&=\sigma_{14}p'+\sigma_{24}q'+\sigma_{44}r'\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{1}{\rho^2}A_6u_5&=\mu_{12}q'+\mu_{14}r'\\ \frac{1}{\rho^2}H_6u_5&=-\mu_{12}p'+\mu_{24}r'\\ \frac{1}{\rho^2}G_6u_5&=-\mu_{14}p'-\mu_{24}q'\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{1}{\rho^2}H_5v_5&=\sigma_{22}p'+\sigma_{23}q'+\sigma_{25}r'\\ \frac{1}{\rho^2}B_5v_5&=\sigma_{23}p'+\sigma_{33}q'+\sigma_{35}r'\\ \frac{1}{\rho^2}F_5v_5&=\sigma_{25}p'+\sigma_{35}q'+\sigma_{55}r'\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{1}{\rho^2}H_6v_5&=\mu_{23}q'+\mu_{25}r'\\ \frac{1}{\rho^2}B_6v_5&=-\mu_{23}p'+\mu_{35}r'\\ \frac{1}{\rho_2}F_6v_5&=-\mu_{25}p'-\mu_{35}q'\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{1}{\rho^2}G_5w_5&=\sigma_{44}p'+\sigma_{45}q'+\sigma_{46}r'\\ \frac{1}{\rho^2}F_5w_5&-\sigma_{45}p'+\sigma_{55}q'+\sigma_{56}r'\\ \frac{1}{p^2}C_5w_5&=\sigma_{46}p'+\sigma_{56}q'+\sigma_{66}r'\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{1}{\rho^2}G_6w_5&=\mu_{45}q'+\mu_{46}r'\\ \frac{1}{\rho^2}F_6w_5&=-\mu_{45}p'+\mu_{56}r'\\ \frac{1}{\rho^2}C_6w_5&=-\mu_{46}p'-\mu_{56}q'\end{aligned}\right\}.$$

The relations involving the quantities A_5, B_5, C_5, F_5, G_5, H_5, are in agreement with the former relations between these quantities. The remaining relations involving A_6, B_6, C_6, F_6, G_6, H_6, can be regarded as providing explicit expressions for this set of non-gremial quantities.

Orthogonal centre of a region in sextuple space.

260. In connection with the locus of the centre of circular curvature of regional geodesics in a sextuple plenary space, we consider the possibility of an orthogonal centre of the region, defining it (on the analogy of the like property for surfaces in a quadruple space *) as the limiting position of the intersection of the orthogonal flats of the region at consecutive points along a regional geodesic.

The equations of the orthogonal flat of the geodesic are

$$\sum(\bar{y}-y)y_1=0,\quad \sum(\bar{y}-y)y_2=0,\quad \sum(\bar{y}-y)y_3=0\,;$$

and therefore, for such intersection with the orthogonal flat at a consecutive point of the geodesic, we associate with them the three additional equations of the type

$$\sum(\bar{y}-y)y_1'-\sum y'y_1=0.$$

* *G.F.D.*, vol. i, § 247.

But

$$y_1'=\eta_1+y_1\alpha_1+y_2\xi_1+y_3\phi_1,$$

and so, at the intersection in question, given by the six equations,

$$\begin{aligned}\sum(\bar{y}-y)y_1'&=\sum(\bar{y}-y)\{\eta_1+y_1\alpha_1+y_2\xi_1+y_3\phi_1\}\\&=\sum(\bar{y}-y)\eta_1\,;\end{aligned}$$

also $\sum y'y_1=u_1$; so that the equation, and the like equations, are

$$\left.\begin{aligned}\sum(\bar{y}-y)\eta_1=u_1\\ \sum(\bar{y}-y)\eta_2=u_2\\ \sum(\bar{y}-y)\eta_3=u_3\end{aligned}\right\},$$

which, with the three earlier equations, should give the required locus.

Now, using the notation (§ 188),

$$\mathfrak{a}=\sum\eta_1^2,\quad \mathfrak{b}=\sum\eta_2^2,\quad \mathfrak{c}=\sum\eta_3^2,\quad \mathfrak{f}=\sum\eta_2\eta_3,\quad \mathfrak{g}=\sum\eta_3\eta_1,\quad \mathfrak{h}=\sum\eta_1\eta_2,$$

the symmetrical determinant of which vanishes, there is a relation

$$\eta_1\bar{\mathfrak{a}}^{\frac{1}{2}}-\eta_2\bar{\mathfrak{b}}^{\frac{1}{2}}+\eta_3\bar{\mathfrak{c}}^{\frac{1}{2}}=0:$$

or, what is the equivalent, the determinant

$$|\,y_1,\ \ y_2,\ \ y_3,\ \ \eta_1,\ \ \eta_2,\ \ \eta_3\,|,$$

of the coefficients of the six quantities $\bar{y}-y$ in the six equations, vanishes. Accordingly, the equations cannot be satisfied simultaneously by finite values of the six quantities, except under one condition.

The exception arises, when the curve is such that a relation

$$u_1\bar{\mathfrak{a}}^{\frac{1}{2}}-u_2\bar{\mathfrak{b}}^{\frac{1}{2}}+u_3\bar{\mathfrak{c}}^{\frac{1}{2}}=0$$

is satisfied; but as relations

$$v_1\bar{\mathfrak{a}}^{\frac{1}{2}}-v_2\bar{\mathfrak{b}}^{\frac{1}{2}}+v_3\bar{\mathfrak{c}}^{\frac{1}{2}}=0,$$

$$u_5\bar{\mathfrak{a}}^{\frac{1}{2}}-v_5\bar{\mathfrak{b}}^{\frac{1}{2}}+w_5\bar{\mathfrak{c}}^{\frac{1}{2}}=0,$$

always are satisfied (*l.c.*), we then should have

$$\begin{vmatrix}u_1, & u_2, & u_3\\ v_1, & v_2, & v_3\\ u_5, & v_5, & w_5\end{vmatrix}=0.$$

The coil would vanish (*l.c.*), and the geodesic would be a tangent to a curve of globular curvature (§ 192).

Assuming this inference not to be fact, we conclude that the equations are not satisfied by finite values of the six quantities $\bar{y}-y$; and we resume the investigation of the intersection, but we change one of the implicit assumptions. We take the orthogonal flat to be represented by the equations

$$\|\ \bar{y}-y,\ \ Y,\ \ l_5,\ \ l_6\ \|=0,$$

so that any point in the flat is given by

$$\bar{y} - y = PY + Ql_5 + Rl_6,$$

P, Q, R, being parametric. We take the orthogonal flat at a point S on the geodesic at a small arc-distance δ from O, this quantity δ being exact; and now, instead of assuming that only small quantities of the first order need be retained in the values of y, Y, l_5, l_6, for the consecutive flat at this point S, we retain second powers of δ in their expression. Thus, at S, we take the new value of y to be

$$= y + y'\delta + \tfrac{1}{2}y''\delta^2 = y + y'\delta + \frac{Y}{2\rho}\delta^2;$$

the new value of Y to be

$$\begin{aligned} &= Y + Y'\delta + \tfrac{1}{2}Y''\delta^2 \\ &= Y + \left(\frac{l_3}{\sigma} - \frac{y'}{\rho}\right)\delta + \tfrac{1}{2}\left\{\frac{l_4}{\sigma\tau} + l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right) - Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - y'\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\}\delta^2; \end{aligned}$$

the new value of l_5 to be

$$\begin{aligned} &= l_5 + l_5'\delta + \tfrac{1}{2}l_5''\delta^2 \\ &= l_5 + \left(\frac{l_6}{\rho_5} - \frac{l_4}{\kappa}\right)\delta + \tfrac{1}{2}\left\{l_6\frac{d}{ds}\left(\frac{1}{\rho_5}\right) - l_5\left(\frac{1}{\kappa^2} + \frac{1}{\rho_5^2}\right) - l_4\frac{d}{ds}\left(\frac{1}{\kappa}\right) + l_3\frac{1}{\kappa\tau}\right\}\delta^2, \end{aligned}$$

(using $1/\rho_5$ for the curvature next after the coil, in the grade of succession); and, finally, the new value of l_6 to be

$$\begin{aligned} &= l_6 + l_6'\delta + \tfrac{1}{2}l_6''\delta^2 \\ &= l_6 - \frac{l_5}{\rho_5}\delta + \tfrac{1}{2}\left\{-l_6\frac{1}{\rho_5^2} - l_5\frac{d}{ds}\left(\frac{1}{\rho_5}\right) + l_4\frac{1}{\kappa\rho_5}\right\}\delta^2, \end{aligned}$$

all up to δ^2 inclusive. Thus the typical space-coordinate of a point in the consecutive orthogonal flat is given by

$$\begin{aligned} \bar{y} - &\left(y + y'\delta + \frac{Y}{2\rho}\delta^2\right) \\ &= \bar{P}Y + \bar{Q}l_5 + \bar{R}l_6 \\ &\quad + \bar{P}\left[\left(\frac{l_3}{\sigma} - \frac{y'}{\rho}\right)\delta + \tfrac{1}{2}\left\{\frac{l_4}{\sigma\tau} + l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right) - Y\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - y'\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\}\delta^2\right] \\ &\quad + \bar{Q}\left[\left(\frac{l_6}{\rho_5} - \frac{l_4}{\kappa}\right)\delta + \tfrac{1}{2}\left\{l_6\frac{d}{ds}\left(\frac{1}{\rho_5}\right) - l_5\left(\frac{1}{\kappa^2} + \frac{1}{\rho_5^2}\right) - l_4\frac{d}{ds}\left(\frac{1}{\kappa}\right) + l_3\frac{1}{\kappa\tau}\right\}\delta^2\right] \\ &\quad + \bar{R}\left[-\frac{l_5}{\rho_5}\delta + \tfrac{1}{2}\left\{-l_6\frac{1}{\rho_5^2} - l_5\frac{d}{ds}\left(\frac{1}{\rho_5}\right) + l_4\frac{1}{\kappa\rho_5}\right\}\delta^2\right], \end{aligned}$$

where the quantities $\bar{P}$, $\bar{Q}$, $\bar{R}$, are parametric, and are equal to P, Q, R, respectively when δ vanishes.

To obtain the point of intersection, we take $\bar{y}$ to be its typical coordinate: the two preceding values of $\bar{y}$ are equal to one another: and thus there are, in all, six equations for the determination of the six parameters P, Q, R, $\bar{P}$, $\bar{Q}$, $\bar{R}$, the equations being linear and homogeneous in the six sets of direction-cosines typically represented by y', l_3, l_4, Y, l_5, l_6.

The terms in y', alone, give

$$\delta - \frac{1}{\rho}\bar{P}\delta - \tfrac{1}{2}\bar{P}\frac{d}{ds}\left(\frac{1}{\rho}\right)\delta^2 = 0,$$

so that, accurately up to the first order,

$$\bar{P} = \rho + \tfrac{1}{2}\rho'\delta.$$

The terms in Y, alone, give

$$P - \bar{P} = \tfrac{1}{2}\frac{\delta^2}{\rho} - \tfrac{1}{2}\bar{P}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\delta^2,$$

and therefore, accurately up to the first order,

$$P = \rho + \tfrac{1}{2}\rho'\delta.$$

The terms in l_3, alone, give

$$\frac{\bar{P}}{\sigma}\delta + \tfrac{1}{2}\bar{P}\frac{d}{ds}\left(\frac{1}{\sigma}\right)\delta^2 + \tfrac{1}{2}\bar{Q}\frac{1}{\kappa\tau}\delta^2 = 0,$$

accurately up to second-order terms inclusive; and we infer that the first two terms in the expression for $\bar{Q}$ are

$$\bar{Q} = -2\frac{\rho\tau\kappa}{\sigma}\frac{1}{\delta} + \frac{\rho\tau\kappa}{\sigma}\left(\frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right).$$

The terms in l_4, alone, give

$$\tfrac{1}{2}\bar{P}\frac{1}{\sigma\tau}\delta^2 - \bar{Q}\frac{1}{\kappa}\delta - \tfrac{1}{2}\bar{Q}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\delta^2 + \tfrac{1}{2}\bar{R}\frac{1}{\kappa\rho_5}\delta^2 = 0,$$

accurately up to second-order terms inclusive; and we infer that the first two terms in the expression for $\bar{R}$ are

$$\bar{R} = -4\frac{\rho\tau\kappa\rho_5}{\sigma}\frac{1}{\delta^2} + 2\frac{\rho\tau\kappa\rho_5}{\sigma}\left(\frac{\kappa'}{\kappa} + \frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right)\frac{1}{\delta}.$$

The terms in l_5, alone, give

$$Q - \bar{Q} = -\tfrac{1}{2}\bar{Q}\left(\frac{1}{\kappa^2} + \frac{1}{\rho_5{}^2}\right)\delta^2 - \bar{R}\left\{\frac{1}{\rho_5}\delta + \tfrac{1}{2}\frac{d}{ds}\left(\frac{1}{\rho_5}\right)\delta^2\right\} = 0,$$

accurately up to second-order terms inclusive; and we infer that the first two terms in the expression for Q are

$$Q = 2\frac{\rho\tau\kappa}{\sigma}\frac{1}{\delta} - \frac{\rho\tau\kappa}{\sigma}\left(2\frac{\rho_5'}{\rho_5} + 2\frac{\kappa'}{\kappa} + \frac{\sigma'}{\sigma} - \frac{\rho'}{\rho}\right).$$

Finally, the terms in l_6, alone, give

$$R-\bar{R}=\bar{Q}\left\{\frac{1}{\rho_5}\delta+\frac{1}{2}\frac{d}{ds}\left(\frac{1}{\rho_5}\right)\delta^2\right\}-\frac{1}{2}\bar{R}\frac{1}{\rho_5{}^2}\delta^2,$$

accurately up to the terms retained; and we infer that the first two terms in R are the same as those in $\bar{R}$, so that

$$R=-4\frac{\rho\kappa\tau\rho_5}{\sigma}\frac{1}{\delta^2}+2\frac{\rho\kappa\tau\rho_5}{\sigma}\left(\frac{\kappa'}{\kappa}+\frac{\sigma'}{\sigma}-\frac{\rho'}{\rho}\right)\frac{1}{\delta}.$$

We thus have the values of the parameters P, Q, R, in the expression

$$\bar{y}-y=PY+Ql_5+Rl_6$$

of the typical coordinate of the orthogonal centre of the region for the direction p', q', r'. It follows that, when the most important terms alone are retained, we have

$$\bar{y}-y=-4\frac{\rho\kappa\tau\rho_5}{\sigma}\frac{1}{\delta^2}l_6,$$

so that the orthogonal centre of the region, belonging to the direction p', q', r', lies at an infinite distance along the sixth (and last) principal line of the geodesic, the plenary space of the region being sextuple.

It was proved that the perpendicular Π (from the selected neighbouring point), drawn upon the tangent flat of the region, is given by

$$\Pi=\frac{\delta^2}{2\rho},$$

as to its most important term. If ρ_0 denote the length of the line from O to the penultimate (infinitely distant) position of the orthogonal centre, we have

$$\rho_0\Pi=2\frac{\kappa\tau\rho_5}{\sigma}.$$

CHAPTER XXII

ORTHOGONAL SURFACES, MINIMAL SURFACES, IN A REGION

Triply orthogonal surfaces in a region: curves of flexure as intersections.

261. The developed theory of systems of triply orthogonal surfaces in a flat is well known *. There is a similar theory of systems of triply orthogonal surfaces in a region, as also of systems of completely orthogonal amplitudes of $n-1$ dimensions in any amplitude of n dimensions.

In the theory of orthogonal surfaces in a region, the main concern relates to the regional properties of the surfaces and not to their spatial properties. When the theory is compared with that of orthogonal surfaces in a flat, the regional normal to a regional surface takes the place of the superficial normal in the flat, and the regional flexure of a geodesic on a regional surface takes the place of the circular curvature of a geodesic on a surface in the flat. Here we shall deal with two results only †, an extension of Dupin's theorem on the intersections of three families of orthogonal surfaces in a flat ‡, and an extension of Bonnet's theorem that the parameter of a family of surfaces constituting their part in a triply orthogonal system in a flat must satisfy a partial differential equation of the third order **.

For the establishment of the extended form of Dupin's theorem, the method used by Puiseux †† will be adopted (with the necessary modifications). The whole configuration is referred to any point O in the region; and it is considered for a small regional range in the vicinity of O. The three families of surfaces are represented by the equations

$$\alpha(p, q, r)=\alpha, \quad \epsilon(p, q, r)=\epsilon, \quad \iota(p, q, r)=\iota,$$

* Reference may be made to the exposition in Darboux's treatise *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes* (1910).

† For a discussion of the main properties of orthogonal configurations in an amplitude of n dimensions, see Bianchi's *Lezioni di geometria differenziale,* (3rd ed.), vol. ii, part ii (1923), chap. xxviii, pp. 621-686.

‡ Dupin, *Développements de géométrie* (1813), pp. 239-242.

** The theorem was first stated by Bonnet in 1862, in his memoir in the *Comptes Rendus,* t. liv (1862), pp. 556, 557. An explicit form of the partial differential equation was obtained by Cayley in 1872; see his *Collected Mathematical Papers,* vol. viii, no. 518, no. 519. For further references see my *Lectures on Differential Geometry,* ch. xi.

†† Liouville, 2me sér., t. viii (1863), p. 336.

where a, ϵ, ι, on the right-hand sides are the parametric constants, the variations of which give rise to the surfaces in the respective families. As the surfaces are orthogonal to one another, we have

$$\sum a\epsilon_1\iota_1=0, \quad \sum a\iota_1 a_1=0, \quad \sum a a_1\epsilon_1=0.$$

From the second and third of those relations, taken in the forms

$$\iota_1\sum a a_1+\iota_2\sum h a_1+\iota_3\sum g a_1=0,$$
$$\epsilon_1\sum a a_1+\epsilon_2\sum h a_1+\epsilon_3\sum g a_1=0,$$

we have

$$\frac{\sum a a_1}{\epsilon_2\iota_3-\epsilon_3\iota_2}=\frac{\sum h a_1}{\epsilon_3\iota_1-\epsilon_1\iota_3}=\frac{\sum g a_1}{\epsilon_1\iota_2-\epsilon_2\iota_1}.$$

We denote the elements, at O, of regional normals at right angles to the families a, ϵ, ι, by dl, dm, dn, respectively; the dilatation of the a-family is denoted by a_l; and thus

$$\Omega a_l\frac{dp}{dl}=\sum a a_1, \quad \Omega a_l\frac{dq}{dl}=\sum h a_1, \quad \Omega a_l\frac{dr}{dl}=\sum g a_1.$$

Again, let p_1', q_1', r_1', denote the direction-variables at O of the curve of intersection of the surfaces ϵ and ι, so that

$$\epsilon_1 p_1'+\epsilon_2 q_1'+\epsilon_3 r_1'=0, \quad \iota_1 p_1'+\iota_2 q_1'+\iota_3 r_1'=0,$$

and therefore

$$\frac{p_1'}{\epsilon_2\iota_3-\epsilon_3\iota_2}=\frac{q_1'}{\epsilon_3\iota_1-\epsilon_1\iota_3}=\frac{r_1'}{\epsilon_1\iota_2-\epsilon_2\iota_1}.$$

The regional arc-relations at O are

$$\sum A p_1'^2=1, \quad \sum A\left(\frac{dp}{dl}\right)^2=1;$$

and therefore the preceding relations give

$$p_1'=\frac{dp}{dl}, \quad q_1'=\frac{dq}{dl}, \quad r_1'=\frac{dr}{dl}.$$

Hence, at O, the directions of the three curves of intersection of the surfaces, which are at right angles to one another in pairs, are the regional normals to the respective surfaces.

This property is descriptive at O; and it belongs to any three surfaces at O merely at right angles to one another. For the extension of Dupin's theorem, we must take account of the ranges in the region in the near vicinity of O; and, in the discussion, the organic succession in each family of the surfaces is used. As the three curves of intersection at O are orthogonal, we use them as parametric curves to which the region is referred; and therefore the regional arc-relation becomes

$$Ap'^2+Bq'^2+Cr'^2=1,$$

while the values of A, B, C, at O will be denoted by A_0, B_0, C_0. The parametric equations of the surfaces in the regional range near O can accordingly be taken in the forms

$$\begin{aligned}
\alpha(p, q, r) &= \alpha = p + \tfrac{1}{2}\alpha_{22}q^2 + \alpha_{23}qr + \tfrac{1}{2}\alpha_{33}r^2 + \dots, \\
\epsilon(p, q, r) &= \epsilon = q + \tfrac{1}{2}\epsilon_{33}r^2 + \epsilon_{23}rp + \tfrac{1}{2}\epsilon_{11}p^2 + \dots, \\
\iota(p, q, r) &= \iota = r + \tfrac{1}{2}\iota_{11}p^2 + \iota_{12}pq + \tfrac{1}{2}\iota_{22}q^2 + \dots,
\end{aligned}$$

the unexpressed terms being of the third and higher orders in the small magnitudes p, q, r.

To utilise the equations of § 196, which give the directions of the curves of regional flexure on a surface, we require (i) certain magnitudes of the region at O, (ii) certain relations expressing the orthogonality of the surfaces at O, and (iii) the general equations satisfied by directions at O on the respective surfaces. We take these in turn.

(i) Because the primary magnitudes F, G, H, of the region vanish, we have (§ 160)

$$0 = B\Delta_{3i} + C\Theta_{2i}, \quad 0 = C\Theta_{1i} + A\Gamma_{3i}, \quad 0 = A\Gamma_{2i} + B\Delta_{1i},$$

for $i = 1, 2, 3$, independently of one another, for the three equations separately. Hence, taking $i = 1$ for the first, $i = 2$ for the second, and $i = 3$ for the third, we have

$$B\Delta_{13} + C\Theta_{12} = 0, \quad C\Theta_{12} + A\Gamma_{23} = 0, \quad A\Gamma_{23} + B\Delta_{13} = 0,$$

and therefore

$$\Gamma_{23} = 0, \quad \Delta_{13} = 0, \quad \Theta_{12} = 0.$$

(ii) As regards the orthogonality of the surfaces, continued beyond O, we have in the vicinity of O,

$$\left.\begin{aligned}
\alpha_1 &= 1 + \dots \\
\alpha_2 &= \alpha_{22}q + \alpha_{23}r + \dots \\
\alpha_3 &= \alpha_{23}q + \alpha_{33}r + \dots
\end{aligned}\right\}, \quad
\left.\begin{aligned}
\epsilon_1 &= \epsilon_{11}p + \epsilon_{13}r + \dots \\
\epsilon_2 &= 1 + \dots \\
\epsilon_3 &= \epsilon_{13}p + \epsilon_{33}r + \dots
\end{aligned}\right\}, \quad
\left.\begin{aligned}
\iota_1 &= \iota_{11}p + \iota_{12}q + \dots \\
\iota_2 &= \iota_{12}p + \iota_{22}q + \dots \\
\iota_3 &= 1 + \dots
\end{aligned}\right\},$$

the unexpressed terms in each value being small compared with the retained terms in that value.

The condition of orthogonality of the surfaces α and ϵ, which is

$$a\alpha_1\epsilon_1 + b\alpha_2\epsilon_2 + c\alpha_3\epsilon_3 = 0$$

in general, becomes

$$a(\epsilon_{11}p + \epsilon_{13}r) + b(\alpha_{22}q + \alpha_{23}r) + \dots \quad = 0$$

on retaining the formally important terms in α_1 and ϵ_1. But along the curve of intersection of these two surfaces, p and q are small quantities of the second order, while r remains of the first order; hence the condition of orthogonality becomes

$$(a\epsilon_{13} + b\alpha_{23})r + \dots \quad = 0.$$

Thus, as a first-order condition of orthogonality in the immediate vicinity of O, and taking

$$a=BC=B_0C_0+\ldots,\quad b=CA=C_0A_0+\ldots,$$

we have

$$B_0\epsilon_{13}+A_0\alpha_{23}=0.$$

Similarly, the first-order condition of orthogonality of the surfaces ϵ and ι in the immediate vicinity of O is found to be

$$C_0\iota_{12}+B_0\epsilon_{13}=0\,;$$

and the like condition from the orthogonality of the surfaces ι and α is found to be

$$A_0\alpha_{23}+C_0\iota_{12}=0.$$

Hence, when the conditions are combined, the complete orthogonality of the surfaces imposes at O the results

$$\alpha_{23}=0,\quad \epsilon_{31}=0,\quad \iota_{12}=0.$$

(iii) Any direction on the α-surface satisfies the relation

$$\alpha_1p'+\alpha_2q'+\alpha_3r'=0,$$

as well as the permanent arc-relation

$$Ap'^2+Bq'^2+Cr'^2=1.$$

Hence all directions at O on the α-surface are given by

$$p'=0,\quad B_0q'^2+C_0r'^2=1.$$

Similarly the directions at O on the ϵ-surface are given by

$$q'=0,\quad A_0p'^2+C_0r'^2=1,$$

and those at O on the ι-surface by

$$r'=0,\quad A_0p'^2+B_0q'^2=1.$$

Now (§ 196) the equation, which gives the directions of the curves of regional flexure on a regional surface $\theta(p,q,r)=0$, is

$$\begin{vmatrix} \vartheta_{11}p'+\vartheta_{12}q'+\vartheta_{13}r', & Ap'+Hq'+Gr', & \theta_1 \\ \vartheta_{12}p'+\vartheta_{22}q'+\vartheta_{23}r', & Hp'+Bq'+Fr', & \theta_2 \\ \vartheta_{13}p'+\vartheta_{23}q'+\vartheta_{33}r', & Gp'+Fq'+Cr', & \theta_3 \end{vmatrix}=0,$$

always in conjunction with $\theta_1p'+\theta_2q'+\theta_3r'=0$ and the permanent arc-relation of the region, the symbols ϑ_{ij} having the values

$$\vartheta_{ij}=\theta_{ij}-\theta_1\Gamma_{ij}-\theta_2\Delta_{ij}-\theta_3\Theta_{ij},$$

for all values of $i, j, = 1, 2, 3$. For the preceding α-surface, in the region as referred to the selected parametric curves, this equation becomes

$$\begin{vmatrix} \bar{a}_{12}q' + \bar{a}_{13}r', & 0, & 1 \\ \bar{a}_{22}q' + \bar{a}_{23}r', & Bq', & 0 \\ \bar{a}_{23}q' + \bar{a}_{33}r', & Cr', & 0 \end{vmatrix} = 0,$$

all values being taken at O. But, as

$$\bar{a}_{ij} = a_{ij} - a_1 \Gamma_{ij} - a_2 \Delta_{ij} - a_3 \Theta_{ij}$$

in general for any surface a, we have (for the surface under consideration)

$$\bar{a}_{23} = a_{23} - \Gamma_{23} = 0$$

at the point O; and therefore the equation for the curves of regional flexure on the α-surface is

$$(C_0 \bar{a}_{22} - B_0 \bar{a}_{33}) q'r' = 0.$$

We have

$$\bar{a}_{22} = a_{22} - \Gamma_{22} = a_{22} - \frac{B_1}{2A},$$

$$\bar{a}_{33} = a_{33} - \Gamma_{33} = a_{33} - \frac{C_1}{2A},$$

the values of $B_1/2A$, $C_1/2A$, being taken at the origin, so that the coefficient of $q'r'$ does not vanish in general. Thus the equation for the directions of regional flexure on the α-surface at O is

$$q'r' = 0,$$

taken with $B_0 q'^2 + C_0 r'^2 = 1$.

The two required directions at O therefore are

$$\left.\begin{matrix} q' = 0 \\ r' = C_0^{-\frac{1}{2}} \end{matrix}\right\}, \qquad \left.\begin{matrix} q' = B_0^{-\frac{1}{2}} \\ r' = 0 \end{matrix}\right\}:$$

that is, they are given by the intersections of the α-surface by the orthogonal ϵ-surface and the orthogonal ι-surface. Similarly for the curves of regional flexure on the ϵ-surface and on the ι-surface.

We therefore infer an extension of Dupin's theorem in the form that, when three families of surfaces in the region are a completely orthogonal system, the curves of intersections of the surfaces are the curves of regional flexure for the surfaces.

Ex. Establish the corresponding extension of Joachimstahl's theorem that, when two regional surfaces intersect everywhere at a constant angle, and when the intersection is a curve of regional flexure for one of the surfaces, the intersection is a curve of regional flexure also for the other surface.

Triply orthogonal surfaces and a partial differential equation of the third order.

262. The establishment of the theorem: that, when a family of regional surfaces constitutes a part of a triply orthogonal system in the region, its parameter must satisfy a partial differential equation of the third order: can be effected by a process analogous to the process devised by Darboux * for such surfaces in a flat.

As before, the triple system is represented by the equations

$$\alpha(p, q, r)=\alpha, \quad \epsilon(p, q, r)=\epsilon, \quad \iota(p, q, r)=\iota;$$

and, no longer specialising the parametric curves as in § 261, we shall write

$$\left.\begin{aligned} a\alpha_1+h\alpha_2+g\alpha_3&=\bar{A}_1\\ h\alpha_1+b\alpha_2+f\alpha_3&=\bar{A}_2\\ g\alpha_1+f\alpha_2+c\alpha_3&=\bar{A}_3\end{aligned}\right\}, \quad \left.\begin{aligned} a\epsilon_1+h\epsilon_2+g\epsilon_3&=\bar{E}_1\\ h\epsilon_1+b\epsilon_2+f\epsilon_3&=\bar{E}_2\\ g\epsilon_1+f\epsilon_2+c\epsilon_3&=\bar{E}_3\end{aligned}\right\}, \quad \left.\begin{aligned} a\iota_1+h\iota_2+g\iota_3&=\bar{I}_1\\ h\iota_1+b\iota_2+f\iota_3&=\bar{I}_2\\ g\iota_1+f\iota_2+c\iota_3&=\bar{I}_3\end{aligned}\right\}.$$

The conditions of orthogonality of the families of surfaces are

$$\begin{aligned} \sum a\epsilon_1\iota_1 &= \sum \bar{E}_1\iota_1 = \sum \bar{I}_1\epsilon_1 = 0,\\ \sum a\iota_1\alpha_1 &= \sum \bar{I}_1\alpha_1 = \sum \bar{A}_1\iota_1 = 0,\\ \sum a\alpha_1\epsilon_1 &= \sum \bar{A}_1\epsilon_1 = \sum \bar{E}_1\alpha_1 = 0,\end{aligned}$$

to be satisfied throughout the whole range.

These conditions are of the first order, in derivatives of the parametric magnitudes α, ϵ, ι. It is necessary to frame similar conditions of the second order and some similar conditions of the third order, to prove the theorem in question.

When the relation $\sum a\alpha_1\epsilon_1=0$ is differentiated with regard to p, we take it in the form

$$\sum \frac{a}{\Omega}\alpha_1\epsilon_1=0,$$

and then

$$\sum \frac{a}{\Omega}(\alpha_1\epsilon_{11}+\epsilon_1\alpha_{11}) - \sum \frac{\alpha_1\epsilon_1}{\Omega}(2a\Gamma_{11}+2h\Gamma_{12}+2g\Gamma_{13})=0,$$

both summations extending over the six terms corresponding to the occurrences of a, b, c, f, g, h. When this equation is re-arranged, it can be made to assume the form

$$\bar{A}_1\bar{\epsilon}_{11}+\bar{A}_2\bar{\epsilon}_{12}+\bar{A}_3\bar{\epsilon}_{13}+\bar{E}_1\bar{\alpha}_{11}+\bar{E}_2\bar{\alpha}_{12}+\bar{E}_3\bar{\alpha}_{13}=0.$$

Differentiations of the same relation with respect to q and to r, with similar transformations, lead to the equations

$$\begin{aligned} \bar{A}_1\bar{\epsilon}_{12}+\bar{A}_2\bar{\epsilon}_{22}+\bar{A}_3\bar{\epsilon}_{23}+\bar{E}_1\bar{\alpha}_{12}+\bar{E}_2\bar{\alpha}_{22}+\bar{E}_3\bar{\alpha}_{23}&=0,\\ \bar{A}_1\bar{\epsilon}_{13}+\bar{A}_2\bar{\epsilon}_{23}+\bar{A}_3\bar{\epsilon}_{33}+\bar{E}_1\bar{\alpha}_{13}+\bar{E}_2\bar{\alpha}_{23}+\bar{E}_3\bar{\alpha}_{33}&=0,\end{aligned}$$

* *Systèmes orthogonaux*, §§ 9-12; cited above in § 261.

respectively. Let these three equations be multiplied by $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$, respectively, and the products be added; then the resulting equation is

$$\sum \bar{I}_1\bar{A}_1\bar{\epsilon}_{11}+\sum \bar{I}_1\bar{E}_1\bar{a}_{11}=0.$$

Proceeding similarly from the first-order conditions $\sum a\epsilon_1\iota_1=0$ and $\sum a\iota_1 a_1=0$, we find similar resulting equations

$$\sum \bar{A}_1\bar{E}_1\bar{\iota}_{11}+\sum \bar{A}_1\bar{I}_1\bar{\epsilon}_{11}=0,$$
$$\sum \bar{E}_1\bar{I}_1\bar{a}_{11}+\sum \bar{E}_1\bar{A}_1\bar{\iota}_{11}=0.$$

When these three equations are combined, we have

$$\sum \bar{E}_1\bar{I}_1\bar{a}_{11}=0, \quad \sum \bar{I}_1\bar{A}_1\bar{\epsilon}_{11}=0, \quad \sum \bar{A}_1\bar{E}_1\bar{\iota}_{11}=0,$$

which are the symmetrical second-order conditions *.

The third-order conditions, affecting a, ϵ, ι, are similarly derivable from these second-order conditions; and they will be developed in connection with a, so as to lead to the special equation of the third order affecting a alone. (There are, of course, the like equations of the same form affecting ϵ alone and ι alone.) We proceed from the relevant second-order condition

$$\sum_{\kappa}\sum_{\mu}\bar{I}_\kappa\bar{E}_\mu\bar{a}_{\kappa\mu}=0,$$

and introducing an operator ϑ_a under the definition

$$\vartheta_a=\bar{A}_1\frac{\partial}{\partial p}+\bar{A}_2\frac{\partial}{\partial q}+\bar{A}_3\frac{\partial}{\partial r},$$

we shall evaluate the relation

$$\vartheta_a\left\{\sum_{\kappa}\sum_{\mu}\bar{I}_\kappa\bar{E}_\mu\bar{a}_{\kappa\mu}\right\}=0.$$

From the relations in § 160, and using (for the region) the double-suffix notation of § 10, we have an equation

$$\frac{\partial}{\partial p}a_{m\mu}=a_{m\mu}\frac{\Omega_1}{\Omega}-\sum_{t}\left[a_{tm}\{1t,\mu\}+a_{t\mu}\{1t,m\}\right],$$

with like expressions for the derivatives with respect to q and to r. Hence

$$\begin{aligned}\frac{\partial\bar{E}_\mu}{\partial p}&=\frac{\partial}{\partial p}\left(\sum_{m}\epsilon_m a_{m\mu}\right)\\
&=\sum_{m}(a_{m\mu}\epsilon_{m1})+\bar{E}_\mu\frac{\Omega_1}{\Omega}-\sum_{t}\sum_{m}\epsilon_m\left[a_{tm}\{1t,\mu\}+a_{t\mu}\{1t,m\}\right]\\
&=\sum_{m}(a_{m\mu}\bar{\epsilon}_{m1})+\sum_{m}\sum_{\lambda}\left[a_{m\mu}\epsilon_\lambda\{m1,\lambda\}\right]+\bar{E}_\mu\frac{\Omega_1}{\Omega}\\
&\qquad-\sum_{t}\sum_{m}\epsilon_m\left[a_{tm}\{1t,\mu\}+a_{t\mu}\{1t,m\}\right];\end{aligned}$$

* The equations

$$\bar{a}_{23}=a_{23}-\Gamma_{23}=0, \quad \bar{\epsilon}_{31}=\epsilon_{31}-\Delta_{31}=0, \quad \bar{\iota}_{12}=\iota_{12}-\Theta_{12}=0,$$

as obtained in § 261 (pp. 228, 229), are the forms of these general conditions with the specialised reference of the region there adopted.

or, because

$$\sum_m \sum_\lambda a_{m\mu}\epsilon_\lambda\{m1, \lambda\} = \sum_\lambda \sum_m a_{\lambda\mu}\epsilon_m\{\lambda 1, m\} = \sum_t \sum_m a_{t\mu}\epsilon_m\{t1, m\},$$

we have

$$\frac{\partial \bar{E}_\mu}{\partial p} = \sum_m (a_{m\mu}\bar{\epsilon}_{m1}) + \bar{E}_\mu \frac{\Omega_1}{\Omega} - \sum_m \sum_t \epsilon_m a_{tm}\{1t, \mu\}$$
$$= \sum_m (a_{m\mu}\bar{\epsilon}_{m1}) + \bar{E}_\mu \frac{\Omega_1}{\Omega} - \sum_t \bar{E}_t\{1t, \mu\}.$$

There are similar expressions for the derivatives of $\bar{E}_\mu$ with respect to q and to r; and therefore

$$\vartheta_a(\bar{E}_\mu) = \sum_m [a_{m\mu}(\bar{A}_1\bar{\epsilon}_{m1} + \bar{A}_2\bar{\epsilon}_{m2} + \bar{A}_3\bar{\epsilon}_{m3})] + \bar{E}_\mu\vartheta_a(\log \Omega) - \sum_r \sum_t [\bar{A}_r\bar{E}_t\{rt, \mu\}]$$
$$= -\sum_m [a_{m\mu}(\bar{E}_1\bar{a}_{m1} + \bar{E}_2\bar{a}_{m2} + \bar{E}_3\bar{a}_{m3})] - \sum_r \sum_t [\bar{A}_r\bar{E}_t\{rt, \mu\}] + \bar{E}_\mu\vartheta_a(\log \Omega),$$

by the second-order results on p. 232.

Similarly we find

$$\vartheta_a(\bar{I}_\kappa) = -\sum_n [a_{n\kappa}(\bar{I}_1\bar{a}_{n1} + \bar{I}_2\bar{a}_{n2} + \bar{I}_3\bar{a}_{n3})] - \sum_s \sum_u [\bar{A}_s\bar{I}_n\{su, \kappa\}] + \bar{I}_\kappa\vartheta_a(\log \Omega).$$

When these values are substituted in the relation

$$0 = \vartheta_a \sum_\mu \sum_\kappa (\bar{I}_\kappa\bar{E}_\mu\bar{a}_{\mu\kappa})$$
$$= \sum_\mu \sum_\kappa [\bar{I}_\kappa\bar{E}_\mu\vartheta_a(\bar{a}_{\mu\kappa})] + \sum_\mu \sum_\kappa [\bar{I}_\kappa\bar{a}_{\mu\kappa}\vartheta_a(\bar{E}_\mu)] + \sum_\mu \sum_\kappa [\bar{E}_\mu\bar{a}_{\mu\kappa}\vartheta_a(\bar{I}_\kappa)],$$

the total coefficient of $\vartheta_a(\log \Omega)$ on the right-hand side

$$= 2\sum_\mu \sum_\kappa \bar{I}_\kappa\bar{E}_\mu\bar{a}_{\mu\kappa} = 0.$$

The total coefficient of $-\bar{I}_\kappa\bar{E}_\mu$, out of the second and the third terms,

$$= \sum_m \sum_n (a_{mn}\bar{a}_{m\mu}\bar{a}_{n\kappa}) + \sum_r \sum_s [\bar{A}_r\bar{a}_{\kappa s}\{r\mu, s\}]$$
$$+ \sum_m \sum_n (a_{mn}\bar{a}_{m\mu}\bar{a}_{n\kappa}) + \sum_r \sum_s [\bar{A}_r\bar{a}_{\mu s}\{r\kappa, s\}]$$
$$= 2\sum_m \sum_n (a_{mn}\bar{a}_{m\mu}\bar{a}_{n\kappa}) + \sum_r \sum_s \bar{A}_r[\bar{a}_{\kappa s}\{r\mu, s\} + \bar{a}_{\mu s}\{r\kappa, s\}]:$$

this quantity we shall denote by $P_{\mu\kappa}$, temporarily. Thus the relation becomes

$$\sum_\mu \sum_\kappa \bar{I}_\kappa\bar{E}_\mu[\vartheta_a(\bar{a}_{\mu\kappa}) - P_{\mu\kappa}] = 0.$$

Next, for the development of the quantity $\vartheta_a(\bar{a}_{\mu\kappa})$, we have

$$\vartheta_a(\bar{a}_{\mu\kappa}) = \bar{A}_1\frac{\partial}{\partial p}(\bar{a}_{\mu\kappa}) + \bar{A}_2\frac{\partial}{\partial q}(\bar{a}_{\mu\kappa}) + \bar{A}_3\frac{\partial}{\partial r}(\bar{a}_{\mu\kappa}).$$

By the results * in § 209, we have

$$\frac{\partial \bar{a}_{\mu\kappa}}{\partial x_t} = \bar{a}_{\mu\kappa t} + a_\mu(\kappa t) + a_\kappa(\mu t)$$
$$+ \tfrac{1}{3}a_1 \left[\frac{dp}{dl} \{(1\mu, \kappa t) + (1\kappa, \mu t)\} \right.$$
$$+ \frac{dq}{dl} \{(2\mu, \kappa t) + (2\kappa, \mu t)\}$$
$$\left. + \frac{dr}{dl} \{(3\mu, \kappa t) + (3\kappa, \mu t)\} \right],$$

with the convention x_1, x_2, x_3, $=p$, q, r, respectively, while dl is an element of arc in the domainal normal to the a-surface, and $\frac{dp}{dl}$, $\frac{dq}{dl}$, $\frac{dr}{dl}$, are the domainal direction-variables of that normal, so that

$$\Omega a_l \frac{dp}{dl} = a a_1 + h a_2 + g a_3 = \bar{A}_1,$$

$$\Omega a_l \frac{dq}{dl} = h a_1 + b a_2 + f a_3 = \bar{A}_2,$$

$$\Omega a_l \frac{dr}{dl} = g a_1 + f a_2 + c a_3 = \bar{A}_3,$$

while, for all values,

$$a_k(ij) = \Gamma_{ij}\bar{a}_{1k} + \Delta_{ij}\bar{a}_{2k} + \Theta_{ij}\bar{a}_{3k} = \sum_m [\bar{a}_{mk}\{ij, m\}].$$

Thus there are three aggregates of terms :

(i), those involving symbols $\bar{a}_{\lambda\mu\nu}$;

(ii), those free from the symbols $\bar{a}_{\lambda\mu\nu}$ and the four-index symbols ;

(iii), those involving the four-index symbols of sphericity.

We take these in turn.

(i) This aggregate

$$= \bar{A}_1\bar{a}_{1\mu\kappa} + \bar{A}_2\bar{a}_{2\mu\kappa} + \bar{A}_3\bar{a}_{3\mu\kappa}$$
$$= \sum_m \sum_n (a_{mn} a_n \bar{a}_{m\mu\kappa}),$$

the summation being for values of m, n, $=1, 2, 3$, independent of one another.

(ii) This aggregate

$$= \bar{A}_1[a_\mu(\kappa 1) + a_\kappa(\mu 1)] + \bar{A}_2[a_\mu(\kappa 2) + a_\kappa(\mu 2)] + \bar{A}_3[a_\mu(\kappa 3) + a_\kappa(\mu 3)]$$
$$= \sum_r \sum_n \sum_s a_{rn} a_n [\bar{a}_{\mu s}\{\kappa r, s\} + \bar{a}_{\kappa s}\{\mu r, s\}]$$
$$= \sum_r \sum_s \bar{A}_r [\bar{a}_{\mu s}\{\kappa r, s\} + \bar{a}_{\kappa s}\{\mu r, s\}].$$

* The cited results relate to a surface $\theta(p, q, r)=0$; the symbols α, in § 209, are quite different from the symbols a in the present discussion.

It is the same as the second double-summation in the magnitude $P_{\mu\kappa}$; and thus, when we frame the complete coefficient of $\bar{I}_\mu \bar{E}_\mu$ in the condition under development, it will cancel this portion of $P_{\mu\kappa}$.

(iii) On the substitution of the stated values of $\frac{dp}{dl}, \frac{dq}{dl}, \frac{dr}{dl}$, this aggregate is found to be

$$\begin{aligned}&= \frac{2}{3\Omega}[\bar{A}_1{}^2 c_{11}(\mu\kappa) + \bar{A}_2{}^2 c_{22}(\mu\kappa) + \bar{A}_3{}^2 c_{33}(\mu\kappa)\\ &\qquad + 2\bar{A}_2\bar{A}_3 c_{23}(\mu\kappa) + 2\bar{A}_3\bar{A}_1 c_{31}(\mu\kappa) + \bar{A}_1\bar{A}_2 c_{12}(\mu\kappa)]\\ &= \frac{2}{3\Omega} K_a(\mu\kappa),\end{aligned}$$

where

$$c_{ii}(\mu\kappa) = (i\mu, \kappa i), \quad 2c_{ij}(\mu\kappa) = (i\mu, \kappa j) + (i\kappa, \mu j).$$

Now let the total coefficient of $I_\kappa E_\mu$ in the equation be collected. We write

$$\bar{A}_{\kappa\mu} = \sum_m \sum_n (a_{mn} a_n \bar{a}_{m\mu\kappa}) - 2\sum_m \sum_n (a_{mn} \bar{a}_{m\mu} \bar{a}_{n\kappa}) + \frac{2}{3\Omega} K_a(\mu\kappa),$$

as the resulting value of the coefficient; the equation becomes

$$\sum_\kappa \sum_\mu \bar{A}_{\kappa\mu} \bar{I}_\kappa \bar{E}_\mu = 0,$$

and it is linear and homogeneous in the six quantities

$$\begin{aligned}\bar{I}_1\bar{E}_1 &= \theta_{11}, & \bar{I}_2\bar{E}_3 + \bar{I}_3\bar{E}_2 &= 2\theta_{23},\\ \bar{I}_2\bar{E}_2 &= \theta_{22}, & \bar{I}_3\bar{E}_1 + \bar{I}_1\bar{E}_3 &= 2\theta_{31},\\ \bar{I}_3\bar{E}_3 &= \theta_{33}, & \bar{I}_1\bar{E}_2 + \bar{I}_2\bar{E}_1 &= 2\theta_{12}.\end{aligned}$$

Another equation, already obtained in the form

$$\sum_\kappa \sum_\mu \bar{I}_\kappa \bar{E}_\mu \bar{a}_{\kappa\mu} = 0,$$

is likewise linear and homogeneous in the same six quantities. Further, again using the double-suffix notation for the primary magnitudes, we have

$$\sum_\kappa \sum_\mu A_{\kappa\mu} \bar{I}_\kappa \bar{E}_\mu = \Omega \sum_\mu \bar{E}_\mu \iota_\mu = 0,$$

thus providing a third similar equation, out of the conditions of orthogonality.

The remaining two conditions of orthogonality are

$$\begin{aligned}a_1\bar{E}_1 + a_2\bar{E}_2 + a_3\bar{E}_3 &= 0,\\ a_1\bar{I}_1 + a_2\bar{I}_2 + a_3\bar{I}_3 &= 0.\end{aligned}$$

Let these be multiplied by $\bar{I}_1$ and $\bar{E}_1$ respectively, and the products be added: also by $\bar{I}_2$ and $\bar{E}_2$ respectively, and the products be added: also by $\bar{I}_3$ and $\bar{E}_3$, respectively, and the products be added. Then, in succession, we obtain relations

$$\begin{aligned}a_1\theta_{11} + a_2\theta_{12} + a_3\theta_{13} &= 0,\\ a_1\theta_{12} + a_2\theta_{22} + a_3\theta_{23} &= 0,\\ a_1\theta_{13} + a_2\theta_{23} + a_3\theta_{33} &= 0,\end{aligned}$$

which, though linear in a_1, a_2, a_3, are linearly independent of one another, *quà* equations in those magnitudes; and these three equations are linear and homogeneous in the same six quantities θ_{ij}.

Thus, in all, there are six equations, linear and homogeneous, in the six quantities θ_{ij}. When these are eliminated, the result can be expressed by the determinantal equation

$$\begin{vmatrix} \bar{A}_{11}, & \bar{A}_{22}, & \bar{A}_{33}, & \bar{A}_{23}, & \bar{A}_{31}, & \bar{A}_{12} \\ \bar{a}_{11}, & \bar{a}_{22}, & \bar{a}_{33}, & \bar{a}_{23}, & \bar{a}_{31}, & \bar{a}_{12} \\ A_{11}, & A_{22}, & A_{33}, & A_{23}, & A_{31}, & A_{12} \\ 2a_1, & 0, & 0, & 0 & a_3, & a_2 \\ 0, & 2a_2, & 0, & a_3, & 0, & a_1 \\ 0, & 0, & 2a_3, & a_2, & a_1, & 0 \end{vmatrix} = 0.$$

This equation involves derivatives of a only, none of ϵ, none of ι; and it involves the magnitudes of the region. It is a partial differential equation of the third order, which must be satisfied by the parametric magnitude $a(p, q, r)$ in order that the surfaces

$$a(p, q, r) = \text{constant}$$

may constitute one set of a system of triply orthogonal surfaces in the region.

The equation is typical, for each of the three families of surfaces in the triply orthogonal system; the respective forms, for the ϵ-surfaces and the ι-surfaces, are obvious.

When the region is a flat, so that the orthogonal systems exist in a triple space that is homaloidal, the resulting equation becomes the equation obtained by Darboux. For, in those circumstances, $A_{ii}=1$, $A_{ij}=0$, for $i, j, =1, 2, 3$; all the quantities Γ, Δ, Θ, vanish, as do all the four-index symbols, so that

$$\bar{a}_{ij} = a_{ij}, \quad K_a(ij) = 0,$$

and thus

$$\bar{A}_{ij} = a_1\bar{a}_{1ij} + a_2\bar{a}_{2ij} + a_3\bar{a}_{3ij} - 2(a_{1i}a_{2j} + a_{2i}a_{2j} + a_{3i}a_{3j});$$

and these sets of values serve to constitute the Darboux equation.

When, in the critical equation, the determinant is expanded and a numerical factor 2 is omitted, it has the form

$$\begin{aligned} \sum \bar{A}_{11}[\quad & a_1a_2a_3(C\bar{a}_{22} - B\bar{a}_{33}) \\ & + (Ba_3^2 - Ca_2^2)a_1\bar{a}_{23} + (Ba_3^2 - 2Fa_2a_3 + Ca_2^2)(a_2\bar{a}_{13} - a_3\bar{a}_{12})] \\ + \sum \bar{A}_{23}[& -\bar{a}_{11}(Ba_3^2a_1 - Ca_2^2a_1 + 2Ga_2^2a_3 - 2Ha_2a_3^2) \\ & + (a_1\bar{a}_{22} - 2a_2\bar{a}_{12})(Aa_3^2 - 2Ga_1a_3 + Ca_1^2) \\ & - (a_1\bar{a}_{33} - 2a_3\bar{a}_{13})(Aa_2^2 - 2Ha_1a_2 + Ba_1^2)] = 0, \end{aligned}$$

where the first summation is cyclical for 11, 22, 33, in the magnitudes $\bar{A}_{ii}$, and the second summation is cyclical for 23, 31, 12, in the magnitudes $\bar{A}_{ij}$.

Ex. 1. Shew that, if T be used to denote the magnitude

$$\left\{\frac{1}{\Omega}\left(\sum a\alpha_1^2\right)\right\}^{-\frac{1}{2}}$$

so that $T\alpha_1 = 1$, and if

$$T_{ij} = \frac{\partial^2 T}{\partial x_i \partial x_j},$$

with the customary convention $x_1, x_2, x_3, = p, q, r$, the equation obtained from the critical third-order equation by substituting T_{ij} for $\bar{A}_{ij}$ is satisfied.

Hence shew that a set of parallel surfaces, associated with the representation of the region with polar geodesics (§ 164) of the region, can be a family of surfaces for a triply orthogonal system *.

Ex. 2. When the α-family of surfaces in a triply orthogonal system is given by an implicit parametric equation

$$\phi(p, q, r, \alpha) = 0,$$

verify that the function ϕ satisfies the similar equation

$$\begin{vmatrix} \Phi_{11}, & \Phi_{22}, & \Phi_{33}, & \Phi_{23}, & \Phi_{31}, & \Phi_{12} \\ \bar{\phi}_{11}, & \bar{\phi}_{22}, & \bar{\phi}_{33}, & \bar{\phi}_{23}, & \bar{\phi}_{31}, & \bar{\phi}_{12} \\ A_{11}, & A_{22}, & A_{33}, & A_{23}, & A_{31}, & A_{12} \\ 2\phi_1, & 0, & 0, & 0, & \phi_3, & \phi_2 \\ 0, & 2\phi_2, & 0, & \phi_3, & 0, & \phi_1 \\ 0, & 0, & 2\phi_3, & \phi_2, & \phi_1, & 0 \end{vmatrix} = 0,$$

where

$$\bar{\phi}_{mn} = \phi_{mn} - \phi_1 \Gamma_{mn} - \phi_2 \Delta_{mn} - \phi_3 \Theta_{mn} = \phi_{mn} - \sum [\phi_r \{mn, r\}];$$

and. with quantities $\bar{\phi}_{ijk}$ defined by the equation

$$\bar{\phi}_{ijk} = \phi_{ijk} - \sum_m [\bar{\phi}_{mi}\{jk, m\} + \bar{\phi}_{mj}\{ki, m\} + \bar{\phi}_{mk}\{ij, m\}]$$
$$- \sum_m \phi_m [\{ijk, m\} + \sum_t \{ti, m\}\{jk, t\}],$$

as in §§ 209, 211, the value of $\Phi_{\mu\kappa}$ is

$$\Phi_{\mu\kappa} = \sum_m \sum_n (a_{mn} \phi_n \bar{\phi}_{m\mu\kappa}) - 2\sum_m \sum_n (a_{mn} \bar{\phi}_{m\mu} \bar{\phi}_{n\kappa}) + \frac{2}{3\Omega} K_\phi(\mu\kappa),$$

in which $K_\phi(\mu\kappa)$ denotes the expression

$$\sum_m \sum_n \sum_i \sum_j \{a_{mi} a_{nj} \phi_i \phi_j c_{mn}(\mu\kappa)\},$$

with the significance of $c_{mn}(\mu\kappa)$ as given on p. 235.

The result is the extension, to a triply orthogonal system in a region, of a theorem due to Darboux † for a triply orthogonal system in a triple homaloidal space.

* This property is the extension, to the region, of the property that a family of parallel surfaces can belong to a triply orthogonal system in a flat (triple homaloidal space).

† *Systèmes orthogonaux* (cited on p. 226), p. 94.

263. When an integral of this critical equation of the third order is known, thus determining one family of a triple set of orthogonal surfaces, the other two families can be constructed by simple quadratures.

Let the minors of $\bar{A}_{11}$, $\bar{A}_{22}$, $\bar{A}_{23}$, in the determinant, which occurs on the left-hand side of the equation, be denoted by m_{11}, m_{22}, m_{33}, respectively, and those of $\bar{A}_{23}$, $\bar{A}_{31}$, $\bar{A}_{12}$, by $2m_{23}$, $2m_{31}$, $2m_{12}$, also respectively. Then, from the five equations

$$\begin{aligned}
&\bar{a}_{11}\theta_{11}+2\bar{a}_{12}\theta_{12}+\bar{a}_{22}\theta_{22}+2\bar{a}_{13}\theta_{13}+2\bar{a}_{23}\theta_{23}+\bar{a}_{33}\theta_{33}=0,\\
&A_{11}\theta_{11}+2A_{12}\theta_{12}+A_{22}\theta_{22}+2A_{13}\theta_{13}+2A_{23}\theta_{23}+A_{33}\theta_{33}=0,\\
&2a_1\theta_{11}+2a_2\theta_{12}+2a_3\theta_{13}=0,\\
&2a_1\theta_{12}+2a_2\theta_{22}+2a_3\theta_{23}=0,\\
&2a_1\theta_{13}+2a_2\theta_{23}+2a_3\theta_{33}=0,
\end{aligned}$$

we have

$$\frac{\theta_{11}}{m_{11}}=\frac{\theta_{12}}{m_{12}}=\frac{\theta_{22}}{m_{22}}=\frac{\theta_{13}}{m_{13}}=\frac{\theta_{23}}{m_{23}}=\frac{\theta_{33}}{m_{33}}.$$

Denoting the common value of these fractions by P, we have

$$\begin{aligned}
\bar{E}_1\bar{I}_1&=Pm_{11}, & \bar{E}_2\bar{I}_3+\bar{E}_3\bar{I}_2&=2Pm_{23},\\
\bar{E}_2\bar{I}_2&=Pm_{22}, & \bar{E}_3\bar{I}_1+\bar{E}_1\bar{I}_3&=2Pm_{31},\\
\bar{E}_3\bar{I}_3&=Pm_{33}, & \bar{E}_1\bar{I}_2+\bar{E}_2\bar{I}_1&=2Pm_{12};
\end{aligned}$$

while, from the last three of the equations, we have

$$\begin{vmatrix} \theta_{11}, & \theta_{12}, & \theta_{13} \\ \theta_{12}, & \theta_{22}, & \theta_{23} \\ \theta_{13}, & \theta_{23}, & \theta_{33} \end{vmatrix}=0,$$

and therefore

$$\begin{vmatrix} m_{11}, & m_{12}, & m_{13} \\ m_{12}, & m_{22}, & m_{23} \\ m_{13}, & m_{23}, & m_{33} \end{vmatrix}=0.$$

When we eliminate the quantities I_1, I_2, I_3, there are three equations

$$\begin{aligned}
m_{33}\bar{E}_2{}^2-2m_{23}\bar{E}_2\bar{E}_3+m_{22}\bar{E}_3{}^2=0,\\
m_{11}\bar{E}_3{}^2-2m_{31}\bar{E}_3\bar{E}_1+m_{33}\bar{E}_1{}^2=0,\\
m_{22}\bar{E}_1{}^2-2m_{12}\bar{E}_1\bar{E}_2+m_{11}\bar{E}_2{}^2=0,
\end{aligned}$$

equations consistent with one another in virtue of the foregoing determinant involving the quantities m_{ij}.

There are also the same set of equations in the magnitudes $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$.

Hence we can take

$$\left.\begin{aligned}
m_{11}\bar{E}_2&=\{m_{12}+(m_{12}{}^2-m_{11}m_{22})^{\frac{1}{2}}\}\bar{E}_1\\
m_{11}\bar{E}_3&=\{m_{13}+(m_{13}{}^2-m_{11}m_{33})^{\frac{1}{2}}\}\bar{E}_1
\end{aligned}\right\},$$

and

$$\left.\begin{aligned} m_{11}\bar{I}_2 &= \{m_{12} - (m_{12}^2 - m_{11}m_{22})^{\frac{1}{2}}\}\bar{I}_1 \\ m_{11}\bar{I}_3 &= \{m_{13} - (m_{13}^2 - m_{11}m_{33})^{\frac{1}{2}}\}\bar{I}_1 \end{aligned}\right\}.$$

Now the general differential equation of the ϵ-family is

$$\epsilon_1\, dp + \epsilon_2\, dq + \epsilon_3\, dr = 0,$$

that is,

$$\bar{E}_1(A\, dp + H\, dq + G\, dr) + \bar{E}_2(H\, dp + B\, dq + F\, dr) + \bar{E}_3(G\, dp + F\, dq + C\, dr) = 0\,;$$

and thus the integral equation of the ϵ-family is obtained by the simple quadrature of the relation

$$\begin{aligned} &m_{11}(A\, dp + H\, dq + G\, dr) + m_{12}(H\, dp + B\, dq + F\, dr) + m_{13}(G\, dp + F\, dq + C\, dr) \\ &\quad + (m_{12}^2 - m_{11}m_{22})^{\frac{1}{2}}(H\, dp + B\, dq + F\, dr) + (m_{13}^2 - m_{11}m_{33})^{\frac{1}{2}}(G\, dp + F\, dq + C\, dr) = 0. \end{aligned}$$

Similarly the integral equation of the ι-family is obtained by the simple quadrature of the relation

$$\begin{aligned} &m_{11}(A\, dp + H\, dq + G\, dr) + m_{12}(H\, dp + B\, dq + F\, dr) + m_{13}(G\, dp + F\, dq + C\, dr) \\ &\quad - (m_{12}^2 - m_{11}m_{22})^{\frac{1}{2}}(H\, dp + B\, dq + F\, dr) - (m_{13}^2 - m_{11}m_{33})^{\frac{1}{2}}(G\, dp + F\, dq + C\, dr) = 0. \end{aligned}$$

The separate conditions of integrability of these two equations are satisfied, in virtue of the critical equation of the third order; and it was, in fact, by expressing the conditions of integrability of the equations that Cayley * was the first to obtain that critical equation when dealing with orthogonal systems in homaloidal triple space.

General equations of minimal surfaces in a region.

264. The characteristic equations of minimal surfaces in a region are obtained by the processes of the calculus of variations applied to an integral representing the area of a general portion of the surface. To obtain any surface in a region, the parametric variables p, q, r, can be made functions of two new (superficial) variables t, u; and the element of superficial arc (being also a regional arc) is given by

$$ds^2 = \sum A\, dp^2 = P\, dt^2 + 2Q\, dt\, du + R\, du^2,$$

where

$$P = \sum A p_t^2, \quad Q = \sum A p_t p_u, \quad R = \sum A p_u^2,$$

the quantities x_t and x_u (for $x = p, q, r$) denoting t- and u-derivatives.

The element of area on the surface is

$$(PR - Q^2)^{\frac{1}{2}}\, dt\, du\,;$$

* *Coll. Math. Papers*, vol. viii, no. 518, no. 519.

and therefore the quantity which is to be made a minimum, in order to obtain a minimal surface, is

$$\iint V\,dt\,du=\iint (PR-Q^2)^{\frac{1}{2}}\,dt\,du.$$

For the present purpose, it is sufficient to form the critical differential equations * rendering this double integral a minimum, there being three dependent variables p, q, r. Let x denote any one of these variables: the three critical equations are

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x_t}\right)+\frac{d}{du}\left(\frac{\partial V}{\partial x_u}\right)-\frac{\partial V}{\partial x}=0.$$

In accordance with the Weierstrass procedure of referring minimal surfaces in a flat to their (conjugate imaginary) nul-lines as parametric curves, let these minimal surfaces be similarly referred, so that we shall have

$$P=\sum Ap_t^2=0,\quad R=\sum Ap_u^2=0,$$

being two general equations satisfied by p, q, r, without any limitation due to the character of the surface. Then $V=iQ$; and thus the triple critical equation becomes

$$\frac{d}{dt}\left(\frac{\partial Q}{\partial x_t}\right)+\frac{d}{du}\left(\frac{\partial Q}{\partial x_u}\right)-\frac{\partial Q}{\partial x}=0.$$

Let the value of Q be substituted, and the equation be formed for the three variables in turn.

When $x=p$, the equation is

$$\frac{d}{dt}(Ap_u+Hq_u+Gr_u)+\frac{d}{du}(Ap_t+Hq_t+Gr_t)-\sum A_1p_tp_u=0.$$

Let

$$\begin{aligned}\bar{p}_{tu}&=p_{tu}+(\Gamma \between p_t,\, q_t,\, r_t \between p_u,\, q_u,\, r_u),\\ \bar{q}_{tu}&=q_{tu}+(\Delta \between p_t,\, q_t,\, r_t \between p_u,\, q_u,\, r_u),\\ \bar{r}_{tu}&=r_{tu}+(\Theta \between p_t,\, q_t,\, r_t \between p_u,\, q_u,\, r_u);\end{aligned}$$

then the foregoing p-critical equation, on dropping a numerical factor 2, becomes

$$A\bar{p}_{tu}+H\bar{q}_{tu}+G\bar{r}_{tu}=0.$$

The q-critical equation and the r-critical equation can similarly be expressed in the respective forms

$$\begin{aligned}H\bar{p}_{tu}+B\bar{q}_{tu}+F\bar{r}_{tu}&=0,\\ G\bar{p}_{tu}+F\bar{q}_{tu}+C\bar{r}_{tu}&=0.\end{aligned}$$

* It is known that the extended Legendre test (securing a minimum, subject to other conditions) is satisfied; and that the Weierstrass test (securing a minimum, for his extensive type of strong variations, also subject to other conditions) is satisfied. The Jacobi test, quantitative as regards range, is particular to each adopted solution of the critical equations. Reference may be made to my *Calculus of Variations*, chaps. ix, x.

Hence we have

$$\bar{p}_{tu}=0,\quad \bar{q}_{tu}=0,\quad \bar{r}_{tu}=0:$$

that is, the critical equations determining the values of p, q, r, as functions of the parameters of the nul-lines on a minimal surface in the region are

$$p_{tu}+\sum \Gamma_{11}p_t p_u=0,\quad q_{tu}+\sum \Delta_{11}p_t p_u=0,\quad r_{tu}+\sum \Theta_{11}p_t u_u=0,$$

while the general relations between these nul-parameters and the regional parameters are

$$\sum A p_t^2=0,\quad \sum A p_u^2=0.$$

Further, we verify, as follows, the property (as for the corresponding property in §§ 18, 165) that the nul-surfaces of the region satisfy the general equations of the minimal surfaces. The original critical p-equation is

$$\frac{\partial V}{\partial p}=\frac{d}{dt}\left(\frac{\partial V}{\partial p_t}\right)+\frac{d}{du}\left(\frac{\partial V}{\partial p_u}\right).$$

Retaining the general parametric curves on the surface, we have

$$\frac{\partial V}{\partial p_t}=\frac{1}{V}\{R(\sum A p_t)-Q(\sum A p_u)\}=\frac{E_p}{V},$$

$$\frac{\partial V}{\partial p_u}=\frac{1}{V}\{P(\sum A p_u)-Q(\sum A p_t)\}=\frac{G_p}{V},$$

$$\frac{\partial V}{\partial p}=\frac{1}{2V}\{R(\sum A_1 p_t^2)-2Q(\sum A_1 p_t p_u)+P(\sum A_1 p_u^2)\}=\frac{F_p}{V};$$

and therefore the critical p-equation can be taken in the form

$$\frac{d}{dt}\left(\frac{E_p}{V}\right)+\frac{d}{du}\left(\frac{G_p}{V}\right)=\frac{1}{V}F_p,$$

or, what is the equivalent,

$$E_p\frac{dV}{dt}+G_p\frac{dV}{du}=\left(\frac{dE_p}{dt}+\frac{dG_p}{du}-F_p\right)V.$$

The other two equations have the similar forms

$$E_q\frac{dV}{dt}+G_q\frac{dV}{du}=\left(\frac{dE_q}{dt}+\frac{dG_q}{du}-F_q\right)V,$$

$$E_r\frac{dV}{dt}+G_r\frac{dV}{du}=\left(\frac{dE_r}{dt}+\frac{dG_r}{du}-F_r\right)V.$$

These three completely general equations of minimal surfaces are satisfied by the particular equation

$$V=0,$$

which implies

$$\frac{dV}{dt}=0,\quad \frac{dV}{du}=0:$$

that is, the equation of the nul-surfaces of the region satisfies the characteristic equations of the minimal surfaces of the region.

265. Consider the minimal surfaces in the Riemann region of constant sphericity $1/\kappa^2$, which has its arc-element in the form

$$D^2\, ds^2 = dp^2 + dq^2 + dr^2,$$

where

$$D = 1 + \frac{1}{4\kappa^2}(p^2 + q^2 + r^2).$$

When any surface is referred to its nul-lines as parametric curves of reference, the regional parameters satisfy the relations

$$p_t^2 + q_t^2 + r_t^2 = 0, \quad p_u^2 + q_u^2 + r_u^2 = 0,$$

t and u being the nul-parameters of reference.

The quantities Γ, Δ, Φ, for the region are

$$\left.\begin{aligned} \Gamma_{11} &= -p\epsilon \\ \Delta_{11} &= \quad q\epsilon \\ \Theta_{11} &= \quad r\epsilon \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_{22} &= \quad p\epsilon \\ \Delta_{22} &= -q\epsilon \\ \Theta_{22} &= \quad r\epsilon \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_{33} &= \quad p\epsilon \\ \Delta_{33} &= \quad q\epsilon \\ \Theta_{33} &= -r\epsilon \end{aligned}\right\},$$

$$\left.\begin{aligned} \Gamma_{23} &= 0 \\ \Delta_{23} &= -r\epsilon \\ \Theta_{23} &= -q\epsilon \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_{31} &= -r\epsilon \\ \Delta_{31} &= 0 \\ \Theta_{31} &= -p\epsilon \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_{12} &= -q\epsilon \\ \Delta_{12} &= -p\epsilon \\ \Theta_{12} &= 0 \end{aligned}\right\},$$

where

$$\epsilon = \frac{1}{2\kappa^2 D}.$$

Thus the partial differential equations, characteristic of minimal surfaces in this region, become *

$$\left.\begin{aligned} 2\kappa^2 D p_{tu} + (-p,\ \ p,\ \ p,\ \ 0,\ -r,\ -q)(p_t, q_t, r_t)(p_u, q_u, r_u) &= 0 \\ 2\kappa^2 D q_{tu} + (\ \ q,\ -q,\ \ q,\ -r,\ \ 0,\ -p)(p_t, q_t, r_t)(p_u, q_u, r_u) &= 0 \\ 2\kappa^2 D r_{tu} + (\ \ r,\ \ r,\ -r,\ -q,\ -p,\ \ 0)(p_t, q_t, r_t)(p_u, q_u, r_u) &= 0 \end{aligned}\right\}.$$

These admit two first integrals, arising out of the equations

$$p_t^2 + q_t^2 + r_t^2 = 0, \quad p_u^2 + q_u^2 + r_u^2 = 0,$$

of the nul-lines of parametric reference on the surface.

* We assume tacitly that the sphericity is not zero, so that ϵ is assumed not to be zero and κ is finite. Otherwise, the characteristic equations of minimal surfaces in the region (which now may be regarded as a flat) become

$$p_{tu} = 0, \quad q_{tu} = 0, \quad r_{tu} = 0,$$

which are the customary equations of minimal surfaces in a triple homaloidal space.

To take account of the equations connected with the nul-lines, let

$$\left.\begin{aligned} p_t &= (1-\lambda^2)F \\ q_t &= i(1+\lambda^2)F \\ r_t &= 2\lambda F \end{aligned}\right\}, \qquad \left.\begin{aligned} p_u &= (1-\mu^2)G \\ q_u &= -i(1+\mu^2)G \\ r_u &= 2\mu G \end{aligned}\right\},$$

with no specific limitations on the quantities λ, F, μ, G, so far as concerns these equations. When these relations are used to construct second derivatives for p, q, r, we have

$$\begin{aligned} p_{tu} &= (1-\lambda^2)F_u - 2F\lambda\lambda_u = (1-\mu^2)G_t - 2G\mu\mu_t, \\ -iq_{tu} &= (1+\lambda^2)F_u + 2F\lambda\lambda_u = -(1+\mu^2)G_t - 2G\mu\mu_t, \\ r_{tu} &= 2\lambda F_u + 2F\lambda_u = 2\mu G_t + 2G\mu_t\ . \end{aligned}$$

When F_u and G_t are eliminated from the three equalities, we find

$$\begin{vmatrix} 1-\lambda^2, & 1-\mu^2, & -F\lambda\lambda_u + G\mu\mu_t \\ 1+\lambda^2, & -1-\mu^2, & F\lambda\lambda_u + G\mu\mu_t \\ 2\lambda\ , & 2\mu\ \ , & F\lambda_u - G\mu_t \end{vmatrix} = 0,$$

and therefore

$$2(1+\lambda\mu)^2(F\lambda_u - G\mu_t) = 0.$$

Hence

$$F\lambda_u - G\mu_t = 0\ ;$$

the third equation then gives

$$\lambda F_u - \mu G_t = 0\ ;$$

and all the equalities are satisfied by taking

$$\left.\begin{aligned} F_u &= -2\mu Z, \quad F\lambda_u = (1+\lambda\mu)Z \\ G_t &= -2\lambda Z, \quad G\mu_t = (1+\lambda\mu)Z \end{aligned}\right\}.$$

The equalities, as giving common values of p_{tu}, of q_{tu}, and of r_{tu}, respectively, lead to the results

$$\begin{aligned} p_{tu} &= -2\ (\lambda+\mu)Z, \\ q_{tu} &= 2i(\lambda-\mu)Z, \\ r_{tu} &= 2\ (1-\lambda\mu)Z. \end{aligned}$$

These values belong to the nul-lines on any surface in the Riemann region of constant sphericity ; and they are not restricted to a minimal surface the equations of which have not been used in their construction.

The first of the characteristic equations of the minimal surface is

$$-2\kappa^2 Dp_{tu} = -pp_tp_u + pq_tq_u + pr_tr_u - r(p_tr_u + p_ur_t) - q(p_tq_u + p_uq_t),$$

and therefore

$$-2\kappa^2(Dp_{tu} - D_tp_u - D_up_t) = p(p_tp_u + q_tq_u + r_tr_u)\ ;$$

and the other two also give

$$-2\kappa^2(Dq_{tu}-D_tq_u-D_uq_t)=q(p_tp_u+q_tq_u+r_tr_u),$$
$$-2\kappa^2(Dr_{tu}-D_tr_u-D_ur_t)=r(p_tp_u+q_tq_u+r_tr_u).$$

Let these equations be multiplied by p, q, r, respectively and the products be added; then we have

$$-2\kappa^2\{D(\textstyle\sum pp_{tu})-4\kappa^2D_tD_u\}=4\kappa^2(D-1)(\textstyle\sum p_tp_u).$$

Now

$$2\kappa^2D_{tu}=(pp_{tu}+qq_{tu}+rr_{tu})+(p_tp_u+q_tq_u+r_tr_u);$$

consequently, for the minimal surface, we have

$$\sum p_tp_u=\frac{2\kappa^2}{2-D}(DD_{tu}-2D_tD_u),\quad \sum pp_{tu}=\frac{4\kappa^2}{2-D}\{(1-D)D_{tu}-D_tD_u\};$$

and the characteristic equations of the minimal surface now assume the form

$$\left.\begin{aligned}Dp_{tu}-D_tp_u-D_up_t&=\frac{p}{D-2}(DD_{tu}-2D_tD_u)\\ Dq_{tu}-D_tq_u-D_uq_t&=\frac{q}{D-2}(DD_{tu}-2D_tD_u)\\ Dr_{tu}-D_tr_u-D_ur_t&=\frac{r}{D-2}(DD_{tu}-2D_tD_u)\end{aligned}\right\}.$$

Various analytical developments are possible: beyond the possession of the two integrals

$$p_t^2+q_t^2+r_t^2=0,\quad p_u^2+q_u^2+r_u^2=0,$$

(easily verified to be integrals, in virtue of the foregoing expression for $\sum p_tp_u$, but not providing any characteristic property of the minimal surface), any further integration requires the resolution of equations of the second order, the primitives of which do not emerge by any of the customary methods.

Ex. 1. Prove that the quantity Z (on p. 243) satisfies the relation

$$\frac{Z}{FG\epsilon}=p(\lambda+\mu)-iq(\lambda-\mu)-r(1-\lambda\mu).$$

Ex. 2. Shew that the characteristic equations are satisfied by $p, q, r, =a_mt+c_mu$, for $m=1, 2, 3$, where the quantities a and c are constants; and express the arc-element on this minimal surface in the form

$$ds^2=\frac{4\kappa^2dt\,du}{(1+tu)^2}.$$

Ex. 3. Just as the catenoid provides a simple special instance of a minimal surface in a flat, so it is natural to enquire what minimal surface in the Riemann region of constant sphericity can be obtained by an equation

$$p^2+q^2=2f(r),$$

for the appropriate form of the function f. It appears that the function f must satisfy an ordinary differential equation of the second order; and even for this special form, the equation is not amenable to the usual methods of integration.

The equation itself will be obtained by another process immediately (§ 266, *Ex.* 2).

Sum of the principal flexures on a regional minimal surface vanishes.

266. A geometrical property of all minimal surfaces can be regarded as an interpretation of the general critical equations: it is the property that the principal radii of regional flexure of the surface are equal and opposite. To establish this result, let $\theta(p, q, r)=0$ be the parametric equation of a minimal surface. The direction-variables p_μ, q_μ, r_μ, of a nul-line on the surface are given by the two equations

$$\theta_1 p_\mu + \theta_2 q_\mu + \theta_3 r_\mu = 0, \quad \sum A p_\mu^2 = 0;$$

and therefore, as t and u are the values of μ for these nul-lines, we have

$$\frac{p_t p_u}{B\theta_3^2 - 2F\theta_3\theta_2 + C\theta_2^2} = \frac{-(q_t r_u + q_u r_t)}{2(A\theta_2\theta_3 - H\theta_1\theta_3 - G\theta_1\theta_2 + F\theta_1^2)}$$

$$= \frac{q_t q_u}{C\theta_1^2 - 2G\theta_1\theta_3 + A\theta_3^2} = \frac{-(r_t p_u + r_u p_t)}{2(B\theta_3\theta_1 - F\theta_2\theta_1 - H\theta_2\theta_3 + G\theta_2^2)}$$

$$= \frac{r_t r_u}{A\theta_2^2 - 2H\theta_2\theta_1 + B\theta_1^2} = \frac{-(p_t q_u + p_u q_t)}{2(C\theta_1\theta_2 - G\theta_3\theta_2 - F\theta_3\theta_1 + H\theta_3^2)}.$$

Now, along any surface $\theta=0$, we have, always,

$$\theta_1 p_{tu} + \theta_2 q_{tu} + \theta_3 r_{tu} + (\theta_{11}, \theta_{22}, \theta_{33}, \theta_{23}, \theta_{31}, \theta_{12} \between p_t, q_t, r_t \between p_u, q_u, r_u) = 0;$$

and therefore, when the surface is minimal, we have

$$(\vartheta_{11}, \vartheta_{22}, \vartheta_{33}, \vartheta_{23}, \vartheta_{31}, \vartheta_{12} \between p_t, q_t, r_t \between p_u, q_u, r_u) = 0.$$

The ratios of the quantities, of which ϑ_{ij} are the coefficients, have just been obtained; when they are substituted, the equation becomes

$$\sum \vartheta_{11}(B\theta_3^2 - 2F\theta_3\theta_2 + C\theta_2^2) = 0.$$

Hence (§ 196), denoting the principal radii of geodesic flexure of the surface by γ_1 and γ_2, we have

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = 0,$$

which is the stated property. It is the extension, to a region, of the property of a minimal surface in a flat that the principal radii of curvature of the superficial geodesics are equal and opposite.

Moreover, the single equation

$$\sum \vartheta_{11}(B\theta_3^2 - 2F\theta_3\theta_2 + C\theta_2^2) = 0$$

is the partial differential equation of the second order of minimal surfaces in the region; the parameters of the region are the three independent variables of the equation.

Ex. 1. When the region is a flat, p, q, r, can be taken as the coordinates in the flat; and then

$$ds^2 = dp^2 + dq^2 + dr^2,$$

so that $A = 1$, $B = 1$, $C = 1$, $F = 0$, $G = 0$, $H = 0$; and all the quantities Γ, Δ, Θ, vanish. The partial differential equation of minimal surfaces in the flat becomes

$$(\theta_2^2 + \theta_3^2)\theta_{11} + (\theta_3^2 + \theta_1^2)\theta_{22} + (\theta_1^2 + \theta_2^2)\theta_{33} - 2\theta_2\theta_3\theta_{23} - 2\theta_3\theta_1\theta_{31} - 2\theta_1\theta_2\theta_{12} = 0.$$

When this equation admits of an integral of the form

$$\theta = \tfrac{1}{2}(p^2 + q^2) + f(r) = 0,$$

we have

$$\theta_1 = p, \quad \theta_2 = q, \quad \theta_3 = f',$$

$$\theta_{11} = 1, \quad \theta_{22} = 1, \quad \theta_{33} = f'', \quad \theta_{23} = 0, \quad \theta_{31} = 0, \quad \theta_{12} = 0;$$

and the equation becomes

$$-f + f'^2 - ff'' = 0.$$

The integral of this equation (on the removal of unessential constants) is

$$-f = \tfrac{1}{2}\cosh^2 r;$$

and the surface is the customary catenoid.

Ex. 2. Let the region be the Riemann region of constant sphericity, the arc-element of which is given by the expression

$$D^2 ds^2 = dp^2 + dq^2 + dr^2,$$

where

$$4\kappa^2(D - 1) = p^2 + q^2 + r^2.$$

The magnitudes for the specific form of the partial differential equation of the second order are given in § 265.

When the equation has an integral of the same form

$$\theta = \tfrac{1}{2}(p^2 + q^2) + f(r) = 0,$$

as in the preceding example, we have

$$\begin{aligned} \vartheta_{11} &= 1 - \epsilon(rf' - p^2 + q^2), & \vartheta_{23} &= q\epsilon(r + f'), \\ \vartheta_{22} &= 1 - \epsilon(rf' + p^2 - q^2), & \vartheta_{31} &= p\epsilon(r + f'), \\ \vartheta_{33} &= \quad f'' + \epsilon(2f + rf'), & \vartheta_{12} &= \quad 2pq\epsilon, \end{aligned}$$

where, now,

$$\frac{1}{2\kappa^2\epsilon} = 1 + \frac{1}{4\kappa^2}(r^2 - 2f).$$

Let these values be substituted in the general equation

$$\sum \vartheta_{11}(B\theta_3^2 - 2F\theta_3\theta_2 + C\theta_2^2) = 0:$$

then, after simple reductions, the equation becomes

$$-f + f'^2 - ff'' = \epsilon\{(2f - f'^2)(2f - rf')\},$$

an ordinary equation of the second order for the determination of f.

When the sphericity is zero, so that the region degenerates into a flat, the value of D is unity; the value of ϵ, being $(2\kappa^2 D)^{-1}$ in general, now is zero; and the equation reduces to that of the catenoid in the preceding example.

SECTION IV : DOMAINS

CHAPTER XXIII

Free Domains : Preliminary : Circular Curvature of Geodesics

Primary magnitudes of a free domain.

267. A curved amplitude of four dimensions is styled a *domain*; when the amplitude is homaloidal, it is styled a *block*: these terms corresponding to curve and line, surface and plane, region and flat, for the least extensive amplitudes. Domains exist in some plenary space that is homaloidal; the dimensions of the plenary space are necessarily more than four in number, are not necessarily the same for all domains, and often remain unspecific. One type of domain is of persistent recurrence in investigations on relativity, mostly with vague references (if any) to a plenary space.

The parameters which represent a domain are taken to be p, q, r, t; and differentiations with respect to them are indicated by means of subscript integers in a quadruple suffix notation. Thus we write

$$\frac{\partial^{i+j+k+l} y}{\partial p^i \partial q^j \partial r^k \partial t^l} = y_{ijkl},$$

for positive integral values of i, j, k, l, including 0. But for the sake of brevity, it usually is convenient to retain the earlier single-suffix notation

$$\frac{\partial y}{\partial p}, \ \frac{\partial y}{\partial q}, \ \frac{\partial y}{\partial r}, \ \frac{\partial y}{\partial t}, = y_1, \ y_2, \ y_3, \ y_4,$$

for first derivatives, and the earlier double-suffix notation

$$\frac{\partial^2 y}{\partial p^2}, \ \frac{\partial^2 y}{\partial p \partial q}, \ \ldots, \ \frac{\partial^2 y}{\partial t^2}, = y_{11}, \ y_{12}, \ \ldots, \ y_{44},$$

for second derivatives; for third derivatives, there is variety of convenience.

The variables specifying a point-position in the plenary space are represented as usual by the typical symbol y; and point-positions in the domain are secured by the expressions of the space-variables in terms of the domainal parameters. The element of domainal arc is given by

$$\begin{aligned} ds^2 = A\,dp^2 &+ 2H\,dp\,dq + 2G\,dp\,dr + 2L\,dp\,dt \\ &+ B\,dq^2 \quad + 2F\,dq\,dr + 2M\,dq\,dt \\ &\qquad\qquad + C\,dr^2 \quad + 2N\,dr\,dt \\ &\qquad\qquad\qquad\qquad + D\,dt^2, \end{aligned}$$

with the customary significance

$$A=\sum y_1^2,\quad H=\sum y_1y_2,\ \dots,\quad D=\sum y_4^2,$$

for the coefficients of the quadratic differential elements; and we write

$$\Omega=\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix}=\sum_i\sum_j\sum_k\sum_l \begin{vmatrix} \frac{\partial y_i}{\partial p}, & \frac{\partial y_j}{\partial q}, & \frac{\partial y_k}{\partial r}, & \frac{\partial y_l}{\partial t} \end{vmatrix}^2.$$

Also we use a, h, ... , to denote the minors of A, H, ... , in Ω, so that

$$a=\frac{\partial\Omega}{\partial A},\quad h=\tfrac{1}{2}\frac{\partial\Omega}{\partial H},$$

and so on.

Instead of the Christoffel symbols $\{\alpha\beta,\gamma\}$, we shall use symbols Γ, Δ, Θ, Φ, according to the tableau

$$\left.\begin{aligned} [\alpha\beta,1]&=\sum y_1y_{\alpha\beta}=A\Gamma_{\alpha\beta}+H\Delta_{\alpha\beta}+G\Theta_{\alpha\beta}+L\Phi_{\alpha\beta}\\ [\alpha\beta,2]&=\sum y_2y_{\alpha\beta}=H\Gamma_{\alpha\beta}+B\Delta_{\alpha\beta}+F\Theta_{\alpha\beta}+M\Phi_{\alpha\beta}\\ [\alpha\beta,3]&=\sum y_3y_{\alpha\beta}=G\Gamma_{\alpha\beta}+F\Delta_{\alpha\beta}+C\Theta_{\alpha\beta}+N\Phi_{\alpha\beta}\\ [\alpha\beta,4]&=\sum y_4y_{\alpha\beta}=L\Gamma_{\alpha\beta}+M\Delta_{\alpha\beta}+N\Theta_{\alpha\beta}+D\Phi_{\alpha\beta} \end{aligned}\right\},$$

for all the combinations α, β, $=1, 2, 3, 4$, including repetitions; and the two sets of symbols are equivalent, through the relations

$$\Gamma_{\alpha\beta}=\{\alpha\beta,1\},\quad \Delta_{\alpha\beta}=\{\alpha\beta,2\},\quad \Theta_{\alpha\beta}=\{\alpha\beta,3\},\quad \Phi_{\alpha\beta}=\{\alpha\beta,4\}.$$

268. The various relations between the first derivatives of the primary magnitudes, and the two Christoffel symbols $[\alpha\beta,\gamma]$ and $\{\alpha\beta,\gamma\}$, are covered by the general equations (§ 12) which, for a domain, become

$$[\alpha\beta,\gamma]=\tfrac{1}{2}\left(\frac{\partial A_{\beta\gamma}}{\partial x_\alpha}+\frac{\partial A_{\alpha\gamma}}{\partial x_\beta}-\frac{\partial A_{\alpha\beta}}{\partial x_\gamma}\right),$$

where $x_1=p$, $x_2=q$, $x_3=r$, $x_4=t$, and where

$$A_{\alpha\beta}=\sum y_\alpha y_\beta,$$

for all the combinations α, β, $=1, 2, 3, 4$. Also

$$\begin{aligned}\frac{\partial A_{ij}}{\partial x_k}&=[ki,j]+[kj,i]\\ &=\Gamma_{ki}A_{j1}+\Delta_{ki}A_{j2}+\Theta_{ki}A_{j3}+\Phi_{ki}A_{j4}\\ &\quad+\Gamma_{kj}A_{i1}+\Delta_{kj}A_{i2}+\Theta_{kj}A_{i3}+\Phi_{kj}A_{i4}.\end{aligned}$$

For the first parametric derivatives of Ω, we have

$$\frac{\partial\Omega}{\partial p}=\sum\frac{\partial\Omega}{\partial A}A_1,$$

the summation being over all the constituents of Ω: substituting, we have

$$\Omega_1=\frac{\partial\Omega}{\partial p}=2\Omega(\Gamma_{11}+\Delta_{12}+\Theta_{13}+\Phi_{14}).$$

Similarly for all the first derivatives: the general result is

$$\frac{\Omega_\mu}{2\Omega}=\Gamma_{1\mu}+\Delta_{2\mu}+\Theta_{3\mu}+\Phi_{4\mu},$$

for $\mu=1, 2, 3, 4$.

For the first derivatives of the first minors of Ω, we proceed (as in § 13) from relations of the type

$$Aa+Hh+Gg+Ll=\Omega.$$

Then

$$\begin{aligned}Aa_\mu+Hh_\mu+Gg_\mu+Ll_\mu&=\Omega_\mu-aA_\mu-hH_\mu-gG_\mu-lL_\mu\\&=\Omega_\mu-a(2\textstyle\sum A\Gamma_{1\mu})\\&\quad-h(\textstyle\sum A\Gamma_{2\mu}+\sum H\Gamma_{1\mu})\\&\quad-g(\textstyle\sum A\Gamma_{3\mu}+\sum G\Gamma_{1\mu})\\&\quad-l(\textstyle\sum A\Gamma_{4\mu}+\sum L\Gamma_{1\mu})\\&=\Omega_\mu-\Omega\Gamma_{1\mu}-(a\textstyle\sum A\Gamma_{1\mu}+h\sum A\Gamma_{2\mu}+g\sum A\Gamma_{3\mu}+l\sum A\Gamma_{4\mu}).\end{aligned}$$

Proceeding similarly from the relations

$$\begin{aligned}Ha+Bh+Fg+Ml&=0,\\Ga+Fh+Cg+Nl&=0,\\La+Mh+Ng+Dl&=0,\end{aligned}$$

with a zero right-hand member in each instance, we have

$$\begin{aligned}Ha_\mu+Bh_\mu+Fg_\mu+Ml_\mu&=-\Omega\Gamma_{2\mu}-(a\textstyle\sum H\Gamma_{1\mu}+h\sum H\Gamma_{2\mu}+g\sum H\Gamma_{3\mu}+l\sum H\Gamma_{4\mu}),\\Ga_\mu+Fh_\mu+Cg_\mu+Nl_\mu&=-\Omega\Gamma_{3\mu}-(a\textstyle\sum G\Gamma_{1\mu}+h\sum G\Gamma_{2\mu}+g\sum G\Gamma_{3\mu}+l\sum G\Gamma_{4\mu}),\\La_\mu+Mh_\mu+Ng_\mu+Dl_\mu&=-\Omega\Gamma_{4\mu}-(a\textstyle\sum L\Gamma_{1\mu}+h\sum L\Gamma_{2\mu}+g\sum L\Gamma_{3\mu}+l\sum L\Gamma_{4\mu}).\end{aligned}$$

Let these four relations, linear in a_μ, h_μ, g_μ, l_μ, be resolved; and, for brevity, write

$$\begin{aligned}a\Psi_{1\mu}+h\Psi_{2\mu}+g\Psi_{3\mu}+l\Psi_{4\mu}&=[\Psi_\mu]_1,\\h\Psi_{1\mu}+b\Psi_{2\mu}+f\Psi_{3\mu}+m\Psi_{4\mu}&=[\Psi_\mu]_2,\\g\Psi_{1\mu}+f\Psi_{2\mu}+c\Psi_{3\mu}+n\Psi_{4\mu}&=[\Psi_\mu]_3,\\l\Psi_{1\mu}+m\Psi_{2\mu}+n\Psi_{3\mu}+d\Psi_{4\mu}&=[\Psi_\mu]_4,\end{aligned}$$

for $\Psi=\Gamma, \Delta, \Theta, \Phi$; then the full tale of relations is

$$\left.\begin{aligned}
-\Omega a_\mu + a\Omega_\mu &= 2[\Gamma_\mu]_1 \\
-\Omega b_\mu + b\Omega_\mu &= 2[\Delta_\mu]_2 \\
-\Omega c_\mu + c\Omega_\mu &= 2[\Theta_\mu]_3 \\
-\Omega d_\mu + d\Omega_\mu &= 2[\Phi_\mu]_4 \\
-\Omega f_\mu + f\Omega_\mu &= [\Delta_\mu]_3 + [\Theta_\mu]_2 \\
-\Omega g_\mu + g\Omega_\mu &= [\Theta_\mu]_1 + [\Gamma_\mu]_3 \\
-\Omega h_\mu + h\Omega_\mu &= [\Gamma_\mu]_2 + [\Delta_\mu]_1 \\
-\Omega l_\mu + l\Omega_\mu &= [\Gamma_\mu]_4 + [\Phi_\mu]_1 \\
-\Omega m_\mu + m\Omega_\mu &= [\Delta_\mu]_4 + [\Phi_\mu]_2 \\
-\Omega n_\mu + n\Omega_\mu &= [\Theta_\mu]_4 + [\Phi_\mu]_3
\end{aligned}\right\} .$$

Types of derived variables.

269. The elements dp, dq, dr, dt, and the associated magnitudes p', q', r', t', are connected with a unilinear extension (a direction) in the domain at the initial point. We shall have to consider not merely curves and directions of curves in a domain, but also surfaces and elements of areas on surfaces (in bilinear extension, constituting superficial orientation in the domain), and regions as well as elements of volumes in regions (in trilinear extension, constituting regional orientation in the domain). Such orientations, whether superficial or regional, require appropriate variables for their expression, exactly as direction-variables p', q', r', t', are required for a direction; and the appropriate variables can be devised in two ways. By one method, they are composed from independent constituent directions, two for a surface (as being bilinear) and three for a region (as being trilinear): by the other method, they are connected with parametric relations among the parameters of the domain, there being one such relation for a region and two such relations for a surface.

We begin with the variables of regional orientation. Let dp, dq, dr, dt; δp, δq, δr, δt; ∂p, ∂q, ∂r, ∂t; denote elements for three directions in the domain which are not complanar, with elements ds, δs, ∂s, of arc in those respective directions: then we take P, Q, R, T, where

$$P, Q, R, T = \left\| \begin{matrix} \dfrac{dp}{ds}, & \dfrac{dq}{ds}, & \dfrac{dr}{ds}, & \dfrac{dt}{ds} \\[2ex] \dfrac{\delta p}{\delta s}, & \dfrac{\delta q}{\delta s}, & \dfrac{\delta r}{\delta s}, & \dfrac{\delta t}{\delta s} \\[2ex] \dfrac{\partial p}{\partial s}, & \dfrac{\partial q}{\partial s}, & \dfrac{\partial r}{\partial s}, & \dfrac{\partial t}{\partial s} \end{matrix} \right\|,$$

as variables of regional orientation, the three directions lying within the orienta-

tion. On the other hand, let a region, having the foregoing orientation at O, be given by the parametric equation

$$\epsilon(p, q, r, t)=0,$$

so that we shall have

$$\epsilon_1 \frac{dp}{ds}+\epsilon_2 \frac{dq}{ds}+\epsilon_3 \frac{dr}{ds}+\epsilon_4 \frac{dt}{ds}=0,$$

$$\epsilon_1 \frac{\delta p}{\delta s}+\epsilon_2 \frac{\delta q}{\delta s}+\epsilon_3 \frac{\delta r}{\delta s}+\epsilon_4 \frac{\delta t}{\delta s}=0,$$

$$\epsilon_1 \frac{\partial p}{\partial s}+\epsilon_2 \frac{\partial q}{\partial s}+\epsilon_3 \frac{\partial r}{\partial s}+\epsilon_4 \frac{\partial t}{\partial s}=0;$$

manifestly, there are the relations

$$\frac{P}{\epsilon_1}=\frac{Q}{\epsilon_2}=\frac{R}{\epsilon_3}=\frac{T}{\epsilon_4}.$$

The common value of these fractions is required.

Let $\widehat{12}$ denote the angle between the directions dp, dq, dr, dt, and $\delta p, \delta q, \delta r, \delta t$; let $\widehat{23}$ denote the angle between $\delta p, \delta q, \delta r, \delta t$, and $\partial p, \partial q, \partial r, \partial t$; and let $\widehat{31}$ denote the angle between dp, dq, dr, dt, and $\partial p, \partial q, \partial r, \partial t$. Then

$$\sum aP^2=\begin{vmatrix} \sum A\left(\frac{dp}{ds}\right)^2, & \sum A\frac{dp}{ds}\frac{\delta p}{\delta s}, & \sum A\frac{dp}{ds}\frac{\partial p}{\partial s} \\ \sum A\frac{dp}{ds}\frac{\delta p}{\delta s}, & \sum A\left(\frac{\delta p}{\delta s}\right)^2, & \sum A\frac{\delta p}{\delta s}\frac{\partial p}{\delta s} \\ \sum A\frac{dp}{ds}\frac{\partial p}{\partial s}, & \sum A\frac{\delta p}{\delta s}\frac{\partial p}{\partial s}, & \sum A\left(\frac{\partial p}{\partial s}\right)^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1, & \cos\widehat{12}, & \cos\widehat{13} \\ \cos\widehat{12}, & 1, & \cos\widehat{23} \\ \cos\widehat{13}, & \cos\widehat{23}, & 1 \end{vmatrix}=\Xi^2,$$

the customary magnitude connected with the solid angle constituted by the three directions composing the orientation.

Again, consider an element of domainal arc dn normal to the region, and let $\frac{dp}{dn}, \frac{dq}{dn}, \frac{dr}{dn}, \frac{dt}{dn}$, be the direction-variables for the direction of this regional normal in the domain: then there are the three relations

$$\frac{dp}{dn}\left(\sum A\frac{dp}{ds}\right)+\frac{dq}{dn}\left(\sum H\frac{dp}{ds}\right)+\frac{dr}{dn}\left(\sum G\frac{dp}{ds}\right)+\frac{dt}{dn}\left(\sum L\frac{dp}{ds}\right)=0,$$

$$\frac{dp}{dn}\left(\sum A\frac{\delta p}{\delta s}\right)+\frac{dq}{dn}\left(\sum H\frac{\delta p}{\delta s}\right)+\frac{dr}{dn}\left(\sum G\frac{\delta p}{\delta s}\right)+\frac{dt}{dn}\left(\sum L\frac{\delta p}{\delta s}\right)=0,$$

$$\frac{dp}{dn}\left(\sum A\frac{\partial p}{\partial s}\right)+\frac{dq}{dn}\left(\sum H\frac{\partial p}{\partial s}\right)+\frac{dr}{dn}\left(\sum G\frac{\partial p}{\partial s}\right)+\frac{dt}{dn}\left(\sum L\frac{\partial p}{\partial s}\right)=0,$$

which express the orthogonality of the normal direction to the whole regional orientation. Now

$$\begin{vmatrix} \sum H\frac{dp}{ds}, & \sum G\frac{dp}{ds}, & \sum L\frac{dp}{ds} \\ \sum H\frac{\delta p}{\delta s}, & \sum G\frac{\delta p}{\delta s}, & \sum L\frac{\delta p}{\delta s} \\ \sum H\frac{\partial p}{\partial s}, & \sum G\frac{\partial p}{\partial s}, & \sum L\frac{\partial p}{\partial s} \end{vmatrix} = aP+hQ+gR+lT,$$

and so for the others; hence there are four equations of the form

$$\frac{dp}{dn}=\lambda(aP+hQ+gR+lT),$$

where λ is a multiplier to be determined, and therefore also

$$\frac{dp}{dn}=\mu(a\epsilon_1+h\epsilon_2+g\epsilon_3+l\epsilon_4),$$

$$\frac{dq}{dn}=\mu(h\epsilon_1+b\epsilon_2+f\epsilon_3+m\epsilon_4),$$

$$\frac{dr}{dn}=\mu(g\epsilon_1+f\epsilon_2+c\epsilon_3+n\epsilon_4),$$

$$\frac{dt}{dn}=\mu(l\epsilon_1+m\epsilon_2+n\epsilon_3+d\epsilon_4),$$

with μ as a multiplier to be determined.

For the determination of μ, we introduce a quantity styled the *regional dilatation* of the parametric region $\epsilon=0$ and we measure it by a quantity ϵ_n, $=\frac{d\epsilon}{dn}$, where $\epsilon+d\epsilon=0$ is the parametric equation of a consecutive region, this quantity ϵ_n manifestly varying from point to point of the domainal region. Thus

$$d\epsilon=\epsilon_1 dp+\epsilon_2 dq+\epsilon_3 dr+\epsilon_4 dt,$$

where the quantities dp, dq, dr, dt, belong to the normal direction in the domain: that is,

$$\epsilon_n=\epsilon_1\frac{dp}{dn}+\epsilon_2\frac{dq}{dn}+\epsilon_3\frac{dr}{dn}+\epsilon_4\frac{dt}{dn}.$$

Consequently, after substitution for the direction-variables of the domainal normal,

$$\epsilon_n=\mu\sum a\epsilon_1^2.$$

Again, the direction-variables of this normal are subject to the permanent domainal relation $\sum A\left(\frac{dp}{dn}\right)^2=1$, so that

$$\begin{aligned}1&=\mu^2\sum A(a\epsilon_1+h\epsilon_2+g\epsilon_3+l\epsilon_4)^2\\&=\mu^2\Omega\sum\epsilon_1(a\epsilon_1+h\epsilon_2+g\epsilon_3+l\epsilon_4)\\&=\mu^2\Omega\sum a\epsilon_1^2.\end{aligned}$$

It follows that

$$\sum a\epsilon_1^2=\Omega\epsilon_n^2,\quad \frac{1}{\mu}=\Omega\epsilon_n,$$

thus obtaining a covariantive expression for ϵ_n and an inferred value of the multiplier μ.

The direction-variables of the domainal normal to the parametric region $\epsilon=0$ therefore become

$$\left.\begin{aligned}\Omega\epsilon_n\frac{dp}{dn}&=a\epsilon_1+h\epsilon_2+g\epsilon_3+l\epsilon_4\\ \Omega\epsilon_n\frac{dq}{dn}&=h\epsilon_1+b\epsilon_2+f\epsilon_3+m\epsilon_4\\ \Omega\epsilon_n\frac{dr}{dn}&=g\epsilon_1+f\epsilon_2+c\epsilon_3+n\epsilon_4\\ \Omega\epsilon_n\frac{dt}{dn}&=l\epsilon_1+m\epsilon_2+n\epsilon_3+d\epsilon_4\end{aligned}\right\},$$

where $\Omega\epsilon_n^2=\sum a\epsilon_1^2$; and they can be expressed in the form

$$\left.\begin{aligned}A\frac{dp}{dn}+H\frac{dq}{dn}+G\frac{dr}{dn}+L\frac{dt}{dn}&=\frac{\epsilon_1}{\epsilon_n}\\ H\frac{dp}{dn}+B\frac{dq}{dn}+F\frac{dr}{dn}+M\frac{dt}{dn}&=\frac{\epsilon_2}{\epsilon_n}\\ G\frac{dp}{dn}+F\frac{dq}{dn}+C\frac{dr}{dn}+N\frac{dt}{dn}&=\frac{\epsilon_3}{\epsilon_n}\\ L\frac{dp}{dn}+M\frac{dq}{dn}+N\frac{dr}{dn}+D\frac{dt}{dn}&=\frac{\epsilon_4}{\epsilon_n}\end{aligned}\right\}.$$

And now we at once have the common value of the fractions in the former relations in the form

$$\frac{P}{\epsilon_1}=\frac{Q}{\epsilon_2}=\frac{R}{\epsilon_3}=\frac{T}{\epsilon_4}=\frac{\Omega^{-\frac{1}{2}}\Xi}{\epsilon_n}.$$

Thus there are the two sets of variables P, Q, R, T, and ϵ_1, ϵ_2, ϵ_3, ϵ_4, for the regional orientation, with the necessary relations which render them equivalent.

We shall require the variations of the regional dilatation ϵ_n and of associated

magnitudes taken in domainal directions. For this end, we frame the parametric variations of ϵ_n. Because

$$\epsilon_n^2 = \sum \frac{a}{\Omega} \epsilon_1^2,$$

we have

$$\epsilon_n \frac{\partial \epsilon_n}{\partial p} = \frac{1}{\Omega} \sum (a\epsilon_1 + h\epsilon_2 + g\epsilon_3 + l\epsilon_4)\epsilon_{11} + \tfrac{1}{2} \sum \epsilon_1^2 \frac{\partial}{\partial p}\left(\frac{a}{\Omega}\right).$$

Let the derivatives of the magnitudes a/Ω, as given in § 268, be substituted, and write

$$\bar{\epsilon}_{ij} = \epsilon_{ij} - \epsilon_1 \Gamma_{ij} - \epsilon_2 \Delta_{ij} - \epsilon_3 \Theta_{ij} - \epsilon_4 \Phi_{ij},$$

for all values 1, 2, 3, 4, of i and j taken independently of one another; then, after reduction, we find

$$\epsilon_n \frac{\partial \epsilon_n}{\partial p} = \frac{1}{\Omega} \sum (a\epsilon_1 + h\epsilon_2 + g\epsilon_3 + l\epsilon_4)\bar{\epsilon}_{11},$$

or, having regard to the results on p. 253,

$$\frac{\partial \epsilon_n}{\partial p} = \bar{\epsilon}_{11} \frac{dp}{dn} + \bar{\epsilon}_{12} \frac{dq}{dn} + \bar{\epsilon}_{13} \frac{dr}{dn} + \bar{\epsilon}_{14} \frac{dt}{dn}.$$

Similarly, we find

$$\frac{\partial \epsilon_n}{\partial q} = \bar{\epsilon}_{12} \frac{dp}{dn} + \bar{\epsilon}_{22} \frac{dq}{dn} + \bar{\epsilon}_{23} \frac{dr}{dn} + \bar{\epsilon}_{24} \frac{dt}{dn},$$

$$\frac{\partial \epsilon_n}{\partial r} = \bar{\epsilon}_{13} \frac{dp}{dn} + \bar{\epsilon}_{23} \frac{dq}{dn} + \bar{\epsilon}_{33} \frac{dr}{dn} + \bar{\epsilon}_{34} \frac{dt}{dn},$$

$$\frac{\partial \epsilon_n}{\partial t} = \bar{\epsilon}_{14} \frac{dp}{dn} + \bar{\epsilon}_{24} \frac{dq}{dn} + \bar{\epsilon}_{34} \frac{dr}{dn} + \bar{\epsilon}_{44} \frac{dt}{dn}.$$

The value of the normal variation of ϵ_n is given by

$$\begin{aligned}\epsilon_{nn} &= \frac{d^2 \epsilon}{dn^2} = \frac{d\epsilon_n}{dn} \\ &= \frac{\partial \epsilon_n}{\partial p}\frac{dp}{dn} + \frac{\partial \epsilon_n}{\partial q}\frac{dq}{dn} + \frac{\partial \epsilon_n}{\partial r}\frac{dr}{dn} + \frac{d\epsilon_n}{dt}\frac{dt}{dn} \\ &= \sum \bar{\epsilon}_{11} \left(\frac{dp}{dn}\right)^2;\end{aligned}$$

and an equivalent form is

$$\Omega^2 \epsilon_n^2 \epsilon_{nn} = (\bar{\epsilon}_{ij} \between \sum a\epsilon_1,\ \sum h\epsilon_1,\ \sum g\epsilon_1,\ \sum l\epsilon_1)^2,$$

where $\sum a\epsilon_1$ denotes $a\epsilon_1 + h\epsilon_2 + g\epsilon_3 + l\epsilon_4$, and similarly for the quantities $\sum h\epsilon_1$, $\sum g\epsilon_1$, $\sum l\epsilon_1$.

Again, along any domainal direction p', q', r', t', whether tangential to the region or not, we have

$$\frac{d\epsilon_n}{ds}=\frac{\partial\epsilon_n}{\partial p}p'+\frac{\partial\epsilon_n}{\partial q}q'+\frac{\partial\epsilon_n}{\partial r}r'+\frac{\partial\epsilon_n}{\partial t}t'=\sum\bar{\epsilon}_{11}\frac{dp}{dn}p'.$$

It is convenient to write

$$\begin{aligned}
\bar{\epsilon}_1&=\bar{\epsilon}_{11}p'+\bar{\epsilon}_{12}q'+\bar{\epsilon}_{13}r'+\bar{\epsilon}_{14}t',\\
\bar{\epsilon}_2&=\bar{\epsilon}_{12}p'+\bar{\epsilon}_{22}q'+\bar{\epsilon}_{23}r'+\bar{\epsilon}_{24}t',\\
\bar{\epsilon}_3&=\bar{\epsilon}_{13}p'+\bar{\epsilon}_{23}q'+\bar{\epsilon}_{33}r'+\bar{\epsilon}_{34}t',\\
\bar{\epsilon}_4&=\bar{\epsilon}_{14}p'+\bar{\epsilon}_{24}q'+\bar{\epsilon}_{34}r'+\bar{\epsilon}_{44}t';
\end{aligned}$$

and then we have

$$\frac{d\epsilon_n}{ds}=\bar{\epsilon}_1\frac{dp}{dn}+\bar{\epsilon}_2\frac{dq}{dn}+\bar{\epsilon}_3\frac{dr}{dn}+\bar{\epsilon}_4\frac{dt}{dn}.$$

There is also the equivalent form

$$\frac{d\epsilon_n}{ds}=\frac{1}{\Omega\epsilon_n}\sum a\epsilon_1\bar{\epsilon}_1,$$

which occurs when the values of the normal direction-variables are inserted.

Now the quantity $\frac{d\epsilon}{ds}$ vanishes everywhere along directions which belong to the region, and consequently $\frac{d}{dn}\left(\frac{d\epsilon}{ds}\right)$ also vanishes everywhere for such directions. It follows that the operators $\frac{d}{ds}$ and $\frac{d}{dn}$, taken along directions, which respectively are tangential and normal to the region and lie within the domain, are not commutative in general; though special exceptions can occur, as for regions defined by the restrictive equations $\epsilon_n{}^2\bar{\epsilon}_{ij}=\epsilon_i\epsilon_j\epsilon_{nn}$.

Superficial orientation-variables.

270. For the variables of superficial orientation, we begin with any two directions in the domain, with dp, dq, dr, dt, and δp, δq, δr, δt, as elements along those directions, and with rudimentary arc-lengths ds and δs; and we denote by υ the angle between the two directions. Then the six variables of the orientation, thus determined, are defined by the relations

$$\left.\begin{aligned}
s_{23}&=\frac{1}{\sin\upsilon}\left(\frac{dq}{ds}\frac{\delta r}{\delta s}-\frac{dr}{ds}\frac{\delta q}{ds}\right), & s_{14}&=\frac{1}{\sin\upsilon}\left(\frac{dp}{ds}\frac{\delta t}{\delta s}-\frac{dt}{ds}\frac{\delta p}{\delta s}\right)\\
s_{31}&=\frac{1}{\sin\upsilon}\left(\frac{dr}{ds}\frac{\delta p}{\delta s}-\frac{dp}{ds}\frac{\delta r}{\delta s}\right), & s_{24}&=\frac{1}{\sin\upsilon}\left(\frac{dq}{ds}\frac{\delta t}{\delta s}-\frac{dt}{ds}\frac{\delta q}{\delta s}\right)\\
s_{12}&=\frac{1}{\sin\upsilon}\left(\frac{dp}{ds}\frac{\delta q}{\delta s}-\frac{dq}{ds}\frac{\delta p}{\delta s}\right), & s_{34}&=\frac{1}{\sin\upsilon}\left(\frac{dr}{ds}\frac{\delta t}{\delta s}-\frac{dt}{ds}\frac{\delta r}{\delta s}\right)
\end{aligned}\right\}$$

which satisfy the identity

$$s_{23}s_{14}+s_{31}s_{24}+s_{12}s_{34}=0,$$

and satisfy also the permanent orientation-relation

$$\sum(AB-H^2)s_{12}{}^2=\sum\sum\sum\sum(A_{ik}A_{jl}-A_{il}A_{jk})s_{ij}s_{kl}=1.$$

The orientation can also be regarded as characteristic of a domainal surface at the point, when the surface is defined by means of two equations

$$\epsilon(p, q, r, t)=0, \quad \omega(p, q, r, t)=0,$$

between the parameters of the domain. The surface thus is the intersection of the two regions defined respectively by $\epsilon=0$ and by $\omega=0$. Let dn be a domainal arc normal to the region $\epsilon=0$, and let $d\nu$ be a domainal arc normal to $\omega=0$: then if ι denote the angle at which the regions intersect, we can take ι as the angle between the two arcs dn and $d\nu$ drawn positively in the domain, so that

$$\cos\iota=\sum A\frac{dp}{dn}\frac{dp}{d\nu}.$$

We write

$$\Omega\epsilon_n{}^2=\sum a\epsilon_1{}^2, \quad \Omega\omega_\nu{}^2=\sum a\omega_1{}^2,$$

so that ϵ_n and ω_ν are the respective normal dilatations of the regions; and thus there are two sets of equations of the forms

$$\Omega\epsilon_n\frac{dp}{dn}=a\epsilon_1+h\epsilon_2+g\epsilon_3+l\epsilon_4,$$

$$\Omega\omega_\nu\frac{dp}{d\nu}=a\omega_1+h\omega_2+g\omega_3+l\omega_4,$$

together with their equivalents of the forms

$$A\frac{dp}{dn}+H\frac{dq}{dn}+G\frac{dr}{dn}+L\frac{dt}{dn}=\frac{\epsilon_1}{\epsilon_n},$$

$$A\frac{dp}{d\nu}+H\frac{dq}{d\nu}+G\frac{dr}{d\nu}+L\frac{dt}{d\nu}=\frac{\omega_1}{\omega_\nu}.$$

Evidently

$$\begin{aligned}\Omega^2\epsilon_n\omega_\nu\cos\iota&=\sum A\{(\sum a\epsilon_1)(\sum a\omega_1)\}\\&=\Omega\sum\{\epsilon_1(\sum a\omega_1)\}\\&=\Omega\sum a\epsilon_1\omega_1,\end{aligned}$$

and so

$$\Omega\epsilon_n\omega_\nu\cos\iota=\sum a\epsilon_1\omega_1,$$

thus expressing ι in terms of the quantities in the parametric equations of the surface.

On the assumption that the earlier construction of the superficial orientation agrees with the parametric derivation of that orientation, we have

$$\left.\begin{aligned}\epsilon_1\frac{dp}{ds}+\epsilon_2\frac{dq}{ds}+\epsilon_3\frac{dr}{ds}+\epsilon_4\frac{dt}{ds}=0\\ \epsilon_1\frac{\delta p}{\delta s}+\epsilon_2\frac{\delta q}{\delta s}+\epsilon_3\frac{\delta r}{\delta s}+\epsilon_4\frac{\delta t}{\delta s}=0\end{aligned}\right\},\qquad \left.\begin{aligned}\omega_1\frac{dp}{ds}+\omega_2\frac{dq}{ds}+\omega_3\frac{dr}{ds}+\omega_4\frac{dt}{ds}=0\\ \omega_1\frac{\delta p}{\delta s}+\omega_2\frac{\delta q}{\delta s}+\omega_3\frac{\delta r}{\delta s}+\omega_4\frac{\delta t}{\delta s}=0\end{aligned}\right\}.$$

Hence

$$\frac{s_{34}}{\begin{vmatrix}\epsilon_1, & \epsilon_2\\ \omega_1, & \omega_2\end{vmatrix}}=\frac{s_{24}}{\begin{vmatrix}\epsilon_3, & \epsilon_1\\ \omega_3, & \omega_1\end{vmatrix}}=\frac{s_{14}}{\begin{vmatrix}\epsilon_2, & \epsilon_3\\ \omega_2, & \omega_3\end{vmatrix}}=\frac{s_{12}}{\begin{vmatrix}\epsilon_3, & \epsilon_4\\ \omega_3, & \omega_4\end{vmatrix}}=\frac{s_{31}}{\begin{vmatrix}\epsilon_2, & \epsilon_4\\ \omega_2, & \omega_4\end{vmatrix}}=\frac{s_{23}}{\begin{vmatrix}\epsilon_1, & \epsilon_4\\ \omega_1, & \omega_4\end{vmatrix}}.$$

Let the common value of these fractions be denoted by μ. We have

$$\sum(cd-n^2)(\epsilon_3\omega_4-\epsilon_4\omega_3)^2=(\sum a\epsilon_1^2)(\sum a\omega_1^2)-(\sum a\epsilon_1\omega_1)^2=\Omega^2\epsilon_n^2\omega_\nu^2\sin^2\iota\,;$$

we had

$$\sum(AB-H^2)s_{12}^2=1\,;$$

and by a property of determinants,

$$cd-n^2=\Omega(AB-H^2).$$

Hence

$$\frac{1}{\mu^2}=\Omega\epsilon_n^2\omega_\nu^2\sin^2\iota.$$

Accordingly, if we take superficial variables by the definition

$$t_{jk}=\frac{\epsilon_j\omega_k-\epsilon_k\omega_j}{\epsilon_n\omega_\nu\sin\iota},$$

these are connected with the variables s_{ij} by the relations

$$\Omega^{\frac{1}{2}}s_{hi}=t_{jk},$$

where $h, i, j, =1, 2, 3$, in cyclical order when $k=4$, and $h, j, k, =1, 2, 3$, in cyclical order when $i=4$. The two sets of variables are equivalent to one another; and the variables t_{ij} satisfy the permanent relation

$$\sum(ab-h^2)t_{12}^2=\Omega.$$

In what precedes, we have considered the domainal normal to each of the regions the equations of which combine to form the equations of the surface. The plane through these domainal normals is a plane orthogonal to the surface.

Ex. Denoting by Θ the quantity $\sum\sum\sum\sum(A_{ik}A_{jl}-A_{il}A_{jk})s_{ij}s_{kl}$ in the foregoing permanent orientation-relation, and by $\bar{s}_{ij}$ the orientation-variables of the orthogonal orientation, for the various values of i and j, prove that

$$\Omega^{\frac{1}{2}}\bar{s}_{ij}=\tfrac{1}{2}\frac{\partial\Theta}{\partial s_{kl}},$$

where i, j, k, l, are the integers 1, 2, 3, 4, in cyclical order.

271. Later, and still in relation to surfaces in the domain, we shall require variations of ϵ and of ω, regarded as functions of the domainal parameters, when these variations are estimated along the two domainal normal directions dn and $d\nu$, separately and also together.

We already have the values of ϵ_n, ϵ_{nn}, and therefore of ω_ν, $\omega_{\nu\nu}$. Now

$$\frac{d\epsilon}{d\nu}=\epsilon_1\frac{dp}{d\nu}+\epsilon_2\frac{dq}{d\nu}+\epsilon_3\frac{dr}{d\nu}+\epsilon_4\frac{dt}{d\nu},$$

and therefore

$$\begin{aligned}\Omega\frac{d\omega}{d\nu}\frac{d\epsilon}{d\nu}&=\sum\epsilon_1\left(\Omega\frac{d\omega}{d\nu}\frac{dp}{d\nu}\right)\\&=\sum\epsilon_1\left(\sum a\omega_1\right)\\&=\sum a\epsilon_1\omega_1=\Omega\frac{d\epsilon}{dn}\frac{d\omega}{d\nu}\cos\iota.\end{aligned}$$

Hence

$$\frac{d\epsilon}{d\nu}=\frac{d\epsilon}{dn}\cos\iota.$$

Similarly, we find

$$\frac{d\omega}{dn}=\frac{d\omega}{d\nu}\cos\iota.$$

Again, we have

$$\begin{aligned}\frac{d^2\epsilon}{d\nu dn}&=\frac{d\epsilon_n}{d\nu}\\&=\frac{dp}{d\nu}\frac{\partial\epsilon_n}{\partial p}+\frac{dq}{d\nu}\frac{\partial\epsilon_n}{\partial q}+\frac{dr}{d\nu}\frac{\partial\epsilon_n}{\partial r}+\frac{dt}{d\nu}\frac{\partial\epsilon_n}{\partial t}\\&=\sum\bar{\epsilon}_{11}\frac{dp}{dn}\frac{dp}{d\nu},\end{aligned}$$

on substituting from § 271; and similarly

$$\frac{d^2\omega}{dnd\nu}=\sum\bar{\omega}_{11}\frac{dp}{dn}\frac{dp}{d\nu}.$$

Next, we have

$$\frac{d\epsilon}{dn}\frac{d\omega}{d\nu}\cos\iota=\sum\frac{a}{\Omega}\epsilon_1\omega_1;$$

and therefore

$$\begin{aligned}&-\frac{\partial\iota}{\partial p}\frac{d\epsilon}{dn}\frac{d\omega}{d\nu}\sin\iota+\frac{\partial}{\partial p}\left(\frac{d\epsilon}{dn}\frac{d\omega}{d\nu}\right)\cos\iota\\&\quad=-\sum\left\{\epsilon_1\omega_1\frac{2}{\Omega}(a\Gamma_{11}+h\Gamma_{12}+g\Gamma_{13}+l\Gamma_{14})\right\}+\sum\frac{a}{\Omega}\epsilon_{11}\omega_1+\sum\frac{a}{\Omega}\epsilon_1\omega_{11}\\&\quad=\frac{d\omega}{d\nu}\left(\bar{\epsilon}_{11}\frac{dp}{d\nu}+\bar{\epsilon}_{12}\frac{dq}{d\nu}+\bar{\epsilon}_{13}\frac{dr}{d\nu}+\bar{\epsilon}_{14}\frac{dt}{d\nu}\right)\\&\qquad+\frac{d\epsilon}{dn}\left(\bar{\omega}_{11}\frac{dp}{dn}+\bar{\omega}_{12}\frac{dq}{dn}+\bar{\omega}_{13}\frac{dr}{dn}+\bar{\omega}_{14}\frac{dt}{dn}\right)\end{aligned}$$

Similarly for the other parametric derivatives of ι. Hence, on reduction,

$$\frac{d\iota}{dn}\frac{d\epsilon}{dn}\frac{d\omega}{d\nu}\sin\iota$$
$$=-\frac{d\omega}{d\nu}\left[\sum\left\{\bar{\epsilon}_{11}\frac{dp}{dn}\left(\frac{dp}{d\nu}-\frac{dp}{dn}\cos\iota\right)\right\}\right]-\frac{d\epsilon}{dn}\left[\sum\left\{\bar{\omega}_{11}\frac{dp}{dn}\left(\frac{dp}{dn}-\frac{dp}{d\nu}\cos\iota\right)\right\}\right],$$

$$\frac{d\iota}{d\nu}\frac{d\epsilon}{dn}\frac{d\omega}{d\nu}\sin\iota$$
$$=-\frac{d\omega}{d\nu}\left[\sum\left\{\bar{\epsilon}_{11}\frac{dp}{d\nu}\left(\frac{dp}{d\nu}-\frac{dp}{dn}\cos\iota\right)\right\}\right]-\frac{d\epsilon}{dn}\left[\sum\bar{\omega}_{11}\frac{dp}{d\nu}\left(\frac{dp}{dn}-\frac{dp}{d\nu}\cos\iota\right)\right].$$

In the same way, by the use of these variations of ι, the inclination of the regions, the following relations are obtained:

$$\frac{d^2\epsilon}{dnd\nu}-\sum\left\{\bar{\epsilon}_{11}\frac{dp}{dn}\frac{dp}{d\nu}\right\}=\frac{\epsilon_n}{\omega_\nu}\sum\left\{\bar{\omega}_{11}\frac{dp}{dn}\left(\frac{dp}{dn}-\frac{dp}{d\nu}\cos\iota\right)\right\},$$
$$\frac{d^2\epsilon}{d\nu^2}-\sum\left\{\bar{\epsilon}_{11}\left(\frac{dp}{d\nu}\right)^2\right\}=\frac{\epsilon_n}{\omega_\nu}\sum\left\{\bar{\omega}_{11}\frac{dp}{d\nu}\left(\frac{dp}{dn}-\frac{dp}{d\nu}\cos\iota\right)\right\},$$
$$\frac{d^2\omega}{d\nu dn}-\sum\left\{\bar{\omega}_{11}\frac{dp}{dn}\frac{dp}{d\nu}\right\}=\frac{\omega_\nu}{\epsilon_n}\sum\left\{\bar{\epsilon}_{11}\frac{dp}{d\nu}\left(\frac{dp}{d\nu}-\frac{dp}{dn}\cos\iota\right)\right\},$$
$$\frac{d^2\omega}{dn^2}-\sum\left\{\bar{\omega}_{11}\left(\frac{dp}{dn}\right)^2\right\}=\frac{\omega_\nu}{\epsilon_n}\sum\left\{\bar{\epsilon}_{11}\frac{dp}{dn}\left(\frac{dp}{d\nu}-\frac{dp}{dn}\cos\iota\right)\right\}.$$

We thus have all the second derivatives of the parametric magnitudes ϵ and ω estimated along the two domainal normals to the respective regions.

Modifications of the quantities of the type $\frac{dp}{dn}-\frac{dp}{d\nu}\cos\iota$ and $\frac{dp}{d\nu}-\frac{dp}{dn}\cos\iota$, appertaining to the orientation orthogonal to the surface, can be constructed similar to those used for $p_1'-p_2'\cos\epsilon$ and $p_2'-p_1'\cos\epsilon$ in the orientation of the surface.

Bi-parametric representation of a surface.

272. Another analytical representation of domainal surfaces, formally distinct from the representation as the intersection of two domainal regions, is provided by postulating the parameters p, q, r, t, as functions of two new independent parameters x, z, in a form

$$p=p(x, z),\quad q=q(x, z),\quad r=r(x, z),\quad t=t(x, z),$$

we write

$$\frac{\partial p}{\partial x}=p_1,\quad \frac{\partial p}{\partial z}=p_2,$$

and similarly for other derivatives: also

$$E=\sum Ap_1^2,\quad J=\sum Ap_1p_2,\quad I=\sum Ap_2^2,$$
$$V^2=EI-J^2=\sum\{(AB-H^2)(p_1q_2-q_1p_2)^2\}:$$

then the arc-element on the surface is

$$ds^2=E\,dx^2+2J\,dxdz+I\,dz^2.$$

Let the two sets of direction-variables $\frac{dx}{ds}, \frac{dz}{ds}$, and $\frac{\delta x}{\delta s}, \frac{\delta z}{\delta s}$, determine the same two directions on the surface in § 270 as are determined by the two sets $\frac{dp}{ds}, \frac{dq}{ds}, \frac{dr}{ds}, \frac{dt}{ds}$, and $\frac{\delta p}{\delta s}, \frac{\delta q}{\delta s}, \frac{\delta r}{\delta s}, \frac{\delta t}{\delta s}$; then, for the orientation-variables of the surface, we have relations of the type

$$s_{12}\sin\bar{\omega}=\frac{dp}{ds}\frac{\delta q}{\delta s}-\frac{dq}{ds}\frac{\delta p}{\delta s}=\begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix}\left(\frac{dx}{ds}\frac{\delta z}{\delta s}-\frac{dz}{ds}\frac{\delta x}{\delta s}\right).$$

But

$$\sin^2\bar{\omega}=\left[\sum A\left(\frac{dp}{ds}\right)^2\right]\left[\sum A\left(\frac{\delta p}{\delta s}\right)^2\right]-\left[\sum A\frac{dp}{ds}\frac{\delta p}{\delta s}\right]^2$$
$$=\sum\left\{(AB-H^2)\left(\frac{dp}{ds}\frac{\delta q}{\delta s}-\frac{dq}{ds}\frac{\delta p}{\delta s}\right)^2\right\}$$
$$=V^2\left(\frac{dx}{ds}\frac{\delta z}{\delta s}-\frac{dz}{ds}\frac{\delta x}{\delta s}\right)^2,$$

the customary expression for the inclination between the parametric lines on the surface, a result to be inferred also from the fact that the inclination of those lines is unaffected by the change of parameters. Hence, as the expression of the orientation-variables of the surface in terms of the new defining equations, we have

$$\frac{s_{12}}{\begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix}}=\frac{s_{23}}{\begin{vmatrix} q_1, & r_1 \\ q_2, & r_2 \end{vmatrix}}=\frac{s_{31}}{\begin{vmatrix} r_1, & p_1 \\ r_2, & p_2 \end{vmatrix}}=\frac{s_{14}}{\begin{vmatrix} p_1, & t_1 \\ p_2, & t_2 \end{vmatrix}}=\frac{s_{24}}{\begin{vmatrix} q_1, & t_1 \\ q_2, & t_2 \end{vmatrix}}=\frac{s_{34}}{\begin{vmatrix} r_1, & t_1 \\ r_2, & t_2 \end{vmatrix}}=\frac{1}{V};$$

and the relation

$$\sum\{(AB-H^2)s_{12}^2\}=\sum\sum\sum\sum(A_{ik}A_{jl}-A_{il}A_{jk})s_{ij}s_{kl}=1$$

still is satisfied.

Thus, corresponding to the three modes of defining a domainal surface: (i), by means of the domainal direction-variables of two guiding directions in the orientation: or (ii), by means of two intersecting parametric regions in the domain: or, (iii), by means of a bi-parametric representation of the domainal parameters, there are three forms of expression for the domainal variables for the orientation of the surface. It is to be noted that, whatever mode of representation

be adopted, the conditions to be satisfied, in order that a domainal direction p', q', r', t', through O may touch the surface having $s_{12}, s_{23}, s_{31}, s_{14}, s_{24}, s_{34}$, as orientation-variables at that point, are

$$\left.\begin{array}{rrrr} & -s_{34}q' & +s_{24}r' & -s_{23}t'=0 \\ s_{34}p' & & -s_{14}r' & -s_{31}t'=0 \\ -s_{24}p' & +s_{14}q' & & -s_{12}t'=0 \\ s_{23}p' & +s_{31}q' & +s_{12}r' & =0 \end{array}\right\}.$$

They are equivalent to two independent relations only, because of the necessary condition

$$s_{23}s_{14}+s_{31}s_{24}+s_{12}s_{34}=0.$$

It further is to be noted that, if p', q', r', t', are direction-variables of a direction not lying in the surface, the magnitudes

$$s_{34}q'-s_{24}r'+s_{23}t', \quad -s_{34}p'+s_{14}r'+s_{31}t', \; s_{24}p'-s_{14}q'+s_{12}t', \; -s_{23}p'-s_{31}q'-s_{12}r',$$

are of the nature of regional variables P, Q, R, T.

Finally, it may be remarked (the remark would apply to the investigations of superficial and of regional properties as well as to those of domainal properties) that, in the discussions which follow, there will appear certain combinations of functions of position (such as primary magnitudes and their derivatives), of the direction-variables of a curve at a point, of the variables P, Q, R, T, of volumetric orientation (shewn, in § 269, to have a geometrical equivalent in the direction-variables of the normal), and of the variables of superficial orientation in any of the three forms that have been indicated. When these combinations represent some geometrical magnitude or quantities related to some geometrical property, they are of the nature of covariantive concomitants belonging to the whole system of quaternariants of the domain. No particular discrimination will be made among the concomitants as regards their types; and there will be no attempt to range them in their respective specific classes. Such discrimination and such ranging become an inevitable necessity in any systematic development of the algebraical theory of quaternariants, with or without its geometrical applications. But this development will not be undertaken at the present stage; all that will be done is to note, occasionally, significant instances of relations connected with that theory.

Tangent block of a domain.

273. The typical equation of a tangent line to the domain through any point O is

$$\frac{\bar{y}-y}{y'}=\lambda,$$

where λ is the current parameter along the line; or, when the value of y' is used, the equation is

$$\bar{y}-y=\lambda\frac{dy}{ds}=y_1\lambda p'+y_2\lambda q'+y_3\lambda r'+y_4\lambda t'$$
$$=\alpha y_1+\beta y_2+\gamma y_3+\delta y_4,$$

with the notational significance for the first parametric derivatives of y. Thus all such tangent lines through the point are such that

$$\| \ \bar{y}-y, \ y_1, \ y_2, \ y_3, \ y_4 \ \|=0,$$

which accordingly are the equations of the tangent block in the domain.

Any point in the tangent block (and so the tangent block itself) can be represented by the four-parameter set of equations

$$\bar{y}-y=\alpha y_1+\beta y_2+\gamma y_3+\delta y_4\,;$$

and any direction l_1, l_2, ... , (typically denoted by l), in the domain can be represented by

$$l=\lambda y_1+\mu y_2+\nu y_3+\kappa y_4.$$

Let Π denote the length of the perpendicular to this block, drawn from a domainal point at a small arc-distance ∇ from the point O; by analysis similar to that already used for the general configuration, and for a surface and a region, we find, up to the second power of ∇ inclusive,

$$X_m\Pi=\tfrac{1}{2}\nabla^2\sum\left\{\left(\frac{\partial^2 y_m}{\partial p^2}-\frac{\partial y_m}{\partial p}\Gamma_{11}-\frac{\partial y_m}{\partial q}\Delta_{11}-\frac{\partial y_m}{\partial r}\Theta_{11}-\frac{\partial y_m}{\partial t}\Phi_{11}\right)p'^2\right\},$$

for all the values of m corresponding to the space-coordinates, the quantities X_1, X_2, ... , denoting the direction-cosines of the perpendicular Π; and we also have

$$\sum X_m\frac{\partial y_m}{\partial p}=0,\quad \sum X_m\frac{\partial y_m}{\partial q}=0,\quad \sum X_m\frac{\partial y_m}{\partial r}=0,\quad \sum X_m\frac{\partial y_m}{\partial t}=0,$$

the last four equations being an analytical expression of the property that the perpendicular Π is at right angles to every direction in the tangent block.

Now it will appear (§ 274) that ρ, the radius of circular curvature of a domainal geodesic in the direction p', q', r', t', and its direction-cosines Y_1, Y_2, ... , are determined by the set of equations

$$\frac{Y_m}{\rho}=\frac{d^2y_m}{ds^2}$$
$$=\sum\left(\frac{\partial^2 y_m}{\partial p^2}-\frac{\partial y_m}{\partial p}\Gamma_{11}-\frac{\partial y_m}{\partial q}\Delta_{11}-\frac{\partial y_m}{\partial r}\Theta_{11}-\frac{\partial y_m}{\partial t}\Phi_{11}\right)p'^2,$$

for all the values of m. Hence we have

$$\frac{Y_m}{\rho}=X_m\frac{2\Pi}{\nabla^2};$$

and therefore

$$\operatorname*{Lim}_{\nabla\to 0}\frac{2\Pi}{\nabla^2}=\frac{1}{\rho},$$

giving an approximation to the length Π of the perpendicular, while

$$X_m=Y_m,$$

for all values of m: that is, the limiting position of the perpendicular on a tangent block, drawn from a contiguous point at an arc-distance ∇ from O, is the direction of the radius of curvature of the geodesic joining the point to O, the limit being attained by the ultimately evanescent value of ∇.

Equations of domainal geodesics: circular curvature.

274. The intrinsic equations of a domainal geodesic are relations which must be satisfied if the magnitude

$$\int\left\{\sum A\left(\frac{dp}{du}\right)^2\right\}^{\frac{1}{2}}du$$

is to be a minimum; and by analysis similar to that used for a surface (§ 93), for a region (§ 161), and for the general configuration (§ 17), it is found that there are four intrinsic equations

$$p''+\sum\Gamma_{11}p'^2=0,\quad q''+\sum\Delta_{11}p'^2=0,\quad r''+\sum\Theta_{11}p'^2=0,\quad t''+\sum\Phi_{11}p'^2=0,$$

which, however, are equivalent to only three independent equations under the retention of the permanent arc-relation

$$\sum Ap'^2=1.$$

Let $1/\rho$ denote the circular curvature of the geodesic at O; and let the direction-cosines of the radius of circular curvature be denoted by Y_1, Y_2, ..., Y being taken as typical of them all. Then

$$Y_m=\rho\frac{d^2y_m}{ds^2},$$

for all the values of m: or, typically,

$$\begin{aligned}\frac{Y}{\rho}&=\frac{d^2y}{ds^2}\\ &=y_1p''+y_2q''+y_3r''+y_4t''+\sum y_{11}p'^2\\ &=\sum(y_{11}-y_1\Gamma_{11}-y_2\Delta_{11}-y_3\Theta_{11}-y_4\Phi_{11})p'^2,\end{aligned}$$

when the geodesic values of p'', q'', r'', t'', are substituted. In accordance with the earlier notation, we write

$$\eta_{\alpha\beta}=y_{\alpha\beta}-y_1\Gamma_{\alpha\beta}-y_2\Delta_{\alpha\beta}-y_3\Theta_{\alpha\beta}-y_4\Phi_{\alpha\beta},$$

for all values α, β, $=1, 2, 3, 4$, in all combinations; and now we have

$$\frac{Y}{\rho}=\sum\eta_{11}p'^2,$$

with the relation

$$\sum Y^2=1,$$

as equations characteristic of the circular curvature of domainal geodesics.

Having regard to the equations (§ 268) which define the magnitudes $\Gamma, \Delta, \Theta, \Phi$, in terms of the quantities $\sum y_\mu y_{\alpha\beta}$, we at once verify the relations

$$\sum y_1\eta_{\alpha\beta}=0, \quad \sum y_2\eta_{\alpha\beta}=0, \quad \sum y_3\eta_{\alpha\beta}=0, \quad \sum y_4\eta_{\alpha\beta}=0.$$

Hence each of the directions, typically represented as to direction-cosines by the magnitude

$$\frac{\eta_{\alpha\beta}}{(\sum \eta_{\alpha\beta}^2)^{\frac{1}{2}}},$$

is orthogonal to the tangent block of the domain; and, of course, the direction of the radius of circular curvature is orthogonal to that block, its direction-cosines satisfying the relations

$$\sum y_1 Y=0, \quad \sum y_2 Y=0, \quad \sum y_3 Y=0, \quad \sum y_4 Y=0.$$

Also we note that, in addition to the tangent block of four dimensions organically related to the whole domain, there is a homaloidal amplitude of five dimensions connected with any domainal geodesic and, for that geodesic, represented by the equations

$$\| \, \bar{y}-y, \quad y_1, \quad y_2, \quad y_3, \quad y_4, \quad Y \, \|=0.$$

The Riemann four-index symbols.

275. It is convenient to introduce the Riemann four-index symbols at this stage, merely modifying the investigation (§ 14) for the general m-fold amplitude for a general domain, and using the conventions

$$x_1=p, \quad x_2=q, \quad x_3=r, \quad x_4=t.$$

In place of the second derivatives y_{ij} of the typical point-variable y, an associated quantity η_{ij} is used, under the definition

$$\eta_{ij}=y_{ij}-y_1\Gamma_{ij}-y_2\Delta_{ij}-y_3\Theta_{ij}-y_4\Phi_{ij},$$

so that also

$$y_{ij}=\eta_{ij}+\sum_\lambda y_\lambda\{ij, \lambda\},$$

with the equivalence of the symbols $\Gamma, \Delta, \Theta, \Phi$, to the Christoffel symbols $\{ab, c\}$.

Because the relation

$$\sum y_\lambda \eta_{\mu\nu}=0$$

holds for each of the combinations $\lambda, \mu, \nu, =1, 2, 3, 4$, repetitions being permissible, we have

$$\sum y_{ik}y_{jl}=\sum \eta_{ik}\eta_{jl}+\sum_\lambda \sum_\mu A_{\lambda\mu}\{ik, \lambda\}\{jl, \mu\}.$$

Now, differentiating the relation

$$A_{il}=\sum y_i y_l$$

with regard to x_j and x_k, we have

$$\frac{\partial^2 A_{il}}{\partial x_j \partial x_k}=\sum\left(\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k}\frac{\partial y}{\partial x_l}+\frac{\partial^3 y}{\partial x_l \partial x_j \partial x_k}\frac{\partial y}{\partial x_i}\right)+\sum\left(y_{ij}y_{kl}+y_{ik}y_{jl}\right),$$

for all combinations i, l, j, k; hence

$$\begin{aligned}\tfrac{1}{2}&\left(\frac{\partial^2 A_{il}}{\partial x_j \partial x_k}-\frac{\partial^2 A_{ik}}{\partial x_j \partial x_l}-\frac{\partial^2 A_{jl}}{\partial x_i \partial x_k}+\frac{\partial^2 A_{jk}}{\partial x_i \partial x_l}\right)\\ &=\sum\left(y_{ik}y_{jl}-y_{jk}y_{il}\right)\\ &=\sum\left(\eta_{ik}\eta_{jl}-\eta_{jk}\eta_{il}\right)+\sum_{\lambda}\sum_{\mu}A_{\lambda\mu}[\{ik,\lambda\}\{jl,\mu\}-\{jk,\lambda\}\{il,\mu\}].\end{aligned}$$

The Riemann four-index symbol (ij, kl) is usually defined as

$$\begin{aligned}(ij, kl)=\tfrac{1}{2}&\left(\frac{\partial^2 A_{il}}{\partial x_j \partial x_k}-\frac{\partial^2 A_{ik}}{\partial x_j \partial x_l}-\frac{\partial^2 A_{jl}}{\partial x_i \partial x_k}+\frac{\partial^2 A_{jk}}{\partial x_i \partial x_l}\right)\\ &+\sum_{\lambda}\sum_{\mu}A_{\lambda\mu}[\{il,\mu\}\{jk,\lambda\}-\{jl,\mu\}\{ik,\lambda\}],\end{aligned}$$

where, in the double summation on the right-hand side, the integers λ and μ can be interchanged; consequently, this four-index symbol is such that

$$(ij, kl)=\sum\left(\eta_{ik}\eta_{jl}-\eta_{il}\eta_{jk}\right),$$

the summation on the right-hand side being for all the quantities η associated with the several point-variables in the plenary space of the domain.

The magnitude, as originally defined, involves only the primary magnitudes of the domain and, explicitly or implicitly, their first and second parametric derivatives; it thus is of the nature of a concomitant of the domain. In the inferred value, it is equal to a quantity which depends upon the magnitudes η, that is, implicitly upon the first and the second parametric derivatives of the space-coordinates, and these magnitudes η are connected with the prime normals of the domainal geodesics; thus, in the inferred form, the spatial quantity, which occurs on the right-hand side, is of the nature of a concomitant of the domain.

The quantities $\eta_{\alpha\beta}$ are unaltered when α and β are interchanged; hence the Riemann four-index symbol satisfies the identical relations

$$\begin{aligned}(ij, kl)&=(ji, lk)=(lk, ji)=(kl, ij)\\ &=-(ij, lk)=-(ji, kl)=-(lk, ij)=-(kl, ji),\\ (ij, kl)&+(ik, lj)+(il, jk)=0.\end{aligned}$$

Thus the symbol vanishes if the integers i and j are equal, and it vanishes if the integers k and l are equal. For a domain, the numbers i, j, k, l, can have the values

1, 2, 3, 4, separately and independently of one another; hence the number of non-vanishing four-index symbols for the domain is twenty-one. There is a single relation among these non-vanishing symbols which, in all its forms, is effectively

$$(12, 34)+(13, 42)+(14, 23)=0\,;$$

consequently, the number of non-vanishing four-index symbols (ij, kl) for a domain, which are linearly independent of one another, is twenty. Obviously, in citing the number of a non-vanishing four-index symbol, we may take $j>i$, $l>k$.

The actual values of the non-vanishing symbols, alike in terms of the primary magnitudes A and their first derivatives as implicitly contained in the symbols Γ, Δ, Θ, Φ, and also in terms of the quantities η, with the convention for the symbol $(\alpha\beta\between\gamma\delta)$ as defined to be

$$(\alpha\beta\between\gamma\delta)=(A, B, C, D, F, G, H, L, M, N\between\Gamma_{\alpha\beta}, \Delta_{\alpha\beta}, \Theta_{\alpha\beta}, \Phi_{\alpha\beta}\between\Gamma_{\gamma\delta}, \Delta_{\gamma\delta}, \Theta_{\gamma\delta}, \Phi_{\gamma\delta}),$$

are as follows:

$$\left.\begin{aligned}
(12, 12)&=\sum(\eta_{11}\eta_{22}-\eta_{12}{}^2)=-\tfrac{1}{2}(A_{22}-2H_{12}+B_{11})-(11\between22)+(12\between12)\\
(13, 13)&=\sum(\eta_{11}\eta_{33}-\eta_{13}{}^2)=-\tfrac{1}{2}(A_{33}-2G_{13}+C_{11})-(11\between33)+(13\between13)\\
(14, 14)&=\sum(\eta_{11}\eta_{44}-\eta_{14}{}^2)=-\tfrac{1}{2}(A_{44}-2L_{14}+D_{11})-(11\between44)+(14\between14)\\
(23, 23)&=\sum(\eta_{22}\eta_{33}-\eta_{23}{}^2)=-\tfrac{1}{2}(B_{33}-2F_{23}+C_{22})-(22\between33)+(23\between23)\\
(24, 24)&=\sum(\eta_{22}\eta_{44}-\eta_{24}{}^2)=-\tfrac{1}{2}(B_{44}-2M_{24}+D_{22})-(22\between44)+(24\between24)\\
(34, 34)&=\sum(\eta_{33}\eta_{44}-\eta_{34}{}^2)=-\tfrac{1}{2}(C_{44}-2N_{34}+D_{33})-(33\between44)+(34\between34)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
(12, 34)&=\sum(\eta_{13}\eta_{24}-\eta_{14}\eta_{23})=\tfrac{1}{2}(F_{14}-G_{24}-M_{13}+L_{23})-(13\between24)+(14\between23)\\
(13, 42)&=\sum(\eta_{14}\eta_{23}-\eta_{12}\eta_{34})=\tfrac{1}{2}(H_{34}-F_{14}-L_{23}+N_{12})-(14\between23)+(12\between34)\\
(14, 23)&=\sum(\eta_{12}\eta_{34}-\eta_{13}\eta_{24})=\tfrac{1}{2}(G_{24}-H_{34}-N_{12}+M_{13})-(12\between34)+(13\between24)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
(12, 31)&=\sum(\eta_{12}\eta_{13}-\eta_{11}\eta_{23})=\tfrac{1}{2}(A_{23}-H_{13}-G_{12}+F_{11})-(12\between13)+(11\between23)\\
(14, 21)&=\sum(\eta_{12}\eta_{14}-\eta_{11}\eta_{24})=\tfrac{1}{2}(A_{24}-H_{14}-L_{12}+M_{11})-(12\between14)+(11\between24)\\
(13, 41)&=\sum(\eta_{13}\eta_{14}-\eta_{11}\eta_{34})=\tfrac{1}{2}(A_{34}-G_{14}-L_{13}+N_{11})-(13\between14)+(11\between34)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
(23, 42)&=\sum(\eta_{23}\eta_{24}-\eta_{22}\eta_{34})=\tfrac{1}{2}(B_{34}-F_{24}-M_{23}+N_{22})-(23\between24)+(22\between34)\\
(21, 32)&=\sum(\eta_{12}\eta_{23}-\eta_{22}\eta_{13})=\tfrac{1}{2}(B_{13}-H_{23}-F_{12}+G_{22})-(12\between23)+(22\between13)\\
(24, 12)&=\sum(\eta_{12}\eta_{24}-\eta_{22}\eta_{14})=\tfrac{1}{2}(B_{14}-H_{24}-M_{12}+L_{22})-(12\between24)+(22\between14)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
(34, 13)&=\sum(\eta_{13}\eta_{34}-\eta_{33}\eta_{14})=\tfrac{1}{2}(C_{14}-G_{34}-N_{13}+L_{33})-(13\between34)+(33\between14)\\
(31, 23)&=\sum(\eta_{13}\eta_{23}-\eta_{33}\eta_{12})=\tfrac{1}{2}(C_{12}-G_{23}-F_{13}+H_{33})-(13\between23)+(33\between12)\\
(32, 43)&=\sum(\eta_{23}\eta_{34}-\eta_{33}\eta_{24})=\tfrac{1}{2}(C_{24}-F_{34}-N_{23}+M_{33})-(23\between34)+(33\between24)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
(41, 24)&=\sum(\eta_{14}\eta_{24}-\eta_{44}\eta_{12})=\tfrac{1}{2}(D_{12}-L_{24}-M_{14}+H_{44})-(14\between24)+(44\between12)\\
(42, 34)&=\sum(\eta_{24}\eta_{34}-\eta_{44}\eta_{23})=\tfrac{1}{2}(D_{23}-M_{34}-N_{24}+F_{44})-(24\between34)+(44\between23)\\
(43, 14)&=\sum(\eta_{14}\eta_{34}-\eta_{44}\eta_{13})=\tfrac{1}{2}(D_{13}-N_{14}-L_{34}+G_{44})-(14\between34)+(44\between13)
\end{aligned}\right\}.$$

To the symbols at the extreme left, a variety of equivalent forms may be given, so that the substituted form might conform to the conventions $j > i$ and $l > k$ in a cited form (ij, kl); but account might then have to be taken also of a possible prefixed sign.

These four-index symbols are the coefficients of the orientation-variables in the Riemann measure of curvature of an amplitude (here, a domain) estimated in any superficial orientation. For those variables, taken to be the quantities s_{ij} of §§ 270, 272, we have

$$\left\{\sum A\left(\frac{dp}{ds}\right)^2\right\}\left\{\sum A\left(\frac{\delta p}{\delta s}\right)^2\right\}-\left\{\sum A\frac{dp}{ds}\frac{\delta p}{\delta s}\right\}^2=\sin^2\bar{\omega},$$

or if we take

$$\frac{dp}{ds}=x_1', \quad \frac{dq}{ds}=x_2', \quad \frac{dr}{ds}=x_3', \quad \frac{dt}{ds}=x_4',$$

$$\frac{\delta p}{\delta s}=z_1', \quad \frac{\delta q}{\delta s}=z_2', \quad \frac{\delta r}{\delta s}=z_3', \quad \frac{\delta t}{\delta s}=z_4',$$

this relation can be written

$$\sum (A_{ik}A_{jl}-A_{il}A_{jk})(x_i'z_j'-x_j'z_i')(x_k'z_l'-x_l'z_k')=\sin^2\bar{\omega}.$$

The Riemann measure of curvature of the domain in the orientation established by the two directions x_1', x_2', x_3', x_4'; and z_1', z_2', z_3', z_4'; is defined to be

$$K=\frac{\sum (ij, kl)\, s_{ij}s_{kl}}{\sum (A_{ik}A_{jl}-A_{il}A_{jk})\, s_{ij}s_{kl}},$$

where, in the summations alike in the numerator and denominator, the full coefficients of s_{ij}^2 have a numerical factor unity when the suffix k, l, is the same as the suffix ij, and the full coefficients of $s_{ij}s_{kl}$ have a numerical factor two when the suffix kl is not the same as the suffix ij.

The significance of the Riemann measure will be established later (§ 312) as the sphericity of the domain in the orientation with the variables s_{ij}.

Geodesic polar coordinates.

276. As a preliminary to the establishment of geodesic polar coordinates, we consider the conditions under which a parametric curve may be a geodesic in the domain. Let it be the curve given by

$$p=\text{variable}, \quad q=\text{constant}, \quad r=\text{constant}, \quad t=\text{constant}.$$

Then the intrinsic equations of this special geodesic become

$$p''+\Gamma_{11}p'^2=0, \quad \Delta_{11}p'^2=0, \quad \Theta_{11}p'^2=0, \quad \Phi_{11}p'^2=0,$$

so that the conditions are

$$\Delta_{11}=0, \quad \Theta_{11}=0, \quad \Phi_{11}=0.$$

In general, we have

$$\begin{aligned}
\tfrac{1}{2}A_1 &= A\Gamma_{11} + H\Delta_{11} + G\Theta_{11} + L\Phi_{11},\\
H_1 - \tfrac{1}{2}A_2 &= H\Gamma_{11} + B\Delta_{11} + F\Theta_{11} + M\Phi_{11},\\
G_1 - \tfrac{1}{2}A_3 &= G\Gamma_{11} + F\Delta_{11} + C\Theta_{11} + N\Phi_{11},\\
L_1 - \tfrac{1}{2}A_4 &= L\Gamma_{11} + M\Delta_{11} + N\Theta_{11} + D\Phi_{11};
\end{aligned}$$

and therefore the three special conditions become

$$\tfrac{1}{2}\frac{A_1}{A} = \frac{1}{H}(H_1 - \tfrac{1}{2}A_2) = \frac{1}{G}(G_1 - \tfrac{1}{2}A_3) = \frac{1}{L}(L_1 - \tfrac{1}{2}A_4),$$

that is,

$$\frac{A_2}{2A} = \frac{1}{A^{\frac{1}{2}}}\frac{\partial}{\partial p}\left(\frac{H}{A^{\frac{1}{2}}}\right), \quad \frac{A_3}{2A} = \frac{1}{A^{\frac{1}{2}}}\frac{\partial}{\partial p}\left(\frac{G}{A^{\frac{1}{2}}}\right), \quad \frac{A_4}{2A} = \frac{1}{A^{\frac{1}{2}}}\frac{\partial}{\partial p}\left(\frac{L}{A^{\frac{1}{2}}}\right).$$

Hence there exists a function of p, q, r, t, which may be denoted by l and is such that

$$A^{\frac{1}{2}} = \frac{\partial l}{\partial p}, \quad \frac{H}{A^{\frac{1}{2}}} = \frac{\partial l}{\partial q}, \quad \frac{G}{A^{\frac{1}{2}}} = \frac{\partial l}{\partial r}, \quad \frac{L}{A^{\frac{1}{2}}} = \frac{\partial l}{\partial t};$$

and the relations, implied by the existence of these four equations, are sufficient to secure that the selected parametric curve satisfies the intrinsic equations of domainal geodesics.

With these values, the domainal element of arc is now given by the expression

$$\begin{aligned}
ds^2 &= \left(\frac{\partial l}{\partial p}\right)^2 dp^2 + 2\frac{\partial l}{\partial p}\frac{\partial l}{\partial q}dp\,dq + 2\frac{\partial l}{\partial p}\frac{\partial l}{\partial r}dp\,dr + 2\frac{\partial l}{\partial p}\frac{\partial l}{\partial t}dp\,dt\\
&\qquad + (B, C, D, N, M, F \between dq, dr, dt)^2\\
&= dl^2 + (B_0, C_0, D_0, N_0, M_0, F_0 \between dq, dr, dt)^2,
\end{aligned}$$

where

$$\left.\begin{aligned}
B_0 &= B - \left(\frac{\partial l}{\partial q}\right)^2, & N_0 &= N - \frac{\partial l}{\partial r}\frac{\partial l}{\partial t}\\
C_0 &= C - \left(\frac{\partial l}{\partial r}\right)^2, & M_0 &= M - \frac{\partial l}{\partial t}\frac{\partial l}{\partial q}\\
D_0 &= D - \left(\frac{\partial l}{\partial t}\right)^2, & F_0 &= F - \frac{\partial l}{\partial q}\frac{\partial l}{\partial r}
\end{aligned}\right\}.$$

Along the special geodesic, we have $dq=0$, $dr=0$, $dt=0$, so that ds becomes equal to dl: that is, the new parameter l is the length of the arc along the geodesic curve. Moreover, the original magnitude A does not vanish, and we have

$$\frac{\partial l}{\partial p} = A^{\frac{1}{2}},$$

so that the complete expression for l must involve p; therefore l can be taken as a domainal parameter, with q, r, t, (or with three quantities, involving q, r, t,

and independent of one another as functions of q, r, t), for the representation of the domain.

A region in a domain can be represented parametrically by a single relation among the parameters. Consider, therefore, the region represented by

$$l = \text{a parametric constant};$$

any arc $d\sigma$, lying in this region, satisfies the equation

$$\frac{\partial l}{\partial p}\frac{dp}{d\sigma}+\frac{\partial l}{\partial q}\frac{dq}{d\sigma}+\frac{\partial l}{\partial r}\frac{dr}{d\sigma}+\frac{\partial l}{\partial t}\frac{dt}{d\sigma}=0,$$

that is, it satisfies the equation

$$A\frac{dp}{d\sigma}+H\frac{dq}{d\sigma}+G\frac{dr}{d\sigma}+H\frac{dt}{d\sigma}=0.$$

Along the geodesic itself, the direction-variables are $p'=A^{-\frac{1}{2}}$, $q'=0$, $r'=0$, $t'=0$; and therefore, if ϕ denotes the inclination between the geodesic and the arc $d\sigma$, we have

$$\cos\phi=\sum Ap'\frac{dp}{d\sigma}=0.$$

Consequently the geodesic, being at right angles to every direction in the region, is orthogonal to the region. It therefore follows that the regions, specified by

$$l = \text{constant}$$

in a domain the arc-element of which is given by an expression

$$ds^2=dl^2+(B_0, C_0, D_0, N_0, M_0, F_0 \between d\phi, d\chi, d\psi)^2,$$

where l, ϕ, χ, ψ, are the four independent parameters, are parallel regions; and the normal geodesic distance between two such regions is the difference in the two values of l by which the regions are specified.

The partial differential equation of the first order, satisfied by geodesically parallel regions in the domain when the arc-element of the domain retains its initial form, can be constructed as follows. The arc-elements given by the two forms

$$(A, B, C, D, F, G, H, L, M, N \between dp, dq, dr, dt)^2,$$
$$dl^2+(B_0, C_0, D_0, N_0, M_0, F_0 \between d\phi, d\chi, d\psi)^2,$$

must be the same for all variations of the parameters. Hence there are four relations of the type

$$A_{ii}=l_i^2+(B_0, C_0, D_0, N_0, M_0, F_0 \between \phi_i, \chi_i, \psi_i)^2,$$

(for $i=1$, 2, 3, 4), with the double-suffix notation of § 10 for the primary magnitudes; and there are ten relations of the type

$$A_{ij}=l_i l_j+(B_0, C_0, D_0, N_0, M_0, F_0 \between \phi_i, \chi_i, \psi_i \between \phi_j, \chi_j, \psi_j),$$

for the combinations $i, j, = 1, 2, 3, 4$. Consequently, we have

$$\begin{vmatrix} A-l_1^2, & H-l_1l_2, & G-l_1l_3, & L-l_1l_4 \\ H-l_1l_2, & B-l_2^2, & F-l_2l_3, & M-l_2l_4 \\ G-l_1l_3, & F-l_2l_3, & C-l_3^2, & N-l_3l_4 \\ L-l_1l_4, & M-l_2l_4, & N-l_3l_4, & D-l_4^2 \end{vmatrix}=0;$$

because the determinant, which results from the substitution of the values of the constituents in the preceding ten relations, vanishes unconditionally. When this determinant, involving the derivatives of l, is expanded, it can be expressed in the form

$$\sum a l_1^2 = \Omega,$$

with the customary notation: that is, the required partial differential equation of the first order satisfied by geodesically parallel regions in the domain is

$$\sum a \left(\frac{\partial \theta}{\partial p}\right)^2 = \Omega.$$

The result is in accord with the earlier result (§ 269) which gives the normal dilatation of a region $\epsilon(p, q, r, t) = 0$ in the form

$$\sum a \epsilon_1^2 = \Omega \epsilon_n^2.$$

Here dn is the domainal element of arc normal to the region; when the region $\epsilon =$ constant belongs to a family of geodesically parallel regions, we have dl as an element of domainal arc orthogonal to the region, so that

$$d\epsilon = dl, \quad dl = dn,$$

and therefore $\epsilon_n = 1$.

Further, the domain manifestly can be referred to parameters which shew the geodesics and their orthogonal regions, the domainal arc then being given by an expression

$$ds^2 = dl^2 + (B, C, D, N, M, F \between dq, dr, dt)^2,$$

where, in general, the magnitudes B, C, D, N, M, F, are functions of l as well as of q, r, t. Such a representation of the arc of the domain will be called *geodesic polar*.

277. In a domainal region, with the arc-element

$$ds_0^2 = (B, C, D, N, M, F \between dq, dr, dt)^2,$$

the right-hand side, *quà* function of dq, dr, dt, can always be expressed in a form

$$d\phi^2 + (E, J, I \between d\chi, d\psi)^2,$$

the quantity l which occurs in the coefficients B, C, D, N, M, F, being regarded as a (parametric) constant; but the transformation is effective solely for the region specified by the assigned value, and a full value of the last expression for all variations in the domain can only lead back to the form for ds^2 already obtained. Such a transformation of ds_0^2 is concerned with regional geodesics. It still remains to enquire whether, in the region, there is another domainal geodesic representation additional to that which has already been obtained: or, what is effectively the same enquiry, whether the domain can have further geodesic representation.

Suppose that, if such a representation can exist, the second polar geodesic is represented by the parametric curve

$$l = \text{constant}, \quad q = \text{variable}, \quad r = \text{constant}, \quad t = \text{constant},$$

so that, as l is constant, the new geodesics will lie in the regions orthogonal to the former polar geodesic. In order that the intrinsic equations of domainal geodesics shall be satisfied, the foregoing values must satisfy

$$l'' + \sum \Gamma_{11} l'^2 = 0, \quad q'' + \sum \Delta_{11} l'^2 = 0, \quad r'' = \sum \Theta_{11} l'^2, \quad t'' = \sum \Phi_{11} l'^2,$$

and therefore we must have

$$\Gamma_{22} = 0, \quad \Theta_{22} = 0, \quad \Phi_{22} = 0.$$

In any representation of a domain, we have

$$\begin{aligned} H_2 - \tfrac{1}{2} B_1 &= A\Gamma_{22} + H\Delta_{22} + G\Theta_{22} + L\Phi_{22}, \\ \tfrac{1}{2} B_2 &= H\Gamma_{22} + B\Delta_{22} + F\Theta_{22} + M\Phi_{22}, \\ F_2 - \tfrac{1}{2} B_3 &= G\Gamma_{22} + F\Delta_{22} + C\Theta_{22} + N\Phi_{22}, \\ M_2 - \tfrac{1}{2} B_4 &= L\Gamma_{22} + M\Delta_{22} + N\Theta_{22} + D\Phi_{22}\,; \end{aligned}$$

in the present representation of the domain, we have

$$H = 0, \quad G = 0, \quad L = 0\,;$$

and therefore, for our immediate purpose, we have

$$B_1 = 0, \quad \tfrac{1}{2} B_2 = B\Delta_{22}, \quad F_2 - \tfrac{1}{2} B_3 = F\Delta_{22}, \quad M_2 - \tfrac{1}{2} B_4 = M\Delta_{22},$$

from the four relations respectively.

Thus B is independent of l; and, because we now have

$$\frac{\partial}{\partial q}\left(\frac{F}{B^{\frac{1}{2}}}\right) = \frac{\partial}{\partial r}(B^{\frac{1}{2}}), \quad \frac{\partial}{\partial q}\left(\frac{M}{B^{\frac{1}{2}}}\right) = \frac{\partial}{\partial t}(B^{\frac{1}{2}}),$$

there must exist a function m, independent of l and involving q, r, t, such that

$$B^{\frac{1}{2}} = \frac{\partial m}{\partial q}, \quad F = \frac{\partial m}{\partial r}\frac{\partial m}{\partial q}, \quad M = \frac{\partial m}{\partial t}\frac{\partial m}{\partial q},$$

making F and M also independent of l. Hence

$$(B, C, D, N, M, F \between dq, dr, dt)^2 = dm^2 + (\bar{C}_0, \bar{N}_0, \bar{D}_0 \between dr, dt)^2,$$

where

$$\bar{C}_0 = C - \left(\frac{dm}{dr}\right)^2, \quad \bar{N}_0 = N - \frac{dm}{dr}\frac{dm}{dt}, \quad \bar{D}_0 = D - \left(\frac{dm}{dt}\right)^2;$$

and the domainal arc can be expressed in the form

$$ds^2 = dl^2 + dm^2 + (\bar{C}_0, \bar{N}_0, \bar{D}_0 \between dr, dt)^2,$$

where $\bar{C}_0$, $\bar{N}_0$, $\bar{D}_0$, are functions of the four parameters.

Consequently, for a domain with its arc expressible in the form

$$ds^2 = dp^2 + dq^2 + (C, N, D \between dr, dt)^2,$$

where the coefficients C, N, D, are functions of the four parameters, the two parametric curves, given by

$$p = \text{variable}, \quad q = \text{constant}, \quad r = \text{constant}, \quad t = \text{constant},$$

$$p = \text{constant}, \quad q = \text{variable}, \quad r = \text{constant}, \quad t = \text{constant},$$

are geodesics at right angles to one another; and thus there is a bipolar geodesic representation of the domain.

Manifestly, the amplitude

$$ds_1^2 = (C, N, D \between dr, dt)^2,$$

with p and q as parametric constants, represents a surface in the domain. At any point on the surface, a tangential direction has a typical direction-cosine

$$y_3 \frac{dr}{ds_1} + y_4 \frac{dt}{ds_1}.$$

The inclination ϕ_p to the parametric geodesic along the p-curve is given by

$$\cos\phi_p = \sum y_1 \left(y_3 \frac{dr}{ds_1} + y_4 \frac{dt}{ds_1}\right) = \frac{dr}{ds_1} \sum y_1 y_3 + \frac{dt}{ds_1} \sum y_1 y_4 = 0,$$

and the inclination ϕ_q to the parametric geodesic along the q-curve is given by

$$\cos\phi_q = \sum y_2 \left(y_3 \frac{dr}{ds_1} + y_4 \frac{dt}{ds_1}\right) = \frac{dr}{ds_1} \sum y_2 y_3 + \frac{dt}{ds_1} \sum y_2 y_4 = 0:$$

that is, every direction in the surface is at right angles to the plane containing the two polar geodesics, and consequently this plane within the tangent block of the domain is the orthogonal plane to the surface.

One analytical representation of this surface is manifestly given by the two equations

$$l = \text{constant}, \quad m = \text{constant};$$

that is, when these two functions l and m are used to provide a parametric representation of a surface, which is orthogonal to the orientation containing the two

domainal geodesics in the bi-polar representation of the domain, they must satisfy the two equations

$$\sum al_1^2 = \Omega, \quad \sum am_1^2 = \Omega;$$

and, as the two geodesics (and therefore the two regions, orthogonal to the two geodesics) are at right angles, there is also (§ 270) the equation

$$\sum al_1 m_1 = 0.$$

Further, there is a surface in the domain given by the arc-representation

$$ds^2 = dl^2 + dm^2,$$

manifestly a surface orthogonal to the preceding surface; and the orientation-variables of this surface can be constructed from the domainal normals to the two regions, l=constant, and m=constant, respectively. For the former, there are (§ 269) four direction-variables of the type

$$\Omega \frac{dp}{dl} = al_1 + hl_2 + gl_3 + ll_4,$$

and, for the latter, there are four similar direction-variables

$$\Omega \frac{dp}{dm} = am_1 + hm_2 + gm_3 + lm_4.$$

Consequently, the orientation-variables in this orientation are of the form

$$\Omega l_{34} = \Omega \left(\frac{dr}{dl}\frac{dt}{dm} - \frac{dr}{dm}\frac{dt}{dl} \right)$$

$$= \left\| \begin{matrix} A, & H, & G, & L \\ H, & B, & F, & M \end{matrix} \right\| \left\| \begin{matrix} l_1, & l_2, & l_3, & l_4 \\ m_1, & m_2, & m_3, & m_4 \end{matrix} \right\|.$$

Now for a surface with an arc-representation $ds^2 = dl^2 + dm^2$, the Riemann sphericity is zero; for, with l and m as the superficial parameters, the quantities A, H, B, of § 88, $=1$, 0, 1, so that the symbol $(12, 12)$, which is the Riemann sphericity of the surface, vanishes. The two curves p=variable, q=constant; and p=constant, q=variable; are domainal geodesics, so that this surface is a surface geodesic to the domain. Thus the sphericity of the domain, in this orientation, must vanish; and therefore (§ 117)

$$\sum \sum \sum \sum (\alpha\beta, \gamma\delta) \theta_{\alpha\beta} \theta_{\gamma\delta} = 0.$$

Hence the earlier surfaces, orthogonal to the two perpendicular geodesics in the bi-polar representation of the domain, are orthogonal to the superficial orientations of zero sphericity; and, if they are given by two parametric equations

$$l(p, q, r, t) = \text{constant}, \quad m(p, q, r, t) = \text{constant},$$

the functions l and m satisfy the equations

$$\sum al_1^2 = \Omega, \quad \sum al_1 m_1 = 0, \quad \sum am_1^2 = \Omega,$$

$$\sum \sum \sum \sum (\alpha\beta, \gamma\delta) \theta_{\alpha\beta} \theta_{\gamma\delta} = 0.$$

Ex. 1. When the arc-element of a domain, given by

$$ds^2 = \sum_i \sum_j A_{ij} dx_i dx_j,$$

with the customary convention $x_1, x_2, x_3, x_4, = p, q, r, t$, is expressible in the form

$$ds^2 = dl^2 + dm^2 + (P, Q, R \between d\phi, d\psi)^2$$

appertaining to the bi-polar representation, prove that the parametric functions l and m satisfy the equations

$$\begin{vmatrix} A_{i\alpha} - l_i l_\alpha - m_i m_\alpha, & A_{i\beta} - l_i l_\beta - m_i m_\beta, & A_{i\gamma} - l_i l_\gamma - m_i m_\gamma \\ A_{j\alpha} - l_j l_\alpha - m_j m_\alpha, & A_{j\beta} - l_j l_\beta - m_j m_\beta, & A_{j\gamma} - l_j l_\gamma - m_j m_\gamma \\ A_{k\alpha} - l_k l_\alpha - m_k m_\alpha, & A_{k\beta} - l_k l_\beta - m_k m_\beta, & A_{k\gamma} - l_k l_\gamma - m_k m_\gamma \end{vmatrix} = 0,$$

for all the values $i, j, k, \alpha, \beta, \gamma, = 1, 2, 3, 4$.

Ex. 2. When the domain is primary, so that its plenary homaloidal space is quintuple, and when the arc of the domain is expressed in the bi-polar form

$$ds^2 = dl^2 + dm^2 + (P, Q, R \between d\phi, d\psi)^2,$$

prove that the product of the four principal circular curvatures of domainal geodesics is equal to

$$\frac{1}{16(PR - Q^2)^3} \begin{vmatrix} P, & Q, & R \\ \dfrac{\partial P}{\partial l}, & \dfrac{\partial Q}{\partial l}, & \dfrac{\partial R}{\partial l} \\ \dfrac{\partial P}{\partial m}, & \dfrac{\partial Q}{\partial m}, & \dfrac{\partial R}{\partial m} \end{vmatrix}.$$

It thus appears that a domain can have two orthogonal families of geodesics; but it cannot have three such orthogonal families of geodesics, unless it is deformable into a flatter amplitude, so that it is not to be regarded as a general domain. Thus even for a primary domain of such a limited type, so that its arc could be expressed in a form

$$ds^2 = dp^2 + dq^2 + dr^2 + D dt^2,$$

the product of the principal circular curvatures of geodesics vanishes; and there are three orientations, orthogonal to one another, which provide vanishing sphericities of the domain. Accordingly, its properties are not of a quite general type and its discussion is omitted.

First derivatives of Christoffel symbols.

278. The first parametric derivatives of the quantities Γ, Δ, Θ, Φ, will be required. Incidentally, it is clear that certain differential conditions among these quantities must be satisfied. For we have (§ 268)

$$\frac{\partial A_{ij}}{\partial x_k} = (\Gamma_{ki} A_{j1} + \Delta_{ki} A_{j2} + \Theta_{ki} A_{j3} + \Phi_{ki} A_{j4}) + (\Gamma_{kj} A_{i1} + \Delta_{kj} A_{i2} + \Theta_{kj} A_{i3} + \Phi_{kj} A_{i4}),$$

and a corresponding expression for $\frac{\partial A_{ij}}{\partial x_l}$, obtained by changing k into l; the identical relations

$$\frac{\partial}{\partial x_l}\left(\frac{\partial A_{ij}}{\partial x_k}\right)=\frac{\partial}{\partial x_k}\left(\frac{\partial A_{ij}}{\partial x_l}\right),$$

must be satisfied for all possible values of i, j, k, l; and their development leads to the indicated conditions.

These conditions can be obtained by a different process, which has the advantage of supplying the values of symmetrical combinations of first-order parametric derivatives and third-order parametric derivatives of the point-variables. On the analogy of the equations, which define the magnitudes $\Gamma_{ij}, \Delta_{ij}, \Theta_{ij}, \Phi_{ij}$, we define * four magnitudes $P_{ijk}, Q_{ijk}, R_{ijk}, S_{ijk}$, by the equations

$$\left.\begin{aligned}\sum y_1 y_{ijk}&=AP_{ijk}+HQ_{ijk}+GR_{ijk}+LS_{ijk}\\ \sum y_2 y_{ijk}&=HP_{ijk}+BQ_{ijk}+FR_{ijk}+MS_{ijk}\\ \sum y_3 y_{ijk}&=GP_{ijk}+FQ_{ijk}+CR_{ijk}+NS_{ijk}\\ \sum y_4 y_{ijk}&=LP_{ijk}+MQ_{ijk}+NR_{ijk}+DS_{ijk}\end{aligned}\right\},$$

for all combinations of values of i, j, k, from the set 1, 2, 3, 4, chosen independently of one another. The four equations define the four magnitudes uniquely; hence, because the left-hand sides remain unaltered when i, j, k are interchanged in a selected combination ijk, the quantities $P_{ijk}, Q_{ijk}, R_{ijk}, S_{ijk}$, similarly remain unaltered in any such interchange.

We differentiate a relation

$$\sum y_1 y_{jk}=A\Gamma_{jk}+H\Delta_{jk}+G\Theta_{jk}+L\Phi_{jk},$$

with respect to x_i, (under the foregoing convention as to the significance of x_1, x_2, x_3, x_4), and we have

$$\begin{aligned}\sum y_1 y_{ijk}+\sum y_{1i} y_{jk}=A\frac{\partial\Gamma_{jk}}{\partial x_i}&+H\frac{\partial\Delta_{jk}}{\partial x_i}+G\frac{\partial\Theta_{jk}}{\partial x_i}+L\frac{\partial\Phi_{jk}}{\partial x_i}\\ &+2\Gamma_{jk}\,(A\Gamma_{1i}+H\Delta_{1i}+G\Theta_{1i}+L\Phi_{1i})\\ &+\Delta_{jk}\{(A\Gamma_{2i}+H\Delta_{2i}+G\Theta_{2i}+L\Phi_{2i})\\ &\qquad+(H\Gamma_{1i}+B\Delta_{1i}+F\Theta_{1i}+M\Phi_{1i})\}\\ &+\Theta_{jk}\{(A\Gamma_{3i}+H\Delta_{3i}+G\Theta_{3i}+L\Phi_{3i})\\ &\qquad+(G\Gamma_{1i}+F\Delta_{1i}+C\Theta_{1i}+N\Phi_{1i})\}\\ &+\Phi_{jk}\{(A\Gamma_{4i}+H\Delta_{4i}+G\Theta_{4i}+L\Phi_{4i})\\ &\qquad+(L\Gamma_{1i}+M\Delta_{1i}+N\Theta_{1i}+D\Phi_{1i})\}\end{aligned}$$

* These magnitudes are the magnitudes $\begin{bmatrix} ijk \\ l \end{bmatrix}$ of § 23, when the n-fold amplitude is a domain: and the analysis, which follows here, is practically the earlier analysis for the value $n=4$.

$$= A\left(\frac{\partial \Gamma_{jk}}{\partial x_i}+\Gamma_{jk}\Gamma_{1i}+\Delta_{jk}\Gamma_{2i}+\Theta_{jk}\Gamma_{3i}+\Phi_{jk}\Gamma_{4i}\right)$$
$$+H\left(\frac{\partial \Delta_{jk}}{\partial x_i}+\Gamma_{jk}\Delta_{1i}+\Delta_{jk}\Delta_{2i}+\Theta_{jk}\Delta_{3i}+\Phi_{jk}\Delta_{4i}\right)$$
$$+G\left(\frac{\partial \Theta_{jk}}{\partial x_i}+\Gamma_{jk}\Theta_{1i}+\Delta_{jk}\Theta_{2i}+\Theta_{jk}\Theta_{3i}+\Phi_{jk}\Theta_{4i}\right)$$
$$+L\left(\frac{\partial \Phi_{jk}}{\partial x_i}+\Gamma_{jk}\Phi_{1i}+\Delta_{jk}\Phi_{2i}+\Theta_{jk}\Phi_{3i}+\Phi_{jk}\Phi_{4i}\right)+(1i \between jk),$$

where, as in § 275,

$$(1i \between jk)=(A, B, C, D, F, G, H, L, M, N \between \Gamma_{1i}, \Delta_{1i}, \Theta_{1i}, \Phi_{1i} \between \Gamma_{jk}, \Delta_{jk}, \Theta_{jk}, \Phi_{jk}).$$

By an earlier result (p. 264), we had

$$\sum y_{1i}y_{jk}=\sum \eta_{1i}\eta_{jk}+(1i, \between jk);$$

hence, when we insert the value of $\sum y_1 y_{ijk}$ in terms of the postulated quantities P_{ijk}, Q_{ijk}, R_{ijk}, S_{ijk}, and when we write

$$\bar{P}_{ijk}=P_{ijk}-\left[\frac{\partial \Gamma_{jk}}{\partial x_i}+\sum_{\mu}\{jk, \mu\}\{i\mu, 1\}\right],$$
$$\bar{Q}_{ijk}=Q_{ijk}-\left[\frac{\partial \Delta_{jk}}{\partial x_i}+\sum_{\mu}\{jk, \mu\}\{i\mu, 2\}\right],$$
$$\bar{R}_{ijk}=R_{ijk}-\left[\frac{\partial \Theta_{jk}}{\partial x_i}+\sum_{\mu}\{jk, \mu\}\{i\mu, 3\}\right],$$
$$\bar{S}_{ijk}=S_{ijk}-\left[\frac{\partial \Phi_{jk}}{\partial x_i}+\sum_{\mu}\{jk, \mu\}\{i\mu, 4\}\right],$$

where the μ-summations on the right-hand sides are for the values $\mu=1, 2, 3, 4$, the foregoing equation becomes

$$A\bar{P}_{ijk}+H\bar{Q}_{ijk}+G\bar{R}_{ijk}+L\bar{S}_{ijk}=-\sum \eta_{1i}\eta_{jk}.$$

Proceeding similarly from the expressions for

$$\sum y_2 y_{ijk}, \quad \sum y_3 y_{ijk}, \quad \sum y_4 y_{ijk},$$

we obtain the respective equations

$$H\bar{P}_{ijk}+B\bar{Q}_{ijk}+F\bar{R}_{ijk}+M\bar{S}_{ijk}=-\sum \eta_{2i}\eta_{jk},$$
$$G\bar{P}_{ijk}+F\bar{Q}_{ijk}+C\bar{R}_{ijk}+N\bar{S}_{ijk}=-\sum \eta_{3i}\eta_{jk},$$
$$L\bar{P}_{ijk}+M\bar{Q}_{ijk}+N\bar{R}_{ijk}+D\bar{S}_{ijk}=-\sum \eta_{4i}\eta_{jk}.$$

Consequently, when we resolve these four equations for the four magnitudes $\bar{P}_{ijk}$, $\bar{Q}_{ijk}$, $\bar{R}_{ijk}$, $\bar{S}_{ijk}$, we have

$$\bar{P}_{ijk} = -\frac{1}{\Omega}\{a(\sum \eta_{1i}\eta_{jk}) + h(\sum \eta_{2i}\eta_{jk}) + g(\sum \eta_{3i}\eta_{jk}) + l(\sum \eta_{4i}\eta_{jk})\},$$

$$\bar{Q}_{ijk} = -\frac{1}{\Omega}\{h(\sum \eta_{1i}\eta_{jk}) + b(\sum \eta_{2i}\eta_{jk}) + f(\sum \eta_{3i}\eta_{jk}) + m(\sum \eta_{4i}\eta_{jk})\},$$

$$\bar{R}_{ijk} = -\frac{1}{\Omega}\{g(\sum \eta_{1i}\eta_{jk}) + f(\sum \eta_{2i}\eta_{jk}) + c(\sum \eta_{3i}\eta_{jk}) + n(\sum \eta_{4i}\eta_{jk})\},$$

$$\bar{S}_{ijk} = -\frac{1}{\Omega}\{l(\sum \eta_{1i}\eta_{jk}) + m(\sum \eta_{2i}\eta_{jk}) + n(\sum \eta_{3i}\eta_{jk}) + d(\sum \eta_{4i}\eta_{jk})\};$$

and we at once deduce values of P_{ijk}, Q_{ijk}, R_{ijk}, S_{ijk}.

Now it was pointed out that the values of these four quantities are unaltered by the interchanges of the symbols i, j, k, in the combination ijk. As the expressions just obtained do not exhibit this formal symmetry under the indicated interchanges, it follows that the equal values of the different deduced expressions lead to relations between the derivatives of Γ, Δ, Θ, Φ. For brevity, we write

$$\sum_{\mu} \{jk, \mu\}\{i\mu, \lambda\} = \Gamma_i(jk),\ \Delta_i(jk),\ \Theta_i(jk),\ \Phi_i(jk),$$

for $\lambda = 1, 2, 3, 4$: also, as in § 10, we denote by $a_{\lambda\mu}$ the minor of $A_{\lambda\mu}$ in Ω, so that $a_{11} = a$, $a_{12} = h$, and so on. Then the relations

$$\begin{aligned}
P_{ijk} &= \frac{\partial \Gamma_{jk}}{\partial x_i} + \Gamma_i(jk) - \frac{1}{\Omega}\sum_{\mu}[a_{1\mu}(\sum \eta_{\mu i}\eta_{jk})] \\
&= \frac{\partial \Gamma_{ki}}{\partial x_j} + \Gamma_j(ki) - \frac{1}{\Omega}\sum_{\mu}[a_{1\mu}(\sum \eta_{\mu j}\eta_{ki})] \\
&= \frac{\partial \Gamma_{ij}}{\partial x_k} + \Gamma_k(ij) - \frac{1}{\Omega}\sum_{\mu}[a_{1\mu}(\sum \eta_{\mu k}\eta_{ij})];
\end{aligned}$$

$$\begin{aligned}
Q_{ijk} &= \frac{\partial \Delta_{jk}}{\partial x_i} + \Delta_i(jk) - \frac{1}{\Omega}\sum_{\mu}[a_{2\mu}(\sum \eta_{\mu i}\eta_{jk})] \\
&= \frac{\partial \Delta_{ki}}{\partial x_j} + \Delta_j(ki) - \frac{1}{\Omega}\sum_{\mu}[a_{2\mu}(\sum \eta_{\mu j}\eta_{ki})] \\
&= \frac{\partial \Delta_{ij}}{\partial x_k} + \Delta_k(ij) - \frac{1}{\Omega}\sum_{\mu}[a_{2\mu}(\sum \eta_{\mu k}\eta_{ij})];
\end{aligned}$$

$$\begin{aligned}
R_{ijk} &= \frac{\partial \Theta_{jk}}{\partial x_i} + \Theta_i(jk) - \frac{1}{\Omega}\sum_{\mu}[a_{3\mu}(\sum \eta_{\mu i}\eta_{jk})] \\
&= \frac{\partial \Theta_{ki}}{\partial x_j} + \Theta_j(ki) - \frac{1}{\Omega}\sum_{\mu}[a_{3\mu}(\sum \eta_{\mu j}\eta_{ki})] \\
&= \frac{\partial \Theta_{ij}}{\partial x_k} + \Theta_k(ij) - \frac{1}{\Omega}\sum_{\mu}[a_{3\mu}(\sum \eta_{\mu k}\eta_{ij})];
\end{aligned}$$

$$S_{ijk} = \frac{\partial \Phi_{jk}}{\partial x_i} + \Phi_i(jk) - \frac{1}{\Omega}\sum_{\mu}[a_{4\mu}(\sum \eta_{\mu i}\eta_{jk})]$$
$$= \frac{\partial \Phi_{ki}}{\partial x_j} + \Phi_j(ki) - \frac{1}{\Omega}\sum_{\mu}[a_{4\mu}(\sum \eta_{\mu j}\eta_{ki})]$$
$$= \frac{\partial \Phi_{ij}}{\partial x_k} + \Phi_k(ij) - \frac{1}{\Omega}\sum_{\mu}[a_{4\mu}(\sum \eta_{\mu k}\eta_{ij})],$$

provide values of P_{ijk}, Q_{ijk}, R_{ijk}, S_{ijk}; and they contain the relations among the first parametric derivatives of Γ, Δ, Θ, Φ.

Expressions for p''', q''', r''', t'''.

279. The intrinsic equations of the domainal geodesic in a direction p', q', r', t', not merely determine the values of the quantities p'', q'', r'', t'', along the geodesic but also lead to the values of the further arc-derivatives of the domainal parameters along the geodesic. We proceed to determine the values of the third arc-derivatives of the parameters.

Differentiating the intrinsic equation

$$-p'' = \sum_j \sum_k \Gamma_{jk} x_j' x_k'$$

along the geodesic, we have formally

$$-p''' = \sum_j \sum_k \left(\sum_i \frac{\partial \Gamma_{jk}}{\partial x_i} x_i'\right) x_j' x_k'$$
$$- \sum_b \sum_a \Gamma_{ab} x_a' \left[\sum_l \sum_m \{lm, b\}\, x_l' x_m'\right]$$
$$- \sum_a \sum_b \Gamma_{ab} x_b' \left[\sum_\lambda \sum_\mu \{\lambda\mu, a\}\, x_\lambda' x_\mu'\right],$$

manifestly a homogeneous quaternary cubic in p', q', r', t', which we take in the form

$$-p''' = \sum_i \sum_j \sum_k \Gamma_{ijk} x_i' x_j' x_k',$$

the summations for i, j, k, being taken for the permutations arising from the values $i, j, k, = 1, 2, 3, 4$. Thus the full coefficient of $x_i'^3$ is Γ_{iii}, the full coefficient of $x_i'^2 x_j'$ is $3\Gamma_{iij}$, and the full coefficient of $x_i' x_j' x_k'$ is $6\Gamma_{ijk}$.

On the right-hand side of the differentiated expression, the full coefficient of the combination of variables $x_i' x_j' x_k'$ is

$$2\frac{\partial \Gamma_{jk}}{\partial x_i} + 2\frac{\partial \Gamma_{ki}}{\partial x_j} + 2\frac{\partial \Gamma_{ij}}{\partial x_k}$$

from the first line in p''',

$$-2\Gamma_i(jk) - 2\Gamma_j(ki) - 2\Gamma_k(ij)$$

from the second line, and also

$$-2\Gamma_i(jk) - 2\Gamma_j(ki) - 2\Gamma_k(ij)$$

from the third line. Let substitution be made for the three derivatives, which

occur irst line, in terms of P_{ijk}; then, when the whole coefficient is collected, we find

$$6\Gamma_{ijk}=6[P_{ijk}-\Gamma_i(jk)-\Gamma_j(ki)-\Gamma_i(kj)]$$
$$+\frac{2}{\Omega}\sum_\mu[a_{1\mu}\sum(\eta_{\mu i}\eta_{jk}+\eta_{\mu j}\eta_{ki}+\eta_{\mu k}\eta_{ij})].$$

The quantities P_{ijk} can be regarded as known in terms of the domainal magnitudes, by their definitions; hence the coefficients Γ_{ijk} can be regarded as similarly known. Also we have

$$P_{ijk}=\Gamma_{ijk}+\Gamma_i(jk)+\Gamma_j(ki)+\Gamma_k(ij)$$
$$-\frac{1}{3\Omega}\sum_\mu[a_{1\mu}\sum(\eta_{\mu i}\eta_{jk}+\eta_{\mu j}\eta_{ki}+\eta_{\mu k}\eta_{ij})].$$

Moreover, when this value of P_{ijk} is substituted in the equations which express the derivatives of Γ, we find

$$\frac{\partial\Gamma_{jk}}{\partial x_i}=\Gamma_{ijk}+\Gamma_j(ki)+\Gamma_k(ij)$$
$$-\frac{1}{3\Omega}\sum_\mu[a_{1\mu}\sum(\eta_{\mu j}\eta_{ki}+\eta_{\mu k}\eta_{ij}-2\eta_{\mu i}\eta_{jk})].$$

But

$$\sum(\eta_{\mu j}\eta_{ki}-\eta_{\mu i}\eta_{jk})=(\mu k, ji),\quad \sum(\eta_{\mu k}\eta_{ij}-\eta_{\mu i}\eta_{jk})=(\mu j, ki),$$

in terms of the Riemann four-index symbols; and therefore

$$\frac{\partial\Gamma_{jk}}{\partial x_i}=\Gamma_{ijk}+\Gamma_j(ki)+\Gamma_k(ij)-\frac{1}{3\Omega}\sum_\mu a_{1\mu}[(\mu j, ki)+(\mu k, ji)],$$

the result holding for all values of i, j, k; and the earlier relations (§ 278) between the derivatives of the quantities, such as

$$\frac{\partial\Gamma_{jk}}{\partial x_i}-\frac{\partial\Gamma_{ki}}{\partial x_j}=\Gamma_j(ki)-\Gamma_i(jk)+\frac{1}{\Omega}\sum_\mu a_{1\mu}(\mu k, ij),$$

are satisfied for all the results, in virtue of the relation

$$(\mu j, ki)+(\mu k, ij)+(\mu i, jk)=0.$$

We proceed similarly to determine q''', r''', t'''; and we find

$$-q'''=\sum\sum\sum\Delta_{ijk}x_i'x_j'x_k',\quad -r'''=\sum\sum\sum\Theta_{ijk}x_i'x_j'x_k',$$
$$-t'''=\sum\sum\sum\Phi_{ijk}x_i'x_j'x_k',$$

where

$$Q_{ijk}=\Delta_{ijk}+\Delta_i(jk)+\Delta_j(ki)+\Delta_k(ij)$$
$$-\frac{1}{3\Omega}\sum_\mu[a_{2\mu}\sum(\eta_{\mu i}\eta_{jk}+\eta_{\mu j}\eta_{ki}+\eta_{\mu k}\eta_{ij})],$$

$$R_{ijk} = \Theta_{ijk} + \Theta_i(jk) + \Theta_j(ki) + \Theta_k(ij)$$
$$-\frac{1}{3\Omega}\sum_\mu [a_{3\mu} \sum (\eta_{\mu i}\eta_{jk} + \eta_{\mu j}\eta_{ki} + \eta_{\mu k}\eta_{ij})],$$
$$S_{ijk} = \Phi_{ijk} + \Phi_i(jk) + \Phi_j(ki) + \Phi_k(ij)$$
$$-\frac{1}{3\Omega}\sum_\mu [a_{4\mu} \sum (\eta_{\mu i}\eta_{jk} + \eta_{\mu j}\eta_{ki} + \eta_{\mu k}\eta_{ij})],$$

while the derivatives of Γ, Δ, Θ, Φ, and the relations between these derivatives, are given by the equations

$$\left.\begin{aligned}
\frac{\partial \Gamma_{jk}}{\partial x_i} &= \Gamma_{ijk} + \Gamma_j(ki) + \Gamma_k(ij) - \frac{1}{3\Omega}\sum_\mu a_{1\mu}[(\mu j, ki) + (\mu k, ji)] \\
\frac{\partial \Delta_{jk}}{\partial x_i} &= \Delta_{ijk} + \Delta_j(ki) + \Delta_k(ij) - \frac{1}{3\Omega}\sum_\mu a_{2\mu}[(\mu j, ki) + (\mu k, ji)] \\
\frac{\partial \Theta_{jk}}{\partial x_i} &= \Theta_{ijk} + \Theta_j(ki) + \Theta_k(ij) - \frac{1}{3\Omega}\sum_\mu a_{3\mu}[(\mu j, ki) + (\mu k, ji)] \\
\frac{\partial \Phi_{jk}}{\partial x_i} &= \Phi_{ijk} + \Phi_j(ki) + \Phi_k(ij) - \frac{1}{3\Omega}\sum_\mu a_{4\mu}[(\mu j, ki) + (\mu k, ji)]
\end{aligned}\right\}.$$

These values are in accord with the corresponding values (§ 23) for the m-fold amplitude, when $m=4$, under the conventions

$$\Gamma_{ij} = \{ij, 1\}, \quad \Delta_{ij} = \{ij, 2\}, \quad \Theta_{ij} = \{ij, 3\}, \quad \Phi_{ij} = \{ij, 4\},$$
$$\Gamma_{ijk} = \{ijk, 1\}, \quad \Delta_{ijk} = \{ijk, 2\}, \quad \Theta_{ijk} = \{ijk, 3\}, \quad \Phi_{ijk} = \{ijk, 4\},$$

and the relations (p. 275, footnote) between the quantities $\begin{bmatrix} ijk \\ \mu \end{bmatrix}$ and P_{ijk}, Q_{ijk}, R_{ijk}, S_{ijk}.

Value of y''' along a geodesic.

280. We now obtain the value of y''', using those results for the derivatives of Γ, Δ, Θ, Φ. We have

$$\eta_{jk} = y_{jk} - y_1\Gamma_{jk} - y_2\Delta_{jk} - y_3\Theta_{jk} - y_4\Phi_{jk},$$

and therefore

$$\begin{aligned}
\frac{\partial \eta_{jk}}{\partial x_i} = y_{ijk} &- \Gamma_{jk}(\eta_{1i} + y_1\Gamma_{1i} + y_2\Delta_{1i} + y_3\Theta_{1i} + y_4\Phi_{1i}) \\
&- \Delta_{jk}(\eta_{2i} + y_1\Gamma_{2i} + y_2\Delta_{2i} + y_3\Theta_{2i} + y_4\Phi_{2i}) \\
&- \Theta_{jk}(\eta_{3i} + y_1\Gamma_{3i} + y_2\Delta_{3i} + y_3\Theta_{3i} + y_4\Phi_{3i}) \\
&- \Phi_{jk}(\eta_{4i} + y_1\Gamma_{4i} + y_2\Delta_{4i} + y_3\Theta_{4i} + y_4\Phi_{4i}) \\
&- y_1\frac{\partial \Gamma_{jk}}{\partial x_i} - y_2\frac{\partial \Delta_{jk}}{\partial x_i} - y_3\frac{\partial \Theta_{jk}}{\partial x_i} - y_4\frac{\partial \Phi_{jk}}{\partial x_i}
\end{aligned}$$

$$=y_{ijk}-\sum_l[\eta_{il}\{jk,\, l\}]$$

$$-y_1\left[\frac{\partial\Gamma_{jk}}{\partial x_i}+\Gamma_i(jk)\right]-y_2\left[\frac{\partial\Delta_{jk}}{\partial x_i}+\Delta_i(jk)\right]-y_3\left[\frac{\partial\Theta_{jk}}{\partial x_i}+\Theta_i(jk)\right]-y_4\left[\frac{\partial\Phi_{jk}}{\partial x_i}+\Phi_i(jk)\right]$$

$$\begin{aligned}&=y_{ijk}-y_1\Gamma_{ijk}-y_2\Delta_{ijk}-y_3\Theta_{ijk}-y_4\Phi_{ijk}-\sum_l\eta_{il}\{jk,\, l\}\\ &\quad -y_1\{\Gamma_i(jk)+\Gamma_j(ki)+\Gamma_k(ij)\}\\ &\quad -y_2\{\Delta_i(jk)+\Delta_j(ki)+\Delta_k(ij)\}\\ &\quad -y_3\{\Theta_i(jk)+\Theta_j(ki)+\Theta_k(ij)\}\\ &\quad -y_4\{\Phi_i(jk)+\Phi_j(ki)+\Phi_k(ij)\}\\ &\quad +\frac{1}{3\Omega}\sum_\mu\sum_\lambda y_\lambda a_{\lambda\mu}[(\mu j,\, ki)+(\mu k,\, ji)],\end{aligned}$$

the middle four lines of which can be expressed in the form

$$-\sum_\lambda\sum_\mu\frac{\partial y}{\partial x_\lambda}[\{jk,\, \mu\}\{i\mu,\, \lambda\}+\{ki,\, \mu\}\{j\mu,\, \lambda\}+\{ij,\, \mu\}\{k\mu,\, \lambda\}],$$

the summations for λ and μ being for the values 1, 2, 3, 4, independently of one another.

When we differentiate the typical relation

$$y''=\sum_j\sum_k\eta_{jk}x_j'x_k'$$

along the geodesic, we have

$$\begin{aligned}y'''=&\sum_j\sum_k\left(\sum_i\frac{\partial\eta_{jk}}{\partial x_i}x_i'\right)x_j'x_k'\\ &-\sum_l\sum_m\eta_{lm}x_l'\left[\sum_a\sum_b\{ab,\, m\}x_a'x_b'\right]\\ &-\sum_l\sum_m\eta_{lm}x_m'\left[\sum_\alpha\sum_\beta\{\alpha\beta,\, l\}x_\alpha'x_\beta'\right],\end{aligned}$$

manifestly a quaternary cubic in p', q', r', t', as variables. We write

$$y'''=\sum_i\sum_j\sum_k\eta_{ijk}x_i'x_j'x_k'$$

with the customary convention as to the variables x'; and, in the expression, the summations for i, j, k, are taken independently of one another, so that the full coefficient of the term in the combination $x_i'x_j'x_k'$, when i, j, k, are different, is $6\eta_{ijk}$.

In the foregoing differentiated expression for y''', the full coefficient of this combination $x_i'x_j'x_k'$ on the right-hand side

$$=2\frac{\partial\eta_{jk}}{\partial x_i}+2\frac{\partial\eta_{ki}}{\partial x_j}+2\frac{\partial\eta_{ij}}{\partial x_k}$$

from the first line,

$$= -2\sum_m [\eta_{im}\{jk,\ m\}+\eta_{jm}\{ki,\ m\}+\eta_{km}\{ij,\ m\}]$$

from the second line, and

$$= -2\sum_l [\eta_{il}\ \{jk,\ l\}+\eta_{jl}\ \{ki,\ l\}+\eta_{kl}\ \{ij,\ l\}]$$

from the third line, the second and the third contributions being equal to one another. Let the foregoing value, obtained for $\dfrac{\partial \eta_{jk}}{\partial x_i}$, and the thence inferred values of the other two terms in the first line, be substituted in this expression; it becomes

$$\begin{aligned}6[y_{ijk} - y_1\Gamma_{ijk} - y_2\Delta_{ijk} - y_3\Theta_{ijk} - y_4\Phi_{ijk}]&\\ -6\sum_l [\eta_{il}\{jk,\ l\}+\eta_{jl}\{ki,\ l\}+\eta_{kl}\{ij,\ l\}]&\\ -6\sum_\lambda\sum_\mu \frac{\partial y}{\partial x_\lambda}[\{jk,\ \mu\}\{i\mu,\ \lambda\}+\{ki,\ \mu\}\{j\mu,\ \lambda\}+\{ij,\ \mu\}\{k\mu,\ \lambda\}]&\\ +\frac{2}{3\Omega}\sum_\lambda\sum_\mu y_\lambda a_{\lambda\mu} T,&\end{aligned}$$

where

$$T=(\mu j,\ ki)+(\mu k,\ ji)+(\mu k,\ ij)+(\mu i,\ kj)+(\mu i,\ jk)+(\mu j,\ ik),$$

a quantity which vanishes identically because of the relation

$$(ab,\ cd)+(ab,\ dc)=0.$$

As the full coefficient is equal to $6\eta_{ijk}$, it follows that

$$\begin{aligned}\eta_{ijk}=y_{ijk} - y_1\Gamma_{ijk} - y_2\Delta_{ijk} - y_3\Theta_{ijk} - y_4\Phi_{ijk}&\\ -\sum_l [\eta_{il}\{jk,\ l\}+\eta_{jl}\{ki,\ l\}+\eta_{kl}\{ij,\ l\}]&\\ -\sum_l\sum_m \frac{\partial y}{\partial x_l}[\{jk,\ m\}\{im,\ l\}+\{ki,\ m\}\{jm,\ l\}+\{ij,\ m\}\{km,\ l\}].&\end{aligned}$$

This value for η_{ijk} is expressed in terms of the known magnitudes of the domain and also of the derivatives of its space-variables alone; and it renders the postulated value of y''' definite, in the form

$$y'''=\sum_i\sum_j\sum_k \eta_{ijk}x_i'x_j'x_k'.$$

Moreover, there is the need of having the parametric derivatives of η_{jk} expressed in terms of η_{ijk} instead of y_{ijk}; manifestly, the typical relation is

$$\begin{aligned}\frac{\partial \eta_{jk}}{\partial x_i}=\eta_{ijk}+\sum_l [\eta_{jl}\{ki,\ l\}+\eta_{kl}\{ji,\ l\}]&\\ +\frac{1}{3\Omega}\sum_l\sum_m y_l a_{lm}[(mj,\ ki)+(mk,\ ji)],&\end{aligned}$$

holding for all values of i, j, k, the summations being for the values $l, m, = 1, 2, 3, 4$, taken independently of one another.

Ex. 1. Verify the relation

$$\sum y_l\eta_{ijk} = -\tfrac{1}{3}\sum(\eta_{il}\eta_{jk}+\eta_{jl}\eta_{kl}+\eta_{kl}\eta_{ij}),$$

for all the values of i, j, k, l.

Ex. 2. Taking y as a function of p, q, r, t, constructing y''' in the form

$$y''' = \sum y_1p''' + 3\sum y_{11}p'p'' + \sum y_{111}p'^3,$$

and substituting the geodesic values of $p'', \ldots, p''', \ldots$, obtain the value of y''' as given in the text.

Ex. 3. Let Q denote the expression for y''', as given by

$$Q = \sum_i\sum_j\sum_k \eta_{ijk}x_i'x_j'x_k',$$

and let

$$Q_i = \sum_j\sum_k \eta_{ijk}x_j'x_k', \quad Q_{ij} = \sum_k \eta_{ijk}x_k'.$$

It is frequently convenient, in connection with the value of y'' which is

$$\sum_i\sum_j \eta_{ij}x_i'x_j',$$

to use the magnitudes η_j, defined by

$$\eta_j = \sum_k \eta_{jk}x_k'.$$

Then, as

$$\begin{aligned}\frac{d\eta_j}{ds} &= \sum_k \frac{d\eta_{jk}}{ds}x_k' - \sum_k\sum_p\sum_q[\eta_{jk}\{pq, k\}x_p'x_q']\\ &= \sum_k \frac{d\eta_{jk}}{ds}x_k' - \sum_l\sum_k\sum_\theta[\eta_{jl}\{k\theta, l\}x_k'x_\theta'],\end{aligned}$$

and as

$$\begin{aligned}\frac{d\eta_{jk}}{ds} = \sum_\theta \eta_{jk\theta}x_\theta' &+ \sum_l\sum_\theta x_\theta'[\eta_{jl}\{k\theta, l\}+\eta_{kl}\{j\theta, l\}]\\ &+\frac{1}{3\Omega}\sum_l\sum_m\sum_\theta y_la_{lm}x_\theta'[(mj, k\theta)+(mk, j\theta)],\end{aligned}$$

the first summation in which is Q_{jk}, we have

$$\begin{aligned}\frac{d\eta_j}{ds} &= \sum_\theta\sum_k \eta_{jk\theta}x_k'x_\theta' + \sum_l\sum_\theta\sum_k[\{j\theta, l\}\eta_{kl}x_\theta'x_k']\\ &\qquad+\frac{1}{3\Omega}\sum_l\sum_m\sum_\theta y_la_{lm}x_\theta'x_k'[(mj, k\theta)+(mk, j\theta)]\\ &= Q_j + \sum_l\sum_\theta\{j\theta, l\}x_\theta'\eta_l\\ &\qquad+\frac{1}{3\Omega}\sum_l\sum_m\sum_\theta y_la_{lm}x_\theta'x_k'[(mj, k\theta)+(mk, j\theta)].\end{aligned}$$

Secondary magnitudes; and the circular curvature of a geodesic.

281. Secondary magnitudes of the domain, immediately associated with the circular curvature of a domainal geodesic, are defined in connection with the typical relation

$$\frac{Y}{\rho}=\sum \eta_{11} p'^2.$$

We take

$$\begin{array}{llll}
\bar{A}=\sum Y\eta_{11}, & & & \\
\bar{H}=\sum Y\eta_{12}, & \bar{B}=\sum Y\eta_{22}, & & \\
\bar{G}=\sum Y\eta_{13}, & \bar{F}=\sum Y\eta_{23}, & \bar{C}=\sum Y\eta_{33}, & \\
\bar{L}=\sum Y\eta_{14}, & \bar{M}=\sum Y\eta_{24}, & \bar{N}=\sum Y\eta_{34}, & \bar{D}=\sum Y\eta_{44},
\end{array}$$

the quantities thus denoted being the secondary magnitudes indicated; and sometimes we write

$$\bar{A}_{ij}=\sum Y\eta_{ij},$$

for all the combinations ij. We can also define the secondary magnitudes by the general relation

$$\bar{A}_{ij}=\sum Y y_{ij},$$

for all the combinations ij; the effect is the same as regards the assigned value, because

$$\eta_{ij}=y_{ij}-y_1\Gamma_{ij}-y_2\Delta_{ij}-y_3\Theta_{ij}-y_4\Phi_{ij},$$

and the relation

$$\sum Y y_k=0$$

is satisfied for the values $k=1, 2, 3, 4$. We also write

$$Y=\begin{vmatrix} \bar{A}, & \bar{H}, & \bar{G}, & \bar{L} \\ \bar{H}, & \bar{B}, & \bar{F}, & \bar{M} \\ \bar{G}, & \bar{F}, & \bar{C}, & \bar{N} \\ \bar{L}, & \bar{M}, & \bar{N}, & \bar{D} \end{vmatrix};$$

and we denote by $\bar{a}$, $\bar{h}$, $\bar{b}$, ... the co-factors of $\bar{A}$, $\bar{H}$, $\bar{B}$, ... in the determinant Y, so that, for example,

$$\bar{a}=\frac{\partial Y}{\partial \bar{A}}, \quad \bar{h}=\tfrac{1}{2}\frac{\partial Y}{\partial \bar{H}},$$

and, generally,

$$\bar{a}_{ii}=\frac{\partial Y}{\partial \bar{A}_{ii}}; \quad \bar{a}_{ij}=\tfrac{1}{2}\frac{\partial Y}{\partial \bar{A}_{ij}}, \text{ with } i\neq j.$$

When the typical equation for the direction-cosines and the length of the

radius of circular curvature is multiplied by Y, and the products are added for all the space-dimensions, then

$$\frac{1}{\rho}=\bar{A}p'^2+2\bar{H}p'q'+\bar{B}q'^2+2\bar{G}p'r'+2\bar{F}q'r'+\bar{C}r'^2$$
$$+2\bar{L}p't'+2\bar{M}q't'+2\bar{N}r't'+\bar{D}t'^2$$
$$=\sum \bar{A}p'^2=\sum_i \sum_j \bar{A}_{ij}x_i'x_j',$$

with the usual conventions $x_1, x_2, x_3, x_4, =p, q, r, t$, for the parameters. We write

$$\left.\begin{aligned} u_1&=Ap'+Hq'+Gr'+Lt' \\ u_2&=Hp'+Bq'+Fr'+Mt' \\ u_3&=Gp'+Fq'+Cr'+Nt' \\ u_4&=Lp'+Mq'+Nr'+Dt' \end{aligned}\right\}, \qquad \left.\begin{aligned} v_1&=\bar{A}p'+\bar{H}q'+\bar{G}r'+\bar{L}t' \\ v_2&=\bar{H}p'+\bar{B}q'+\bar{F}r'+\bar{M}t' \\ v_3&=\bar{G}p'+\bar{F}q'+\bar{C}r'+\bar{N}t' \\ v_4&=\bar{L}p'+\bar{M}q'+\bar{N}r'+\bar{D}t' \end{aligned}\right\};$$

and we note that

$$u_1p'+u_2q'+u_3r'+u_4t'=1, \quad v_1p'+v_2q'+v_3r'+v_4t'=\frac{1}{\rho}.$$

These secondary magnitudes $\bar{A}_{ij}$ involve the typical direction-cosine Y, which itself involves the direction-variables p', q', r', t', of the geodesic unless the plenary space of the domain is quintuple; and therefore the secondary magnitudes themselves involve these direction-variables, save under the one exception. Thus the expression for $1/\rho$ is only apparently a quaternary quadratic in p', q', r', t'; the coefficients in the quadratic form are, in fact, implicit functions of these variables.

Already, in connection with the Riemann four-index symbol, certain combinations of the type

$$\sum \sum \eta_{ij}\eta_{kl}$$

have been introduced. The combinations, there found useful, were restricted to a select number. For the expression of the circular curvature, all such combinations are required, though they collect in sets for some of the terms. The full expression for $1/\rho^2$, the simplest form of explicit expression in terms of the specific direction-variables of the geodesic, is

$$\frac{1}{\rho^2}=\sum \left(\frac{Y}{\rho}\right)^2=\sum \left(\sum_i \sum_j \eta_{ij}x_i'x_j'\right)^2,$$

where the un-integered summation is to be effected for all the plenary space range. In this expression, the full coefficient of $x_i'^4$ is

$$\sum \eta_{ii}^2,$$

with the same significance for the un-integered sign of summation; the full coefficient of $x_i'^3x_j'$, with i and j unequal, is

$$4\sum \eta_{ii}\eta_{ij};$$

the full coefficient of $x_i'^2 x_j'^2$, with i and j unequal, is

$$4\sum \eta_{ij}^2 + 2\sum \eta_{ii}\eta_{jj},$$

a quantity with which the Riemann four-index symbol (ij, ij), $= \sum (\eta_{ii}\eta_{jj} - \eta_{ij}^2)$, is associated; the full coefficient of $x_i'^2 x_j' x_k'$, with i, j, k, distinct from one another, is

$$4\sum \eta_{ii}\eta_{jk} + 8\sum \eta_{ij}\eta_{ik},$$

a quantity with which the Riemann four-index symbol (ij, ik),

$$= \sum (\eta_{ii}\eta_{jk} - \eta_{ij}\eta_{ik}),$$

is associated; and the full coefficient of the only four-variable combination $p'q'r't'$ is

$$8\sum \eta_{12}\eta_{34} + 8\sum \eta_{13}\eta_{24} + 8\sum \eta_{14}\eta_{23},$$

with which the Riemann four-index symbols (12, 34), (13, 42), (14, 23) are associated, their sum in turn being zero (§ 275). The results for all these coefficients (except the last, which is particular) hold for the values i, j, k, $=1, 2, 3, 4$, unequal values being chosen.

The partial derivatives of $1/\rho$ with respect to the direction-variables p', q', r', t', will be needed: they can be deduced from the foregoing expression for $1/\rho^2$. We have

$$\frac{\partial}{\partial p'}\left(\frac{1}{\rho^2}\right) = \frac{\partial}{\partial p'}\sum \{(\sum \eta_{11} p'^2)^2\},$$

the inner summation being over the four variables, and the outer over the space-range: that is,

$$\begin{aligned}\frac{\partial}{\partial p'}\left(\frac{1}{\rho^2}\right) &= 4\sum \{(\sum \eta_{11} p'^2)(\eta_{11}p' + \eta_{12}q' + \eta_{13}r' + \eta_{14}t')\} \\ &= 4\sum \frac{Y}{\rho}(\eta_{11}p' + \eta_{12}q' + \eta_{13}r' + \eta_{14}t') \\ &= \frac{4}{\rho}(\bar{A}p' + \bar{H}q' + \bar{G}r' + \bar{L}t') = \frac{4}{\rho} v_1;\end{aligned}$$

and so for the others. The full set of results is

$$\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right) = 2v_1,\quad \frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right) = 2v_2,\quad \frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right) = 2v_3,\quad \frac{\partial}{\partial t'}\left(\frac{1}{\rho}\right) = 2v_4.$$

As $1/\rho^2$ is a homogeneous form in p', q', r', t', of algebraical order four, and as it is not the exact square of a quadratic form in those variables unless the plenary space of the domain is quintuple, we can regard $1/\rho$ as a homogeneous form, not rational, of order two in p', q', r', t', satisfying the Euler relation

$$p'\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right) + q'\frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right) + r'\frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right) + t'\frac{\partial}{\partial t'}\left(\frac{1}{\rho}\right) = \frac{2}{\rho}.$$

Further, we can obtain expressions, rational in p', q', r', t', for the secondary magnitudes $\bar{A}/\rho$. Because

$$\frac{Y}{\rho}=\sum_i\sum_j \eta_{ij}x_i'x_j',$$

and because $\bar{A}_{kl}=\sum Y\eta_{kl}$, for all values of k and l, we have

$$\frac{\bar{A}_{kl}}{\rho}=\sum\frac{Y}{\rho}\eta_{kl}=\sum_i\sum_j(\sum\eta_{ij}\eta_{kl})x_i'x_j',$$

the inner un-integered summation extending over the whole range of the plenary space of the domain.

These results can be used, in association with § 297, for the determination of magnitudes of grade higher than $\bar{A}_{ij}$, in the same way as the corresponding results for a region have been used for the like purpose (§§ 254, 258).

Ex. Because $\bar{A}_{kl}/\rho$ is a rational homogeneous integral function of p', q', r', t', of order two, and because $1/\rho$ is a non-rational homogeneous function of those variables of order two, it follows that $\bar{A}_{kl}$ is a non-rational non-integral function of those variables, homogeneous and of net order zero; hence, by Euler's theorem,

$$p'\frac{\partial\bar{A}_{kl}}{\partial p'}+q'\frac{\partial A_{kl}}{\partial q'}+r'\frac{\partial A_{kl}}{\partial r'}+t'\frac{\partial A_{kl}}{\partial t'}=0.$$

This result can be verified at once; for $\bar{A}_{kl}/\rho$ obviously satisfies

$$p'\frac{\partial}{\partial p'}\left(\frac{\bar{A}_{kl}}{\rho}\right)+q'\frac{\partial}{\partial q'}\left(\frac{\bar{A}_{kl}}{\rho}\right)+r'\frac{\partial}{\partial r'}\left(\frac{\bar{A}_{kl}}{\rho}\right)+t'\frac{\partial}{\partial t'}\left(\frac{\bar{A}_{kl}}{\rho}\right)=2\frac{\bar{A}_{kl}}{\rho},$$

being a homogeneous integral rational function of p', q', r', t', of degree two. When the equation

$$\sum p'\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right)=\frac{2}{\rho}$$

in the text is used, the required relation follows.

The result, thus verified, enables the form

$$\frac{1}{\rho}=\sum\sum\bar{A}_{ij}x_i'x_j'$$

to be used in connection with the cited equation; the Euler operator $\sum p'\frac{\partial}{\partial p'}$, acting on each coefficient $\bar{A}_{ij}$, gives zero as the result for the coefficient, the form thus being the same as though $\bar{A}_{ij}$ were independent of p', q', r', t', which is the actual fact only when the plenary space of the domain is quintuple.

282. Before passing, it is desirable to indicate at once some limitations on the range of analysis required when the plenary space of the domain is quintuple. Implicit limitations have already occurred; a return to the subject of primary domains will be made later (Chapter XXXI).

When that plenary space is quintuple, a set of coordinate axes is easily obtained through the properties of the tangent block of a domain. That block contains four leading lines which, being the tangents to the parametric curves, do not lie in one flat; and the prime normal to any domainal geodesic is at right angles to each of these lines, because the relations $\sum Yy_i=0$, are satisfied for $i=1, 2, 3, 4$. In a quintuple space, there can be only one direction at right angles to each of four lines which do not lie in one flat; and therefore the prime normal of any domainal geodesic is unique for the domain, being the same in direction for all geodesic normals. Hence we can take the direction of this normal to the region, and the (four) directions of the four parametric curves at O, as a set of spatial axes of reference for the domain in the quintuple space.

Any directed quantity can be represented in terms of its components along these axes: consequently, a magnitude such as y_{ij} can be represented as a linear function of

$$y_1, \quad y_2, \quad y_3, \quad y_4, \quad Y,$$

(these quantities being proportional to the typical direction-cosines of the respective axes), with appropriate coefficients. When we take

$$y_{ij}=y_1p_{ij}+y_2q_{ij}+y_3r_{ij}+y_4t_{ij}+Y\bar{\omega}_{ij},$$

the quantities p_{ij}, q_{ij}, r_{ij}, t_{ij}, $\bar{\omega}_{ij}$, being the same throughout the five equations in the plenary space, are obtained at once. Multiply by y_λ, and add the results; we have

$$\begin{aligned}A_{1\lambda}\Gamma_{ij}&+A_{2\lambda}\Delta_{ij}+A_{3\lambda}\Theta_{ij}+A_{4\lambda}\Phi_{ij}\\&=\sum y_\lambda y_{ij}=A_{1\lambda}p_{ij}+A_{2\lambda}q_{ij}+A_{3\lambda}r_{ij}+A_{4\lambda}t_{ij},\end{aligned}$$

holding for $\lambda=1, 2, 3, 4$, so that

$$p_{ij}=\Gamma_{ij}, \quad q_{ij}=\Delta_{ij}, \quad r_{ij}=\Theta_{ij}, \quad t_{ij}=\Phi_{ij}.$$

Again, multiply by Y, and add the results; we have

$$\bar{A}_{ij}=\sum Yy_{ij}=\bar{\omega}_{ij}.$$

Thus the relation becomes

$$y_{ij}=y_1\Gamma_{ij}+y_2\Delta_{ij}+y_3\Theta_{ij}+y_4\Phi_{ij}+Y\bar{A}_{ij},$$

or, what is the equivalent,

$$\eta_{ij}=Y\bar{A}_{ij},$$

for all the values $i, j, =1, 2, 3, 4$.

These equations may be regarded as the set of partial differential equations of the second order, satisfied by the five space-coordinates of any point in a domain in quintuple space.

Without dwelling on the simplifications of the analysis for the domain, which arise when it is a primary domain so that its plenary space is quintuple, two particular results in connection with that quintuple space may be noted.

(i) The Riemann four-index symbols become expressible in terms of the secondary magnitudes $\bar{A}$ alone. Manifestly

$$\sum \eta_{\alpha\beta}\eta_{\gamma\delta} = \bar{A}_{\alpha\beta}\bar{A}_{\gamma\delta}\sum Y^2 = \bar{A}_{\alpha\beta}\bar{A}_{\gamma\delta},$$

for all values of α, β, γ, δ; and therefore the Riemann four-index symbol is

$$(ij, kl) = \sum (\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}) = \bar{A}_{ik}\bar{A}_{jl} - \bar{A}_{il}\bar{A}_{jk}.$$

The Riemann measure of curvature of the domain, when the domain exists in a plenary quintuple space, and when the measure is taken (as in § 270) for the domainal orientation determined by two directions x_1', x_2', x_3', x_4', and z_1', z_2', z_3', z_4', with orientation-variables

$$s_{ij} = x_i'z_j' - x_j'z_i',$$

now becomes

$$K = \frac{\sum (\bar{A}_{ik}\bar{A}_{jl} - \bar{A}_{il}\bar{A}_{jk}) s_{ij}s_{kl}}{\sum (A_{ik}A_{jl} - A_{il}A_{jk}) s_{ij}s_{kl}},$$

both summations being over the set of orientation-variables.

(ii) There are Mainardi-Codazzi relations among the secondary magnitudes $\bar{A}_{ij}$, which are functions solely of position when the domain lies in a quintuple space. They are analogous to the relations which hold for a surface in triple space * and for a region † in quadruple space.

Let the relation

$$\eta_{jk} = Y\bar{A}_{jk}$$

be differentiated with respect to x_i; then

$$\begin{aligned} Y\frac{\partial \bar{A}_{jk}}{\partial x_i} + \bar{A}_{jk}\frac{\partial Y}{\partial x_i} = \frac{\partial \eta_{jk}}{\partial x_i} & \\ = \eta_{ijk} + \sum_l [\eta_{jl}\{ki, l\} + \eta_{kl}\{ji, l\}] & \\ + \frac{1}{3\Omega}\sum_l \sum_m a_{lm}y_l[(mj, ki) + (mk, ji)], & \end{aligned}$$

by the formula in § 280. Let this relation be multiplied by Y, and the results be added for the five space-dimensions; we have

$$\frac{\partial \bar{A}_{jk}}{\partial x_i} = \sum Y\eta_{ijk} + \sum_l [\bar{A}_{jl}\{ki, l\} + \bar{A}_{kl}\{ji, l\}].$$

In this equation, let the numbers i, j, k, be changed, each to the next, in cyclical order; and, in the resulting equation, let this cyclical change be effected again.

* *Lectures on Differential Geometry*, § 35.

† *G.F.D.*, vol. ii, § 279.

Then, successively, we obtain equations

$$\frac{\partial \bar{A}_{ki}}{\partial x_j} = \sum Y\eta_{ijk} + \sum_l [\bar{A}_{kl}\{ij, l\} + \bar{A}_{il}\{kj, l\}],$$

$$\frac{\partial \bar{A}_{ij}}{\partial x_k} = \sum Y\eta_{ijk} + \sum_l [\bar{A}_{il}\{jk, l\} + \bar{A}_{jl}\{ik, l\}];$$

and therefore we infer the equalities

$$\frac{\partial \bar{A}_{jk}}{\partial x_i} + \sum_l \bar{A}_{il}\{jk, l\} = \frac{\partial \bar{A}_{ki}}{\partial x_j} + \sum_l \bar{A}_{jl}\{ki, l\} = \frac{\partial \bar{A}_{ij}}{\partial x_k} + \sum_l \bar{A}_{kl}\{ij, l\},$$

which are the Mainardi-Codazzi relations for a primary domain. They are in accord with the like relations (§ 81) for a general primary amplitude.

Ex. Obtain the value of $\dfrac{\partial Y}{\partial x_k}$ in the form

$$\begin{vmatrix} -\dfrac{\partial Y}{\partial x_k}, & \bar{A}_{1k}, & \bar{A}_{2k}, & \bar{A}_{3k}, & \bar{A}_{4k} \\ y_1, & A, & H, & G, & L \\ y_2, & H, & B, & F, & M \\ y_3, & G, & F, & C, & N \\ y_4, & L, & M, & N, & D \end{vmatrix} = 0,$$

valid for $k = 1, 2, 3, 4$; and deduce the relation

$$\begin{aligned} -\frac{dY}{ds} = \frac{1}{\Omega}[& (av_1 + hv_2 + gv_3 + lv_4)\, y_1 \\ & + (hv_1 + bv_2 + fv_3 + mv_4) y_2 \\ & + (gv_1 + fv_2 + cv_3 + nv_4) y_3 \\ & + (lv_1 + mv_2 + nv_3 + dv_4) y_4]. \end{aligned}$$

Note. The former value holds for a domain in quintuple space: the latter relation holds for a domain in any plenary space.

CHAPTER XXIV

Domainal Geodesics: Gremial Curvatures

General property of the tangent block of a domain.

283. The tangent line of a domainal geodesic lies in the tangent block of the domain. The second principal line of the geodesic, its prime normal, is orthogonal to that block. The three principal lines of the geodesic, next in the grade of its successive curvatures, (being the binormal, the trinormal, and the quartinormal), also lie in the block.

To establish this property, arguments similar to those used for a general amplitude (§§ 33, 39, 44), for a surface (§ 95), and for a region (§§ 169, 170), can be adduced. It may be established otherwise, as follows.

Connected with the block, there are $4N$ magnitudes of the type $\dfrac{\partial y}{\partial x_i}$, where $i=1, 2, 3, 4$, and y is a space-variable typical of all the N space-coordinates of any position. Also, connected with the domain for the purpose of specifying directions, there are four magnitudes p', q', r', t'; but, as they are subject to the permanent condition $\sum Ap'^2=1$, they can count only as three disposable magnitudes. Thus, in all, there are $4N+3$ disposable magnitudes.

The direction-cosines of the tangent, N in number, are typified by the form

$$l_1=y'=y_1p'+y_2q'+y_3r'+y_4t'.$$

Thus N magnitudes are required for the determination of the N quantities l_1; the condition

$$1=\sum l_1^2=\sum Ap'^2$$

is already satisfied, and so there is no diminution in the number of magnitudes, required from the stock of $4N+3$ for the determination of l_1. That number accordingly is N.

The direction-cosines of the binormal, N in number, are typified by the form

$$l_3=\lambda y_1+\mu y_2+\nu y_3+\varpi y_4.$$

Thus N magnitudes are required for the determination of the N quantities l_3. The external universal relation $\sum l_3^2=1$ diminishes the number of independent quantities l_3 by unity, so that only $N-1$ magnitudes will be required. But there is the intrinsic relation $\sum l_3l_1=0$ in the geodesic system, a relation which demands, in effect, one more of the magnitudes. Accordingly, the total demand on the stock of $4N+3$ magnitudes, made for the determination of the direction of the binormal, is N.

The direction-cosines of the trinormal, N in number, are typified by the form

$$l_4 = \alpha y_1 + \beta y_2 + \gamma y_3 + \delta y_4.$$

Thus N magnitudes are required for the determination of the N direction-cosines l_4. The external universal relation $\sum l_4^2 = 1$ diminishes the demand on the stock of available magnitudes by unity, as it effectively reduces the number of unconditional quantities l_4 to $N-1$ only: so that only $N-1$ magnitudes of the stock will be required. But there are two relations

$$\sum l_4 l_1 = 0, \quad \sum l_4 l_3 = 0,$$

in the geodesic system: and these make two demands on magnitudes. Accordingly, the total demand on the stock of $4N+3$ magnitudes, made by this determination of the trinormal, is $N+1$.

Similarly, as the last line to be included, the direction-cosines of the quartinormal, N in number, are typified by the form

$$l_5 = \epsilon y_1 + \eta y_2 + \iota y_3 + \omega y_4.$$

Thus N magnitudes are required for the determination of the N direction-cosines l_5; as these are lessened by one unit because of the external universal relation $\sum l_5^2 = 1$, only $N-1$ magnitudes would thus be required. But there are three relations

$$\sum l_5 l_1 = 0, \quad \sum l_5 l_3 = 0, \quad \sum l_5 l_4 = 0,$$

in the orthogonal system of the geodesic; and these make three demands on magnitudes. Accordingly, the total demand on the stock of $4N+3$ magnitudes, made by this determination of the quartinormal, is $N+2$.

Thus the total number of net demands, being N for the tangent, N for the binormal, $N+1$ for the trinormal, and $N+2$ for the quartinormal, is $4N+3$, the same as the number of magnitudes in the available stock. The number thus is exactly sufficient to satisfy the total number of independent demands.

As regards the actual determinations, the combination necessary for y' $(=l_1)$ is settled from the beginning, by the assignment of a direction of the geodesic. For l_3, l_4, l_5, the sets of quantities $\alpha, \beta, \gamma, \delta$; $\lambda, \mu, \nu, \varpi$; $\epsilon, \eta, \iota, \omega$; are determined in association with the postulated forms, by the Frenet equations

$$\frac{dY}{ds} = \frac{l_3}{\sigma} - \frac{y'}{\rho},$$

$$\frac{dl_3}{ds} = \frac{l_4}{\tau} - \frac{Y}{\sigma},$$

$$\frac{dl_4}{ds} = \frac{l_5}{\kappa} - \frac{l_3}{\tau},$$

in which Y and ρ can be regarded as given externally, and from which also values of σ, τ, κ, can be deduced.

Thus there is established the property that the three selected principal lines of the domainal geodesic lie within the tangent block of the domain.

The precise determination of the specific coefficients of y_1, y_2, y_3, y_4, in the expression of the typical direction-cosines of the binormal, the trinormal, and the quartinormal, will now be effected.

Binormal of a geodesic: the torsion.

284. For the typical direction-cosine l_3 of the binormal of a geodesic, the Frenet equation is

$$\frac{l_3}{\sigma}=\frac{y'}{\rho}+Y';$$

and the postulated form for l_3, belonging to a domainal geodesic, is

$$l_3=\lambda y_1+\mu y_2+\nu y_3+\varpi y_4.$$

Now we have

$$\begin{aligned}\sum y_\iota y'&=\sum y_i(y_1p'+y_2q'+y_3r'+y_4t')\\&=A_{\iota 1}p'+A_{i2}q'+A_{i3}r'+A_{i4}t'=u_\iota,\end{aligned}$$

for the values $i=1, 2, 3, 4$. Also, as $\sum Yy_\iota=0$, we have

$$\begin{aligned}\sum y_iY'&=-\sum Yy_i'\\&=-\sum Y(y_{1\iota}p'+y_{2\iota}q'+y_{3\iota}r'+y_{4i}t')\\&=-(\bar{A}_{\iota 1}p'+\bar{A}_{\iota 2}q'+\bar{A}_{\iota 3}r'+\bar{A}_{\iota 4}t')=v_i,\end{aligned}$$

for the same range of values of i.

Multiply the Frenet equation by y_i, and add the result for all the dimensions of the plenary space: then

$$\begin{aligned}&\frac{1}{\sigma}(A_{\iota 1}\lambda+A_{i2}\mu+A_{\iota 3}\nu+A_{\iota 4}\varpi)\\&\qquad=\frac{1}{\sigma}\sum y_1l_3\\&\qquad=\frac{1}{\rho}u_\iota-v_\iota,\end{aligned}$$

holding for $i=1, 2, 3, 4$. Let

$$\left.\begin{aligned}\sum av_1&=av_1+hv_2+gv_3+lv_4=\bar{v}_1\\\sum hv_1&=hv_1+bv_2+fv_3+mv_4=\bar{v}_2\\\sum gv_1&=gv_1+fv_2+cv_3+nv_4=\bar{v}_3\\\sum lv_1&=lv_1+mv_2+nv_3+dv_4=\bar{v}_4\end{aligned}\right\},$$

the corresponding relations connected with u_1, u_2, u_3, u_4, being

$$\sum au_1=\Omega p',\quad \sum hu_1=\Omega q',\quad \sum gu_1=\Omega r',\quad \sum lu_1=\Omega t'.$$

When the foregoing equations are resolved for λ, μ, ν, ϖ, they give

$$\frac{\lambda}{\sigma}=\frac{p'}{\rho}-\frac{\bar{v}_1}{\Omega},\quad \frac{\mu}{\sigma}=\frac{q'}{\rho}-\frac{\bar{v}_2}{\Omega},\quad \frac{\nu}{\sigma}=\frac{r'}{\rho}-\frac{\bar{v}_3}{\Omega},\quad \frac{\varpi}{\sigma}=\frac{t'}{\rho}-\frac{\bar{v}_4}{\Omega},$$

thus determining the parameters λ, μ, ν, ϖ, in

$$l_3=\lambda y_1+\mu y_2+\nu y_3+\varpi y_4.$$

Let these values of the parameters be now substituted in the postulated value of l_3, and let the result be compared with the original Frenet equation

$$\frac{l_3}{\sigma}=\frac{y'}{\rho}+Y'.$$

Because $y'=y_1p'+y_2q'+y_3r'+y_4t'$, we at once infer

$$Y'=-\frac{1}{\Omega}(y_1\bar{v}_1+y_2\bar{v}_2+y_3\bar{v}_3+y_4\bar{v}_4),$$

a result which will often be used.

Further, we have

$$\begin{aligned}\sum Y'^2&=\frac{1}{\Omega^2}\sum(y_1\bar{v}_1+y_2\bar{v}_2+y_3\bar{v}_3+y_4\bar{v}_4)^2\\&=\frac{1}{\Omega^2}(A, B, C, D, F, G, H, L, M, N \between \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)^2.\end{aligned}$$

Now

$$\begin{aligned}A\bar{v}_1+H\bar{v}_2+G\bar{v}_3+L\bar{v}_4&=\Omega v_1,\\H\bar{v}_1+B\bar{v}_2+F\bar{v}_3+M\bar{v}_4&=\Omega v_2,\\G\bar{v}_1+F\bar{v}_2+C\bar{v}_3+N\bar{v}_4&=\Omega v_3,\\L\bar{v}_1+M\bar{v}_2+N\bar{v}_3+D\bar{v}_4&=\Omega v_4;\end{aligned}$$

and therefore

$$\begin{aligned}\sum Y'^2&=\frac{1}{\Omega}(v_1\bar{v}_1+v_2\bar{v}_2+v_3\bar{v}_3+v_4\bar{v}_4)\\&=\frac{1}{\Omega}(a, b, c, d, f, g, h, l, m, n \between v_1, v_2, v_3, v_4)^2\\&=\frac{1}{\Omega}\sum a v_1^2.\end{aligned}$$

Squaring the Frenet equation in the form

$$Y'=\frac{l_3}{\sigma}-\frac{y'}{\rho},$$

adding for all the space-dimensions, and using the relation $\sum l_3y'=0$, we have

$$\sum Y'^2=\frac{1}{\sigma^2}+\frac{1}{\rho^2};$$

and therefore

$$\Omega\left(\frac{1}{\sigma^2}+\frac{1}{\rho^2}\right)=\sum a v_1{}^2,$$

an equation giving a value for the torsion of a domainal geodesic, the circular curvature being known. The result can be expressed in the equivalent form

$$-\Omega\left(\frac{1}{\sigma^2}+\frac{1}{\rho^2}\right)=\begin{vmatrix} A, & H, & G, & L, & v_1 \\ H, & B, & F, & M, & v_2 \\ G, & F, & C, & N, & v_3 \\ L, & M, & N, & D, & v_4 \\ v_1, & v_2, & v_3, & v_4, & 0 \end{vmatrix}.$$

Further, we have

$$\sum a u_1{}^2=\Omega,$$

$$\sum a u_1 v_1=\Omega(p'v_1+q'v_2+r'v_3+t'v_4)=\frac{\Omega}{\rho};$$

and therefore

$$\frac{\Omega^2}{\sigma^2}=\Omega\,.\,\Omega\left(\frac{1}{\sigma^2}+\frac{1}{\rho^2}\right)-\left(\frac{\Omega}{\rho}\right)^2$$

$$=(\textstyle\sum a u_1{}^2)(\sum a v_1{}^2)-(\sum a u_1 v_1)^2$$

$$=\textstyle\sum (ab-h^2)(u_1v_2-u_2v_1)^2.$$

But

$$ab-h^2=\Omega(CD-N^2),$$

and so for the other minors of the second order in Ω. Consequently

$$\frac{\Omega}{\sigma^2}=\sum (CD-N^2)(u_1v_2-u_2v_1)^2,$$

an expression for the torsion alone; and it can be taken in the equivalent form

$$\frac{\Omega}{\sigma^2}=\begin{vmatrix} A, & H, & G, & L, & u_1, & v_1 \\ H, & B, & F, & M, & u_2, & v_2 \\ G, & F, & C, & N, & u_3, & v_3 \\ L, & M, & N, & D, & u_4, & v_4 \\ u_1, & u_2, & u_3, & u_4, & 0, & 0 \\ v_1, & v_2, & v_3, & v_4, & 0, & 0 \end{vmatrix}.$$

Ex. 1. The verification, that the directions represented analytically by the typical direction-cosines

$$y'=y_1p'+y_2q'+y_3r'+y_4t', \quad l_3=y_1\lambda+y_2\mu+y_3\nu+y_4\varpi,$$

with the foregoing values of λ, μ, ν, ϖ, are at right angles to one another, is immediate.

For

$$\sum y' l_3 = \sum (y_1 p' + y_2 q' + y_3 r' + y_4 t')(y_1 \lambda + y_2 \mu + y_3 \nu + y_4 \varpi)$$
$$= \lambda u_1 + \mu u_2 + \nu u_3 + \varpi u_4$$
$$= \sigma \sum \left\{ u_1 \left(\frac{p'}{\rho} - \frac{\bar{v}_1}{\Omega} \right) \right\}.$$

Now

$$\sum u_1 p' = 1, \quad \sum u_1 \bar{v}_1 = \sum a u_1 v_1 = \frac{\Omega}{\rho},$$

as in the text; and therefore

$$\sum y' l_3 = 0,$$

thus verifying the orthogonality.

Ex. 2. There are various summations, over the dimensions of the plenary space and connected with the domain, which are placed here for subsequent reference, the verifications in each instance being direct:

(i) $\sum y_i l_3 = \sigma \left(\frac{u_i}{\rho} - v_i \right)$;

(ii) $\sum y_i Y' = -v_i$;

(iii) $\sum \eta_{ij} l_3 = 0$;

(iv) $\sum y_{ij} l_3 = \sigma \sum_k \left[\left(\frac{u_k}{\rho} - v_k \right) \{ij, k\} \right]$;

(v) $\sum y_i y''' = -\frac{v_i}{\rho}$;

(vi) $\sum y_{ij} y' = \sum_k [u_k \{ij, k\}]$;

(vii) $\sum y_{ij} Y' = -\sum_k [v_k \{ij, k\}]$;

for all the values 1, 2, 3, 4, of i, j, k.

Ex. 3. Defining the quantities $\gamma_i, \delta_i, \theta_i, \phi_i$, by the equations

$$\gamma_i = \Gamma_{i1} p' + \Gamma_{i2} q' + \Gamma_{i3} r' + \Gamma_{i4} t',$$
$$\delta_i = \Delta_{i1} p' + \Delta_{i2} q' + \Delta_{i3} r' + \Delta_{i4} t'$$
$$\theta_i = \Theta_{i1} p' + \Theta_{i2} q' + \Theta_{i3} r' + \Theta_{i4} t',$$
$$\phi_i = \Phi_{i1} p' + \Phi_{i2} q' + \Phi_{i3} r' + \Phi_{i4} t',$$

so that we have

$$-p'' = \gamma_1 p' + \gamma_2 q' + \gamma_3 r' + \gamma_4 t',$$

with like values for q'', r'', t'', we easily verify the following relations:

(i) $\sum y_k y_\mu' = A_{1k} \gamma_\mu + A_{2k} \delta_\mu + A_{3k} \theta_\mu + A_{4k} \phi_\mu$;

(ii) $\sum y' y_\mu' = u_1 \gamma_\mu + u_2 \delta_\mu + u_3 \theta_\mu + u_4 \phi_\mu$;

(iii) $-\sum Y' y_\mu' = v_1 \gamma_\mu + v_2 \delta_\mu + v_3 \theta_\mu + v_4 \phi_\mu$;

(iv) $\sum l_3 y_\mu' = \sigma \left[\left(\frac{u_1}{\rho} - v_1 \right) \gamma_\mu + \left(\frac{u_2}{\rho} - v_2 \right) \delta_\mu + \left(\frac{u_3}{\rho} - v_3 \right) \theta_\mu + \left(\frac{u_4}{\rho} - v_4 \right) \phi_\mu \right]$.

Further, as

$$u_\mu = \sum y_\mu y',$$

we have

$$\frac{du_\mu}{ds} = \sum y_\mu y'' + \sum y_\mu' y'$$

$$= \frac{1}{\rho} \sum y_\mu Y + u_1\gamma_\mu + u_2\delta_\mu + u_3\theta_\mu + u_4\phi_\mu$$

$$= u_1\gamma_\mu + u_2\delta_\mu + u_3\theta_\mu + u_4\phi_\mu,$$

for all the values $\mu = 1, 2, 3, 4$.

Expression for $\frac{d}{ds}\left(\frac{1}{\rho}\right)$ *along a geodesic.*

285. From the formulæ

$$\rho y'' = Y, \quad \frac{l_3}{\sigma} - \frac{y'}{\rho} = Y',$$

we have

$$\frac{l_3}{\rho\sigma} = y''' + \frac{\rho'}{\rho^2} Y + \frac{y'}{\rho^2} = y''' - Y \frac{d}{ds}\left(\frac{1}{\rho}\right) + \frac{y'}{\rho^2};$$

hence, on multiplying by Y, adding for all the space-dimensions, and using the relations $\sum Yl_3 = 0$, $\sum Y^2 = 1$, $\sum Yy' = 0$, it follows that

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum Yy'''.$$

A value of y''' has been obtained in the form

$$y''' = \sum \sum \sum \eta_{ijk} x_i' x_j' x_k',$$

where

$$\eta_{ijk} = y_{ijk} - \sum_l [\eta_{il}\{jk, l\} + \eta_{jl}\{ki, l\} + \eta_{kl}\{ij, l\}]$$

$$- y_1 \Gamma_{ijk} - y_2 \Delta_{ijk} - y_3 \Theta_{ijk} - y_4 \Phi_{ijk}$$

$$- \sum_l \sum_m \frac{\partial y}{\partial x_l} [\{jk, m\}\{im, l\} + \{ki, m\}\{jm, l\} + \{ij, m\}\{km, l\}].$$

Let new quantities e_{ijk}, E_{ijk}, obviously of the third order in the parametric derivatives, be defined by the relations

$$e_{ijk} = \sum Y \eta_{ijk}, \quad E_{ijk} = \sum Y y_{ijk};$$

obviously, they are connected by the equation

$$e_{ijk} = E_{ijk} - \sum_l [\bar{A}_{il}\{jk, l\} + \bar{A}_{jl}\{ki, l\} + \bar{A}_{kl}\{ij, l\}].$$

Then we have

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=\sum\sum\sum e_{ijk}x_i'x_j'x_k',$$

thus expressing the arc-derivative of the circular curvature of a domainal geodesic, taken along the geodesic, apparently as a quaternary cubic in p', q', r', t'. It is to be noted that, though η_{ijk} is a function of position only, Y depends upon the direction-variables of the geodesic: hence e_{ijk} implicitly involves those direction-variables.

We had quantities u_1, u_2, u_3, u_4, derived from $\sum Ap'^2$, and quantities v_1, v_2, v_3, v_4, derived from $\sum \bar{A}p'^2$; it is necessary to have quantities, denoted by w_1, w_2, w_3, w_4, and derived from $\sum e_{111}p'^3$, according to the definitions

$$\begin{aligned}w_i&=\tfrac{1}{3}\frac{\partial}{\partial x_i'}\sum\sum\sum e_{ijk}x_i'x_j'x_k'\\&=e_{i11}p'^2+2e_{i12}p'q'+e_{i22}q'^2+2e_{i13}p'r'+2e_{i23}q'r'+e_{i33}r'^2\\&\qquad+2e_{i14}p't'+2e_{i24}q't'+2e_{i34}r't'+e_{i44}t'^2,\end{aligned}$$

for $i=1, 2, 3, 4$; also we write

$$w_{ij}=e_{ij1}p'+e_{ij2}q'+e_{ij3}r'+e_{ij4}t',$$

for values of i, j, $=1, 2, 3, 4$, in all combinations, independently of one another. Then we have

$$w_i=\sum_j w_{ij}x_j'.$$

Next, for the arc-derivatives of the secondary magnitudes $\bar{A}$, we use the definition

$$\bar{A}_{ij}=\sum Yy_{ij},$$

so that

$$\bar{A}_{ij}'=\frac{d\bar{A}_{ij}}{ds}=\sum\left\{Y\left(\sum_k y_{ijk}x_k'\right)\right\}+\sum y_{ij}Y'.$$

Now (§ 284, *Ex.* 2, vii)

$$\sum y_{ij}Y'=-\sum_k v_k\{ij,k\};$$

and

$$\sum Y\left(\sum_k y_{ijk}x_k'\right)=\sum_k E_{ijk}x_k'.$$

We substitute for the magnitudes E_{ijk} in terms of the magnitudes e_{ijk}. For the full expression of the sum, we use the notation, already stated (on p. 296) for quantities γ_m, δ_m, θ_m, ϕ_m, in the forms

$$\gamma_m=\sum_k \Gamma_{km}x_k',\quad \delta_m=\sum_k \Delta_{km}x_k',\quad \theta_m=\sum \Theta_{km}x_k',\quad \phi_m=\sum \Phi_{km}x_k';$$

and we have

$$\sum_k E_{ijk}x_k' = w_{ij} + \gamma_j \bar{A}_{i1} + \delta_j \bar{A}_{i2} + \theta_j \bar{A}_{i3} + \phi_j \bar{A}_{i4}$$
$$+ \gamma_i \bar{A}_{j1} + \delta_i \bar{A}_{j2} + \theta_i \bar{A}_{j3} + \phi_i \bar{A}_{j4}$$
$$+ \Gamma_{ij} v_1 + \Delta_{ij} v_2 + \Theta_{ij} v_3 + \Phi_{ij} v_4.$$

As the last line cancels the term arising from $\sum y_{ij} Y'$, it follows that

$$\frac{d\bar{A}_{ij}}{ds} = w_{ij} + \gamma_j \bar{A}_{i1} + \delta_j \bar{A}_{i2} + \theta_j \bar{A}_{i3} + \phi_j \bar{A}_{i4}$$
$$+ \gamma_i \bar{A}_{j1} + \delta_i \bar{A}_{j2} + \theta_i \bar{A}_{j3} + \phi_i \bar{A}_{j4}$$
$$= w_{ij} + \sum_k \sum_l x_k' [\bar{A}_{il}\{kj, l\} + \bar{A}_{jl}\{ki, l\}],$$

reverting to the Christoffel symbols.

Again, we have

$$v_i = \sum_j \bar{A}_{ij} x_j',$$

and therefore

$$\frac{dv_i}{ds} = \sum_j \frac{d\bar{A}_{ij}}{ds} x_j' + \sum_j \bar{A}_{ij} x_j''.$$

When we substitute for the arc-derivatives of the quantities $\bar{A}_{ij}$, the aggregate of terms in w_{ij}

$$= \sum_j w_{ij} x_j' = w_i;$$

the terms in $\gamma_j, \delta_j, \theta_j, \phi_j$, give an aggregate

$$= \bar{A}_{i1}\left(\sum_j \gamma_j x_j'\right) + \bar{A}_{i2}\left(\sum_j \delta_j x_j'\right) + \bar{A}_{i3}\left(\sum_j \theta_j x_j'\right) + \bar{A}_{i4}\left(\sum_j \phi_j x_j'\right)$$
$$= -\bar{A}_{i1}p'' - \bar{A}_{i2}q'' - \bar{A}_{i3}r'' - \bar{A}_{i4}t''$$
$$= -\sum_j \bar{A}_{ij} x_j'';$$

and the terms in $\gamma_i, \delta_i, \theta_i, \phi_i$, give an aggregate

$$= \gamma_i(\sum \bar{A}_{j1} x_j') + \delta_i(\sum \bar{A}_{j2} x_j') + \theta_i(\sum \bar{A}_{j3} x_j') + \phi_i(\sum \bar{A}_{j4} x_j')$$
$$= \gamma_i v_1 + \delta_i v_2 + \theta_i v_3 + \phi_i v_4.$$

Hence, when substitution is effected,

$$\frac{dv_i}{ds} = w_i + \gamma_i v_1 + \delta_i v_2 + \theta_i v_3 + \phi_i v_4,$$

holding for all the values $i = 1, 2, 3, 4$.

As a verification, we note that, by taking

$$\frac{1}{\rho} = \sum_i v_i x_i',$$

we have

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum_i \frac{dv_i}{ds} x_i' + \sum_i v_i x_i'' = \sum w_i x_i',$$

which is effectively the formal value for the left-hand side.

Quantities $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$, were defined (§ 284) by the equations

$$\bar{v}_1 = av_1 + hv_2 + gv_3 + lv_4$$

and three of like form, so that

$$\frac{\bar{v}_1}{\Omega} = \frac{a}{\Omega}v_1 + \frac{h}{\Omega}v_2 + \frac{g}{\Omega}v_3 + \frac{l}{\Omega}v_4.$$

Hence

$$\begin{aligned}\frac{d}{ds}\left(\frac{\bar{v}_1}{\Omega}\right) = \frac{1}{\Omega}(aw_1 + hw_2 + gw_3 + lw_4) \\ + \frac{v_1}{\Omega}(a\gamma_1 + h\gamma_2 + g\gamma_3 + l\gamma_4) \\ + \frac{v_2}{\Omega}(a\delta_1 + h\delta_2 + g\delta_3 + l\delta_4) \\ + \frac{v_3}{\Omega}(a\theta_1 + h\theta_2 + g\theta_3 + l\theta_4) \\ + \frac{v_4}{\Omega}(a\phi_1 + h\phi_2 + g\phi_3 + l\phi_4) \\ + v_1\frac{d}{ds}\left(\frac{a}{\Omega}\right) + v_2\frac{d}{ds}\left(\frac{h}{\Omega}\right) + v_3\frac{d}{ds}\left(\frac{g}{\Omega}\right) + v_4\frac{d}{ds}\left(\frac{l}{\Omega}\right).\end{aligned}$$

The parametric derivatives of a/Ω, h/Ω, g/Ω, l/Ω, have been obtained (§ 268), so that

$$\begin{aligned}-\Omega\frac{d}{ds}\left(\frac{a}{\Omega}\right) &= 2(a\gamma_1 + h\gamma_2 + g\gamma_3 + l\gamma_4), \\ -\Omega\frac{d}{ds}\left(\frac{h}{\Omega}\right) &= (a\delta_1 + h\delta_2 + g\delta_3 + l\delta_4) + (h\gamma_1 + b\gamma_2 + f\gamma_3 + m\gamma_4), \\ -\Omega\frac{d}{ds}\left(\frac{g}{\Omega}\right) &= (a\theta_1 + h\theta_2 + g\theta_3 + l\theta_4) + (g\gamma_1 + f\gamma_2 + c\gamma_3 + n\gamma_4), \\ -\Omega\frac{d}{ds}\left(\frac{l}{\Omega}\right) &= (a\phi_1 + h\phi_2 + g\phi_3 + l\phi_4) + (l\gamma_1 + m\gamma_2 + n\gamma_3 + d\gamma_4):\end{aligned}$$

thus the last five lines in the expression for $\frac{d}{ds}\left(\frac{v_1}{\Omega}\right)$ become

$$\begin{aligned}= -\frac{v_1}{\Omega}(a\gamma_1 + h\gamma_2 + g\gamma_3 + l\gamma_4) \\ -\frac{v_2}{\Omega}(h\gamma_1 + b\gamma_2 + f\gamma_3 + m\gamma_4) \\ -\frac{v_3}{\Omega}(g\gamma_1 + f\gamma_2 + c\gamma_3 + n\gamma_4) \\ -\frac{v_4}{\Omega}(l\gamma_1 + m\gamma_2 + n\gamma_3 + d\gamma_4) \\ = -\frac{1}{\Omega}(\gamma_1\bar{v}_1 + \gamma_2\bar{v}_2 + \gamma_3\bar{v}_3 + \gamma_4\bar{v}_4).\end{aligned}$$

Consequently, we have

$$\frac{d}{ds}\left(\frac{\bar{v}_1}{\Omega}\right)=\frac{1}{\Omega}[(aw_1+hw_2+gw_3+lw_4)-(\gamma_1\bar{v}_1+\gamma_2\bar{v}_2+\gamma_3\bar{v}_3+\gamma_4\bar{v}_4)].$$

Similarly for the derivatives of $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$. When we write

$$\left.\begin{aligned}\bar{w}_1&=aw_1+hw_2+gw_3+lw_4\\ \bar{w}_2&=hw_1+bw_2+fw_3+mw_4\\ \bar{w}_3&=gw_1+fw_2+cw_3+nw_4\\ \bar{w}_4&=lw_1+mw_2+nw_3+dw_4\end{aligned}\right\},$$

the derivatives of $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$, are given in the scheme

$$\left.\begin{aligned}\Omega\frac{d}{ds}\left(\frac{\bar{v}_1}{\Omega}\right)&=\bar{w}_1-(\gamma_1\bar{v}_1+\gamma_2\bar{v}_2+\gamma_3\bar{v}_3+\gamma_4\bar{v}_4)\\ \Omega\frac{d}{ds}\left(\frac{\bar{v}_2}{\Omega}\right)&=\bar{w}_2-(\delta_1\bar{v}_1+\delta_2\bar{v}_2+\delta_3\bar{v}_3+\delta_4\bar{v}_4)\\ \Omega\frac{d}{ds}\left(\frac{\bar{v}_3}{\Omega}\right)&=\bar{w}_3-(\theta_1\bar{v}_1+\theta_2\bar{v}_2+\theta_3\bar{v}_3+\theta_4\bar{v}_4)\\ \Omega\frac{d}{ds}\left(\frac{\bar{v}_4}{\Omega}\right)&=\bar{w}_4-(\phi_1\bar{v}_1+\phi_2\bar{v}_2+\phi_3\bar{v}_3+\phi_4\bar{v}_4)\end{aligned}\right\}.$$

Ex. 1. From the relation

$$\sum Yy_1=0,$$

we have, by differentiation,

$$\begin{aligned}\sum y_1Y'&=-\sum Yy_1'\\ &=-\sum Y(y_{11}p'+y_{12}q'+y_{13}r'+y_{14}t')\\ &=-(\bar{A}p'+\bar{H}q'+\bar{G}r'+\bar{L}t')=-v_1\,;\end{aligned}$$

and therefore, by another differentiation along the geodesic arc,

$$\sum y_1Y''+\sum y_1'Y'=-\frac{dv_1}{ds}=-(w_1+\gamma_1v_1+\delta_1v_2+\theta_1v_3+\phi_1v_4).$$

Also we have had (§ 284, *Ex.* 3, iii)

$$\sum Y'y_1'=-(\gamma_1v_1+\delta_1v_2+\theta_1v_3+\phi_1v_4)\,;$$

and therefore

$$\sum y_1Y''=-w_1.$$

Generally, for $j=1, 2, 3, 4$, we have

$$\sum y_jY''=-w_j.$$

Further, from

$$\frac{d^2}{ds^2}(y_jY)=0,$$

we deduce

$$\sum Yy_j''=w_j+2(\gamma_jv_1+\delta_jv_2+\theta_jv_3+\phi_jv_4).$$

Ex. 2. From the relation

$$\frac{l_3}{\sigma}=Y'+\frac{y'}{\rho},$$

we have

$$\sum l_3 Y'=\frac{1}{\sigma};$$

hence, by differentiation,

$$\sum l_3 Y''+\sum Y'\frac{dl_3}{ds}=\frac{d}{ds}\left(\frac{1}{\sigma}\right).$$

Now

$$\sum Y'\frac{dl_3}{ds}=\sum\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right)=0,$$

because of the relations

$$\sum l_3 l_4=0,\quad \sum y' l_4=0,\quad \sum l_3 Y=0,\quad \sum y' Y=0;$$

and therefore

$$\sum l_3 Y''=\frac{d}{ds}\left(\frac{1}{\sigma}\right).$$

We have

$$l_3=y_1\lambda+y_2\mu+y_3\nu+y_4\varpi,$$

where

$$\frac{\lambda}{\sigma}=\frac{p'}{\rho}-\frac{v_1}{\Omega},$$

and so for μ, ν, ϖ; hence

$$\begin{aligned}\sum l_3 Y''&=\lambda\sum y_1 Y''+\mu\sum y_2 Y''+\nu\sum y_3 Y''+\varpi\sum y_4 Y''\\&=-(\lambda w_1+\mu w_2+\nu w_3+\varpi w_4).\end{aligned}$$

Consequently

$$\begin{aligned}\frac{1}{\sigma}\frac{d}{ds}\left(\frac{1}{\sigma}\right)&=-\frac{1}{\rho}(w_1 p'+w_2 q'+w_3 r'+w_4 t')+\frac{1}{\Omega}(w_1\bar{v}_1+w_2\bar{v}_2+w_3\bar{v}_3+w_4\bar{v}_4)\\&=-\frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{1}{\Omega}\sum av_1 w_1:\end{aligned}$$

that is,

$$\frac{1}{\Omega}\sum av_1 w_1=\frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{1}{\sigma}\frac{d}{ds}\left(\frac{1}{\sigma}\right).$$

Note. The result can be established by direct differentiation of

$$\frac{1}{\Omega}\sum av_1^2=\frac{1}{\rho^2}+\frac{1}{\sigma^2}.$$

Ex. 3. Differentiating the relation

$$\frac{l_3}{\sigma}=\frac{y'}{\rho}+Y'$$

along the geodesic arc, we have

$$\begin{aligned}\frac{1}{\sigma}\left(\frac{l_4}{\tau}-\frac{Y}{\sigma}\right)+l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right)&=y'\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{y''}{\rho}+Y''\\&=y'\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{Y}{\rho^2}+Y''.\end{aligned}$$

Multiply by y', and add for all the space-dimensions: then

$$\sum y'Y'' = -\frac{d}{ds}\left(\frac{1}{\rho}\right).$$

Similarly after multiplication by Y and addition: after multiplication by l_3 and addition: and after multiplication by l_4, followed by addition; we successively find

$$-\sum YY'' = \frac{1}{\rho^2} + \frac{1}{\sigma^2}, \quad \sum l_3 Y'' = \frac{d}{ds}\left(\frac{1}{\sigma}\right), \quad \sum l_4 Y'' = \frac{1}{\sigma\tau}.$$

Ex. 4. Obtain the results:

$$\sum y'Y''' = -\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right);$$

$$\sum YY''' = -\tfrac{3}{2}\frac{d}{ds}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right);$$

$$\sum l_3 Y''' = \frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2}\right);$$

$$\sum l_4 Y''' = \sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right);$$

$$\sum l_5 Y''' = \frac{1}{\sigma\tau\kappa}.$$

Deduce values for the quantities

$$\sum y'Y'''', \quad \sum YY'''', \quad \sum l_3 Y'''', \quad \sum l_4 Y''''.$$

Trinormal of a geodesic: the tilt.

286. As the trinormal of a domainal geodesic lies in the tangent block of the domain, the typical direction-cosine l_4 is of the form

$$l_4 = y_1\alpha + y_2\beta + y_3\gamma + y_4\delta,$$

where α, β, γ, δ, have to be determined.

The trinormal is at right angles to the tangent to the geodesic, so that

$$\sum y' l_4 = 0:$$

that is,

$$\alpha u_1 + \beta u_2 + \gamma u_3 + \delta u_4 = 0.$$

The trinormal is at right angles to the binormal to the geodesic, so that $\sum l_3 l_4 = 0$: or, as

$$l_3 = \frac{\sigma}{\rho} y' + \sigma Y',$$

and as the condition $\sum y' l_4 = 0$ has been used, the new condition will be satisfied by a relation $\sum l_4 Y' = 0$; that is, by § 284,

$$\sum (y_1\alpha + y_2\beta + y_3\gamma + y_4\delta)(y_1\bar{v}_1 + y_2\bar{v}_2 + y_3\bar{v}_3 + y_4\bar{v}_4) = 0,$$

and therefore, by the definition of $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{v}_4$,

$$\alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4 = 0.$$

Again, as always for a geodesic in any configuration, the transformed Frenet equation for the tilt is (§ 8)

$$\frac{l_4}{\tau}=y'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)+Y\sigma\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)+Y'\sigma'+Y''\sigma.$$

In this equation, substitute the postulated value of l_4, multiply by y_1, and add for all the like equations for the space-dimensions; then, using the relations (§ 285, *Ex.* 1)

$$\sum y_1 Y'=-v_1,\quad \sum y_1 Y''=-w_1,$$

we have

$$\frac{1}{\tau}(A\alpha+H\beta+G\gamma+L\delta)=u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_1\sigma'-w_1\sigma.$$

Multiplying by y_2, y_3, y_4, and proceeding in similar fashion, we obtain, in the respective instances, the equations

$$\frac{1}{\tau}(H\alpha+B\beta+F\gamma+M\delta)=u_2\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_2\sigma'-w_2\sigma,$$

$$\frac{1}{\tau}(G\alpha+F\beta+C\gamma+N\delta)=u_3\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_3\sigma'-w_3\sigma.$$

$$\frac{1}{\tau}(L\alpha+M\beta+N\gamma+D\delta)=u_4\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_4\sigma'-w_4\sigma,$$

Thus there are four linear non-homogeneous equations in the quantities α, β, γ, δ; when resolved, they give

$$\left.\begin{aligned}
\frac{\Omega}{\tau}\alpha&=\Omega p'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\bar{v}_1\sigma'-\bar{w}_1\sigma\\
\frac{\Omega}{\tau}\beta&=\Omega q'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\bar{v}_2\sigma'-\bar{w}_2\sigma\\
\frac{\Omega}{\tau}\gamma&=\Omega r'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\bar{v}_3\sigma'-\bar{w}_3\sigma\\
\frac{\Omega}{\tau}\delta&=\Omega t'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-\bar{v}_4\sigma'-\bar{w}_4\sigma
\end{aligned}\right\},$$

so that the parametric expression for l_4 is known.

Let these equations, which give the values of α, β, γ, δ, be multiplied by y_1, y_2, y_3, y_4, and the results be added; then as

$$y_1p'+y_2q'+y_3r'+y_4t'=y',$$

$$y_1\bar{v}_1+y_2\bar{v}_2+y_3\bar{v}_3+y_4\bar{v}_4=-\Omega Y',$$

the result becomes

$$\frac{\Omega}{\tau}l_4=\Omega y'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)+\Omega\sigma'Y'-\sigma(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3+y_4\bar{w}_4).$$

When this is compared with the foregoing modified Frenet equation for l_4, the equality of the two values requires the relation

$$Y'' + Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) = -\frac{1}{\Omega}(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3+y_4\bar{w}_4),$$

which therefore can be regarded as providing an expression for Y'', analogous in form to the cited expression for Y'.

We note that the modified Frenet equation for l_4 can be taken in the equivalent form (§ 8)

$$Y'' = -y'\frac{d}{ds}\left(\frac{1}{\rho}\right) - Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) + l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right) + \frac{l_4}{\sigma\tau},$$

which expresses Y'' as a linear combination of y', Y, l_3, l_4, so that Y'' is typical of a vector lying in the osculating block of the geodesic.

We can obtain an expression for the tilt. By the value of l_4 which has been obtained, it follows that

$$\frac{l_4}{\sigma\tau} - \frac{\sigma'}{\sigma}Y' - \frac{y'}{\sigma}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) = -\frac{1}{\Omega}(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3+y_4\bar{w}_4),$$

that is,

$$\frac{l_4}{\sigma\tau} - \frac{\sigma'}{\sigma^2}l_3 + \frac{\rho'}{\rho^2}y' = -\frac{1}{\Omega}(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3+y_4\bar{w}_4).$$

Let this equation be squared, and the results be added for all the space-dimensions. As

$$\sum l_4^2 = 1,\quad \sum l_3^2 = 1,\quad \sum y'^2 = 1,\quad \sum l_3 l_4 = 0,\quad \sum y' l_4 = 0,\quad \sum y' l_3 = 0,$$

the new left-hand side

$$= \frac{1}{\sigma^2\tau^2} + \frac{\sigma'^2}{\sigma^4} + \frac{\rho'^2}{\rho^4}.$$

The new right-hand side

$$\begin{aligned} &= \frac{1}{\Omega^2}\sum(y_1\bar{w}_1+y_2\bar{w}_2+y_3\bar{w}_3+y_4\bar{w}_4)^2 \\ &= \frac{1}{\Omega^2}\sum A\bar{w}_1^2 \\ &= \frac{1}{\Omega}(w_1\bar{w}_1+w_2\bar{w}_2+w_3\bar{w}_3+w_4\bar{w}_4) \\ &= \frac{1}{\Omega}\sum a w_1^2\,; \end{aligned}$$

and therefore

$$\frac{1}{\sigma^2\tau^2} = \frac{1}{\Omega}\sum a w_1^2 - \frac{\sigma'^2}{\sigma^4} - \frac{\rho'^2}{\rho^4}.$$

As covariantive expressions are known (§ 285) for σ'/σ^2 and ρ'/ρ^2, we thus obtain a covariantive expression for the tilt.

Ex. 1. (i) When the values obtained for $\alpha, \beta, \gamma, \delta$, are substituted in the relation

$$\alpha u_1 + \beta u_2 + \gamma u_3 + \delta u_4 = 0,$$

as given in the text, then, because

$$\sum u_1 p' = 1,$$

$$\sum u_1 \bar{v}_1 = \sum u_1 (av_1 + hv_2 + gv_3 + lv_4)$$

$$= \Omega (v_1 p' + v_2 q' + v_3 r' + v_4 t') = \frac{\Omega}{\rho},$$

$$\sum u_1 \bar{w}_1 = \sum u_1 (aw_1 + hw_2 + gw_3 + lw_4)$$

$$= \Omega (w_1 p' + w_2 q' + w_3 r' + w_4 t') = \Omega \frac{d}{ds}\left(\frac{1}{\rho}\right),$$

we have

$$\frac{\Omega}{\tau} (\alpha u_1 + \beta u_2 + \gamma u_3 + \delta u_4) = \Omega \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \sigma' \frac{\Omega}{\rho} - \sigma \Omega \frac{d}{ds}\left(\frac{1}{\rho}\right) = 0,$$

so that the condition is satisfied.

(ii) Let the values obtained for $\alpha, \beta, \gamma, \delta$, be substituted in the relation

$$\alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4 = 0,$$

also given in the text. Because

$$\sum v_1 p' = \frac{1}{\rho},$$

$$\sum v_1 \bar{v}_1 = \sum a v_1^2 = \Omega \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right),$$

the relation becomes

$$\frac{\Omega}{\rho} \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \sigma' \Omega \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) - \sigma (v_1 \bar{w}_1 + v_2 \bar{w}_2 + v_3 \bar{w}_3 + v_4 \bar{w}_4) = 0\,;$$

and therefore

$$\sum a v_1 w_1 = -\Omega \left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right).$$

We thus have the geometrical expression of a covariant of the system of concomitants.

Note. It is convenient to collect the results of this type thus far obtained, being

$$\sum a u_1^2 = \Omega\,;$$

$$\sum a u_1 v_1 = \frac{\Omega}{\rho}, \quad \sum a v_1^2 = \Omega \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right);$$

$$\sum a u_1 w_1 = \Omega \frac{d}{ds}\left(\frac{1}{\rho}\right), \quad \sum a v_1 w_1 = -\Omega \left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right), \quad \sum a w_1^2 = \Omega \left(\frac{1}{\sigma^2 \tau^2} + \frac{\sigma'^2}{\sigma^4} + \frac{\rho'^2}{\rho^4}\right).$$

Ex. 2. Let the relation

$$Y'' + Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) = -\frac{1}{\Omega}(y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3 + y_4\bar{w}_4)$$

be squared, and let the results be added for all the space-dimensions. Because

$$\sum Y''Y = -\sum Y'^2 = -\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right),$$

and, as in the text

$$\sum (y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3 + y_4\bar{w}_4)^2 = \Omega \sum a w_1^2,$$

we find

$$\sum Y''^2 - \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2 = \frac{1}{\Omega}\sum a w_1^2$$

$$= \frac{\sigma'^2}{\sigma^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\sigma^2\tau^2},$$

so that

$$\sum Y''^2 = \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2 + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4} + \frac{1}{\sigma^2\tau^2}.$$

Ex. 3. (i) Let the relation

$$Y'' + Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) = -\frac{1}{\Omega}(y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3 + y_4\bar{w}_4)$$

be multiplied by y_i, and the results be added for the space-dimensions; then

$$\sum y_i Y'' = -\frac{1}{\Omega}(A_{i1}\bar{w}_1 + A_{i2}\bar{w}_2 + A_{i3}\bar{w}_3 + A_{i4}\bar{w}_4)$$

$$= -w_i,$$

as before (§ 285, *Ex.* 1).

(ii) When the same relation is multiplied by η_{ij} and the results are added similarly, we have

$$\sum \eta_{ij} Y'' = -\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\bar{A}_{ij}.$$

(iii) Let the same relation be multiplied by y_{ij} and the results be added similarly. Because

$$\sum Y y_{ij} = \bar{A}_{ij}, \quad \sum y_k y_{ij} = A_{1k}\Gamma_{ij} + A_{2k}\Delta_{ij} + A_{3k}\Theta_{ij} + A_{4k}\Phi_{ij},$$

we find

$$\begin{aligned}\sum Y'' y_{ij} + \bar{A}_{ij}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) = -\frac{1}{\Omega}[&\bar{w}_1(A\Gamma_{ij} + H\Delta_{ij} + G\Theta_{ij} + L\Phi_{ij}) \\ &+ \bar{w}_2(H\Gamma_{ij} + B\Delta_{ij} + F\Theta_{ij} + M\Phi_{ij}) \\ &+ \bar{w}_3(G\Gamma_{ij} + F\Delta_{ij} + C\Theta_{ij} + N\Phi_{ij}) \\ &+ \bar{w}_4(L\Gamma_{ij} + M\Delta_{ij} + N\Theta_{ij} + D\Phi_{ij})] \\ = -(\Gamma_{ij}w_1 + \Delta_{ij}w_2 + \Theta_{ij}w_3 + \Phi_{ij}w_4)&\end{aligned}$$

When this equation is taken for $j=1, 2, 3, 4$, in succession, the value of i remaining the same; and when these equations are multiplied by p', q', r', t', respectively, and the products are added; we find

$$\sum Y''y_i' = -\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_i - (\gamma_i w_1 + \delta_i w_2 + \theta_i w_3 + \phi_i w_4),$$

with the former significance (p. 296) for the symbols γ_i, δ_i, θ_i, ϕ_i.

Ex. 4. When the two equations

$$\frac{l_3}{\sigma} - \frac{y'}{\rho} = -\frac{1}{\Omega}(y_1\bar{v}_1 + y_2\bar{v}_2 + y_3\bar{v}_3 + y_4\bar{v}_4),$$

$$\frac{l_4}{\sigma\tau} - \frac{\sigma'}{\sigma^2}l_3 + \frac{\rho'}{\rho^2}y' = -\frac{1}{\Omega}(y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3 + y_4\bar{w}_4),$$

are combined, by taking the product of the left-hand sides and the product of the right-hand sides, and summing for the space-dimensions, we have

$$-\frac{\sigma'}{\sigma^3} - \frac{\rho'}{\rho^3} = \frac{1}{\Omega^2}\sum A\bar{v}_1\bar{w}_1$$

$$= \frac{1}{\Omega}(v_1\bar{w}_1 + v_2\bar{w}_2 + v_3\bar{w}_3 + v_4\bar{w}_4) = \frac{1}{\Omega}\sum av_1w_1,$$

in accordance with the result in § 285, *Ex.* 2.

Expression for $\dfrac{d^2}{ds^2}\left(\dfrac{1}{\rho}\right)$ *along a geodesic.*

287. To obtain the expression for the magnitude $\dfrac{d^2}{ds^2}\left(\dfrac{1}{\rho}\right)$, which will be taken in the form

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k \sum_l f_{ijkl}x_i'x_j'x_k'x_l',$$

we can proceed from

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k e_{ijk}x_i'x_j'x_k',$$

by arc-differentiation. We begin by obtaining the arc-derivative of the general coefficient e_{ijk}, the value of which (§ 285) is

$$e_{ijk} = \sum Yy_{ijk} - \sum_a [\bar{A}_{ia}\{jk, a\} + \bar{A}_{ja}\{ki, a\} + \bar{A}_{ka}\{ij, a\}].$$

The value of Y' is (§ 284)

$$Y' = -\frac{1}{\Omega}\sum_r \sum_\mu a_{ra}v_a y_r,$$

and so

$$\sum Y'y_{ijk} = -\frac{1}{\Omega}\sum_r \sum_a [a_{ra}v_r(\sum y_r y_{ijk})]$$

$$= -\frac{1}{\Omega}\sum_r \sum_a a_{ra}v_r(A_{1r}P_{ijk} + A_{2r}Q_{ijk} + A_{3r}R_{ijk} + A_{4r}S_{ijk})$$

$$= -(v_1P_{ijk} + v_2Q_{ijk} + v_3R_{ijk} + v_4S_{ijk});$$

when the values of P_{ijk}, Q_{ijk}, R_{ijk}, S_{ijk}, as given in § 279 in terms of the quantities $\{ijk, l\}$ are inserted, we find

$$\begin{aligned}\sum Y' y_{ijk} = &-\sum_a v_a \{ijk, a\}\\ &-\sum_\lambda \sum_a v_a [\{jk, \lambda\}\{i\lambda, a\} + \{ki, \lambda\}\{j\lambda, a\} + \{ij, \lambda\}\{k\lambda, a\}]\\ &+\frac{1}{3\Omega}\sum_\lambda \sum_a v_\lambda a_{\lambda a} [\sum (\eta_{ai}\eta_{jk} + \eta_{aj}\eta_{ki} + \eta_{ak}\eta_{ij})].\end{aligned}$$

Also, we write

$$\sum Y y_{\alpha\beta\gamma\delta} = F_{\alpha\beta\gamma\delta};$$

and therefore

$$\frac{d}{ds}(\sum Y y_{ijk}) = \sum_l F_{ijkl} x_l' + \sum Y' y_{ijk},$$

the value of the second term being inserted.

Next, we have (§ 285)

$$\frac{d\bar{A}_{mn}}{ds} = \sum_l (e_{lmn} x_l') + \sum_\beta \sum_l [\bar{A}_{n\beta}\{lm, \beta\} + \bar{A}_{m\beta}\{ln, \beta\}] x_l',$$

for all values of m and n; and therefore

$$\frac{d\bar{A}_{ia}}{ds}\{jk, a\} + \frac{d\bar{A}_{ja}}{ds}\{ki, a\} + \frac{d\bar{A}_{ka}}{ds}\{ij, a\},$$

which is a portion of the derivative of the second term in the value of e_{ijk},

$$\begin{aligned}= &\sum_l [e_{ila}\{jk, a\} + e_{jla}\{ki, a\} + e_{kla}\{ij, a\}] x_l'\\ &+\sum_l \sum_\beta x_l' [\bar{A}_{ia}\{jk, a\} + \bar{A}_{ja}\{ki, a\} + \bar{A}_{ka}\{ij, a\}]\{la, \beta\}\\ &+\sum_l \sum_\beta x_l' \bar{A}_{a\beta} [\{il, \beta\}\{jk, a\} + \{jl, \beta\}\{ki, a\} + \{kl, \beta\}\{ij, a\}].\end{aligned}$$

Also, we have (§ 279)

$$\begin{aligned}\frac{\partial}{\partial x_l}\{jk, a\} = \{jkl, a\} &+ \sum_\gamma [\{lj, \gamma\}\{k\gamma, a\} + \{lk, \gamma\}\{j\gamma, a\}]\\ &-\frac{1}{3\Omega}\sum_m a_{ma}[(mj, kl) + (mk, jl)];\end{aligned}$$

and therefore

$$\bar{A}_{ia}\frac{d}{ds}\{jk, a\} + \bar{A}_{ja}\frac{d}{ds}\{ki, a\} + \bar{A}_{ka}\frac{d}{ds}\{ij, a\},$$

which is the remaining portion of the derivative of the second summation in the cited expression (§ 285) for e_{ijk},

$$= \sum_l x_l'[\bar{A}_{ia}\{jkl, a\} + \bar{A}_{ja}\{kil, a\} + \bar{A}_{ka}\{ijl, a\}]$$
$$+ \sum_l x_l'\bar{A}_{ia}\left[\sum_\gamma \{lj, \gamma\}\{k\gamma, a\} + \{lk, \gamma\}\{j\gamma, a\}\right]$$
$$+ \sum_l x_l'\bar{A}_{ja}\left[\sum_\gamma \{lk, \gamma\}\{i\gamma, a\} + \{li, \gamma\}\{k\gamma, a\}\right]$$
$$+ \sum_l x_l'\bar{A}_{ka}\left[\sum_\gamma \{li, \gamma\}\{j\gamma, a\} + \{lj, \gamma\}\{i\gamma, a\}\right]$$
$$- \frac{1}{3\Omega}\sum_l \sum_m x_l' a_{ma}[\bar{A}_{ia}(mj, kl) + \bar{A}_{ia}(mk, jl)$$
$$+ \bar{A}_{ja}(mk, il) + \bar{A}_{ja}(mi, kl) + \bar{A}_{ka}(mi, jl) + \bar{A}_{ka}(mj, il)].$$

Let these results be substituted; then, after some re-arrangement, we find the value of e_{ijk}', the arc-derivative of e_{ijk}, to be given by

$$\frac{d}{ds}e_{ijk} - \sum_l F_{ijkl}x_l' = -\sum_l x_l' \sum_a [\bar{A}_{ia}\{jkl, a\} + \bar{A}_{ja}\{kli, a\} + \bar{A}_{ka}\{lij, a\} + \bar{A}_{la}\{ijk, a\}]$$
$$- \sum_l x_l' \sum_a [e_{ila}\{jk, a\} + e_{jla}\{ki, a\} + e_{kla}\{ij, a\}]$$
$$- \sum_l x_l' \sum_a \sum_\beta \bar{A}_{ia}[\{jk, \beta\}\{l\beta, a\} + \{kl, \beta\}\{j\beta, a\} + \{lj, \beta\}\{k\beta, a\}]$$
$$- \sum_l x_l' \sum_a \sum_\beta \bar{A}_{ja}[\{kl, \beta\}\{i\beta, a\} + \{li, \beta\}\{k\beta, a\} + \{ik, \beta\}\{l\beta, a\}]$$
$$- \sum_l x_l' \sum_a \sum_\beta \bar{A}_{ka}[\{li, \beta\}\{j\beta, a\} + \{ij, \beta\}\{l\beta, a\} + \{jl, \beta\}\{i\beta, a\}]$$
$$- \sum_l x_l' \sum_a \sum_\beta \bar{A}_{la}[\{ij, \beta\}\{k\beta, a\} + \{jk, \beta\}\{i\beta, a\} + \{ki, \beta\}\{j\beta, a\}]$$
$$- \sum_l x_l' \sum_a \sum_\beta \bar{A}_{a\beta}[\{il, \beta\}\{jk, a\} + \{jl, \beta\}\{ki, a\} + \{kl, \beta\}\{ij, a\}]$$
$$- \frac{1}{3\Omega}\sum_l \sum_m \sum_a x_l' a_{ma}[\quad \bar{A}_{ia}\sum (\eta_{mk}\eta_{jl} + \eta_{mj}\eta_{kl} - 2\eta_{jk}\eta_{ml})$$
$$+ \bar{A}_{ja}\sum (\eta_{mi}\eta_{kl} + \eta_{mk}\eta_{il} - 2\eta_{ki}\eta_{ml})$$
$$+ \bar{A}_{ka}\sum (\eta_{mj}\eta_{il} + \eta_{mi}\eta_{jl} - 2\eta_{ij}\eta_{ml})]$$
$$+ \frac{1}{3\Omega}\sum_m \sum_a v_a a_{ma}[\sum (\eta_{mi}\eta_{jk} + \eta_{mj}\eta_{ki} + \eta_{mk}\eta_{ij})].$$

Also

$$\frac{d}{ds}(x_i'x_j'x_k') = -x_j'x_k' \sum_m \sum_n [(mn, i)x_m'x_n'] - x_k'x_i' \sum_m \sum_n [(mn, j)x_m'x_n']$$
$$- x_i'x_j' \sum_m \sum_n [\{mn, k)x_m'x_n'].$$

We thus have the separate parts of the derivative of the magnitude

$$\frac{6}{i!\,j!\,k!}e_{ijk}x_i'x_j'x_k',$$

the representative term in the arc-derivative of the circular curvature.

To select the coefficient of $x_i'x_j'x_k'x_l'$ in the quantity

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

we have two sets of terms: viz. those arising from the terms

$$e_{ijk}'x_i'x_j'x_k', \quad e_{jkl}'x_j'x_k'x_l', \quad e_{kli}'x_k'x_l'x_i', \quad e_{lij}'x_l'x_i'x_j';$$

and those arising from

$$e_{ijk}\frac{d}{ds}(x_i'x_j'x_k'), \quad e_{jkl}\frac{d}{ds}(x_j'x_k'x_l'), \quad e_{kli}\frac{d}{ds}(x_k'x_l'x_i'), \quad e_{lij}\frac{d}{ds}(x_l'x_i'x_j').$$

The aggregate coefficient of $x_i'x_j'x_k'x_l'$, account being taken of the fact that each of the coefficients e_{ijk}, e_{jkl}, e_{kli}, e_{lij}, has its numerical coefficient (which is 6 in the most general combination when i, j, k, l, are different from one another), in the latter set is

$$-12\sum_a [\ e_{jka}\{il,\ a\} + e_{kia}\{jl, a\} + e_{ija}\{kl, a\}$$
$$+ e_{ila}\{jk, a\} + e_{jla}\{ki, a\} + e_{kla}\{ij, a\}].$$

The aggregate coefficient in the former set, with the same account of the numerical coefficient, is

$$\begin{aligned}
&24F_{ijkl}\\
&\quad -12\sum_a [e_{ila}\{jk, a\} + e_{jla}\{ki, a\} + e_{kla}\{ij, a\}\\
&\qquad\qquad + e_{jka}\{il, a\} + e_{kia}\{jl, a\} + e_{ija}\{kl, a\}]\\
&\quad -24\sum_a [\bar{A}_{ia}\{jkl, a\} + \bar{A}_{ja}\{kli, a\} + \bar{A}_{ka}\{lij, a\} + \bar{A}_{la}\{ijk, a\}]\\
&\quad -24\sum_a\sum_\beta [\bar{A}_{ia}(jkl, a, \beta) + \bar{A}_{ja}(kli, a, \beta) + \bar{A}_{ka}(lij, a, \beta) + \bar{A}_{la}(ijk, a, \beta)]\\
&\quad -24\sum_a\sum_\beta \bar{A}_{a\beta}[\{il, a\}\{jk, \beta\} + \{jl, a\}\{ki, \beta\} + \{kl, a\}\{ij, \beta\}]\\
&\quad +\frac{2}{\Omega}\sum_m\sum_a a_{ma}[\bar{A}_{ia}(m; jkl) + \bar{A}_{ja}(m, kli) + \bar{A}_{ka}(m, lij) + \bar{A}_{la}(m, ijk)],
\end{aligned}$$

where the symbols (jkl, a, β) and (m, jkl) have the significance

$$(jkl, a, \beta) = \{j\beta, a\}\{kl, \beta\} + \{k\beta, a\}\{lj, \beta\} + \{l\beta, a\}\{kj, \beta\},$$
$$(m, jkl) = \sum (\eta_{mj}\eta_{kl} + \eta_{mk}\eta_{lj} + \eta_{ml}\eta_{jk}).$$

The aggregate of terms, arising from the eighth, ninth, and tenth lines in the expression for e_{ijk}' on p. 310, and from the corresponding quantities in the expressions for e_{jkl}', e_{kli}', e_{lij}', proves to be zero; so there is no such set of terms in the complete aggregate.

This complete aggregate is to be the full coefficient of the combination $x_i' x_j' x_k' x_l'$ in

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = \sum_i \sum_j \sum_k \sum_l f_{ijkl} x_i' x_j' x_k' x_l',$$

and therefore is equal to $24 f_{ijkl}$, when i, j, k, l, are different from one another; consequently

$$\begin{aligned} f_{ijkl} = & \sum Y y_{ijkl} \\ & - \sum_a [e_{ila}\{jk, a\} + e_{jla}\{ki, a\} + e_{kla}\{ij, a\} \\ & \qquad + e_{jka}\{il, a\} + e_{kia}\{jl, a\} + e_{ija}\{kl, a\}] \\ & - \sum_a [\bar{A}_{ia}\{jkl, a\} + \bar{A}_{ja}\{kli, a\} + \bar{A}_{ka}\{lij, a\} + \bar{A}_{la}\{ijk, a\}] \\ & - \sum_a \sum_\beta [\bar{A}_{ia}(jkl, a, \beta) + \bar{A}_{ja}(kli, a, \beta) + \bar{A}_{ka}(lij, a, \beta) + \bar{A}_{la}(ijk, a, \beta)] \\ & - \sum_a \sum_\beta \bar{A}_{a\beta}[\{il, a\}\{jk, \beta\} + \{jl, a\}\{ki, \beta\} + \{kl, a\}\{ij, \beta\}] \\ & + \frac{1}{12\Omega} \sum_m \sum_a a_{ma}[\bar{A}_{ia}(m, jkl) + \bar{A}_{ja}(m, kli) + \bar{A}_{ka}(m, lij) + \bar{A}_{la}(m, ijk)], \end{aligned}$$

thus giving the value of the typical coefficient in the expression.

Using this value of f_{ijkl} in place of F_{ijkl}, which is equal to $\sum Y y_{ijkl}$, and now using θ as the summation-number instead of l in the former expression for e_{ijk}', we can change that expression to the form

$$\begin{aligned} \frac{d}{ds}(e_{ijk}) = & \sum_\theta f_{ijk\theta}\, x_\theta' \\ & + \sum_\theta \sum_a x_\theta' [e_{jka}\{i\theta, a\} + e_{kia}\{j\theta, a\} + e_{ija}\{k\theta, a\}] \\ & - \frac{1}{3\Omega} \sum_\theta \sum_m \sum_a x_\theta' a_{ma} [\bar{A}_{ia}(mj, k\theta) + \bar{A}_{ja}(mk, i\theta) + \bar{A}_{ka}(mi, j\theta) \\ & \qquad + \bar{A}_{ia}(mk, j\theta) + \bar{A}_{ja}(mi, k\theta) + \bar{A}_{ka}(mj, i\theta)] \\ & + \frac{1}{4\Omega} \sum_m \sum_a a_{ma} v_a(m, ijk) \\ & - \frac{1}{12\Omega} \sum_\theta \sum_m \sum_a x_\theta' a_{ma} [\bar{A}_{ia}(m, jk\theta) + \bar{A}_{ja}(m, ki\theta) + \bar{A}_{ka}(m, ij\theta)]. \end{aligned}$$

It is to be remarked that, although the summations for integers such as m, a, θ, in this expression are implied as to be taken only for the values 1, 2, 3, 4, because the number of parameters which determine position in the domain is four, the summations can equally be taken (and the result will equally apply) for a general n-fold amplitude with the range of values $1, 2, \ldots, n$.

In connection with the last statement, it may also be remarked that, in accord with the general result as regards e_{ijk}', there occurs the special instance when the configuration is a surface in homaloidal triple space *.

* See my *Lectures on Differential Geometry*, § 41, p. 59.

Quartinormal of a geodesic; the coil.

288. The fourth normal (the quartinormal) of a domainal geodesic lies in the tangent block of the domain; and therefore its typical direction-cosine l_5 can be expressed in the form

$$l_5 = y_1\epsilon + y_2\eta + y_3\iota + y_4\omega,$$

where the parameters ϵ, η, ι, ω, to be determined, are the same for all the direction-cosines of the line.

This fourth normal is at right angles to the tangent to the geodesic, with a typical direction-cosine y'; hence $\sum l_5 y' = 0$, and so

$$\epsilon u_1 + \eta u_2 + \iota u_3 + \omega u_4 = 0,$$

because $\sum y' y_i = u_\iota$.

It is at right angles to the binormal of the geodesic, with a typical direction-cosine l_3; hence $\sum l_5 l_3 = 0$, that is.

$$\epsilon \sum y_1 l_3 + \eta \sum y_2 l_3 + \iota \sum y_3 l_3 + \omega \sum y_4 l_3 = 0.$$

But (§ 284, *Ex.* 2)

$$\sum y_\iota l_3 = \frac{\sigma}{\rho} u_\iota - \sigma v_\iota,$$

for $i = 1, 2, 3, 4$; hence, taking account of the first condition, this condition connected with the trinormal becomes

$$\epsilon v_1 + \eta v_2 + \iota v_3 + \omega v_4 = 0.$$

The fourth normal is at right angles to the trinormal of the geodesic, with a typical direction-cosine l_4; hence $\sum l_5 l_4 = 0$, that is,

$$\epsilon \sum y_1 l_4 + \eta \sum y_2 l_4 + \iota \sum y_3 l_4 + \omega \sum y_4 l_4 = 0.$$

But (§ 286)

$$\frac{1}{\tau} \sum y_i l_4 = u_\iota \frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - v_\iota \sigma' - w_\iota \sigma,$$

for $i = 1, 2, 3, 4$; hence, taking account both of the first condition and of the second condition, the new condition becomes

$$\epsilon w_1 + \eta w_2 + \iota w_3 + \omega w_4 = 0.$$

These three homogeneous linear relations among ϵ, η, ι, ω, can be resolved; and they give

$$\frac{1}{\epsilon}\begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ w_2, & w_3, & w_4 \end{vmatrix} = \frac{-1}{\eta}\begin{vmatrix} u_3, & u_4, & u_1 \\ v_3, & v_4, & v_1 \\ w_3, & w_4, & w_1 \end{vmatrix} = \frac{1}{\iota}\begin{vmatrix} u_4, & u_1, & u_2 \\ v_4, & v_1, & v_2 \\ w_4, & w_1, & w_2 \end{vmatrix} = \frac{-1}{\omega}\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ w_1, & w_2, & w_3 \end{vmatrix} = E,$$

where the common value of the four quantities can be determined from the condition

$$1 = \sum l_5^2 = \sum A\epsilon^2.$$

Hence

$$E^2 = \sum A\epsilon^2 E^2 = \sum A \begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ w_2, & w_3, & w_4 \end{vmatrix}^2;$$

and therefore

$$\Omega^2 E^2 = \sum \begin{vmatrix} b, & f, & m \\ f, & c, & n \\ m, & n, & d \end{vmatrix} \begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ w_2, & w_3, & w_4 \end{vmatrix}^2$$

$$= \begin{vmatrix} \sum a u_1^2, & \sum a u_1 v_1, & \sum a u_1 w_1 \\ \sum a u_1 v_1, & \sum a v_1^2, & \sum a v_1 w_1 \\ \sum a u_1 w_1, & \sum a v_1 w_1, & \sum a w_1^2 \end{vmatrix}.$$

Now we have established the relations

$$\sum a u_1^2 = \Omega, \quad \sum a u_1 v_1 = \Omega \frac{1}{\rho}, \quad \sum a u_1 w_1 = -\Omega \frac{\rho'}{\rho^2},$$

$$\sum a v_1^2 = \Omega \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right),$$

$$\sum a v_1 w_1 = -\Omega \left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right),$$

$$\sum a w_1^2 = \Omega \left(\frac{1}{\sigma^2 \tau^2} + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4}\right).$$

When these values are substituted for the several constituents, the determinant is found to be

$$= \frac{\Omega^3}{\sigma^4 \tau^2};$$

and therefore

$$E = \frac{\Omega^{\frac{1}{2}}}{\sigma^2 \tau}.$$

The values of ϵ, η, ι, ω, now are known; and thus the expression for l_5 becomes

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2 \tau} l_5 = \begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix},$$

giving the direction-cosines of the quartinormal.

As yet, no use (explicit or implicit) has been made of the Frenet equation

for the direction of the quartinormal and the magnitude of the coil. Differentiating the relation

$$Y'' + Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) = \frac{l_4}{\sigma\tau} + l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right) - y'\frac{d}{ds}\left(\frac{1}{\rho}\right),$$

which (§ 8) effectively is the Frenet equation for the trinormal, and using the Frenet equations

$$l_4' = \frac{l_5}{\kappa} - \frac{l_3}{\tau}, \quad l_3' = \frac{l_4}{\tau} - \frac{Y}{\sigma}, \quad Y' = \frac{l_3}{\sigma} - \frac{y'}{\rho}, \quad y'' = \frac{Y}{\rho},$$

we have, after re-arrangements,

$$Y''' = \frac{l_5}{\sigma\tau\kappa} + l_4\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right) + l_3\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)\right\}$$
$$+3Y\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right) - y'\left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) - \frac{1}{\rho}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\right\},$$

which really is the Frenet system adapted (§ 8) to a geodesic in any configuration, Y taking the place of l_2.

Other forms can be given to this expression for Y'''. It can be used at once to derive a covariantive expression for the coil. Multiplying by l_5, summing for all the equations, and using the relations

$$\sum l_5 l_4 = 0, \quad \sum l_5 l_3 = 0, \quad \sum l_5 Y = 0, \quad \sum l_5 y' = 0,$$

we have

$$\sum l_5 Y''' = \frac{1}{\sigma\tau\kappa}.$$

Now multiply the foregoing determinantal equation for l_5 throughout by Y''', and use the last result. We had

$$\sum y_1 Y'' = -w_1,$$

$$\sum y_1' Y'' = -\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_1 - (\gamma_1 w_1 + \delta_1 w_2 + \theta_1 w_3 + \phi_1 w_4),$$

so that, differentiating the former, we have

$$\sum y_1 Y''' = -\frac{dw_1}{ds} + \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_1 + (\gamma_1 w_1 + \delta_1 w_2 + \theta_1 w_3 + \phi_1 w_4),$$

or using a symbol z_i, such that

$$z_\iota = \frac{dw_i}{ds} - \gamma_\iota w_1 - \delta_\iota w_2 - \theta_\iota w_3 - \phi_\iota w_4,$$

we have

$$\sum y_1 Y''' = -z_1 + \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_1;$$

and, generally,

$$\sum y_i Y''' = -z_i + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) v_i.$$

Hence

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^3 \tau^2 \kappa} = \begin{vmatrix} \sum y_1 Y''', & \sum y_2 Y''', & \sum y_3 Y''', & \sum y_4 Y''' \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}$$

$$= \begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix}.$$

Ex. 1. As regards the quantity z_i (for $i=1, 2, 3, 4$), it is convenient to introduce magnitudes $\bar{z}_i$, analogous to the magnitudes $\bar{v}_i$ of § 284 and $\bar{w}_i$ of § 285, and defined by the relations

$$\bar{z}_i = a_{1i} z_1 + a_{2i} z_2 + a_{3i} z_3 + a_{4i} z_4.$$

Then it is easy to verify that, for all the values 1, 2, 3, 4, of the numeral j, we have

$$\Omega \frac{d}{ds}\left(\frac{\bar{w}_j}{\Omega}\right) = \bar{z}_j - \gamma_j \bar{w}_1 - \delta_j \bar{w}_2 - \theta_j \bar{w}_3 - \phi_j \bar{w}_4,$$

analogous to the like results for $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$.

Ex. 2. We have

$$\sum_i x_i' z_i = \sum_i \left(\frac{dw_i}{ds} x_i'\right) - w_1 \left(\sum_i \gamma_i x_i'\right) - w_2 \left(\sum_i \delta_i x_i'\right) - w_3 \left(\sum_i \theta_i x_i'\right) - w_4 \left(\sum_i \phi_i x_i'\right)$$

$$= \sum_i \left(\frac{dw_i}{ds} x_i'\right) + \sum_i (w_i x_i'')$$

$$= \frac{d}{ds}\left(\sum x_i' w_i\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right).$$

Now

$$\sum_i x_i' \left(\sum_j \sum_k \sum_l f_{ijkl} x_j' x_k' x_l'\right) = \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right).$$

But we cannot infer that the quantities

$$z_i - \sum_j \sum_k \sum_l f_{ijkl} x_j' x_k' x_l',$$

for $i=1, 2, 3, 4$, are zero; their values depend linearly upon the Riemann four-index symbols, and they are not used in the immediately succeeding investigations.

289. The preceding equation connecting l_5 and Y''' can be written in the slightly modified form

$$\frac{l_5}{\sigma\tau\kappa}+l_4\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+l_3\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-y'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)$$
$$=Y'''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y'-3Y\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right).$$

Multiply the equation throughout by y_j and add for the system. As the sum of the right-hand sides

$$=-z_j+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_j-\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)v_j=-z_j,$$

the result is to give

$$\begin{aligned}&A_{1j}\left[\frac{\epsilon}{\sigma\tau\kappa}+\alpha\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\lambda\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-p'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)\right]\\
+&A_{2j}\left[\frac{\eta}{\sigma\tau\kappa}+\beta\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\mu\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-q'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)\right]\\
+&A_{3j}\left[\frac{\iota}{\sigma\tau\kappa}+\gamma\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\nu\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-r'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)\right]\\
+&A_{4j}\left[\frac{\omega}{\sigma\tau\kappa}+\delta\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\varpi\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-t'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)\right]=-z_j,\end{aligned}$$

holding for $j=1, 2, 3, 4$. Consequently

$$\left.\begin{aligned}\frac{\epsilon}{\sigma\tau\kappa}+\alpha\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\lambda\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-p'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)&=-\frac{1}{\Omega}\bar{z}_1\\
\frac{\eta}{\sigma\tau\kappa}+\beta\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\mu\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-q'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)&=-\frac{1}{\Omega}\bar{z}_2\\
\frac{\iota}{\sigma\tau\kappa}+\gamma\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\nu\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-r'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)&=-\frac{1}{\Omega}\bar{z}_3\\
\frac{\omega}{\sigma\tau\kappa}+\delta\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\varpi\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-t'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)&=-\frac{1}{\Omega}\bar{z}_4\end{aligned}\right\},$$

thus providing explicit values of ϵ, η, ι, ω, the domainal parameters in l_5, after the substitution of the values of α, β, γ, δ, the domainal parameters in l_4, and the values of λ, μ, ν, ϖ, the domainal parameters in l_3.

The results, which thus express $\bar{z}_1$, $\bar{z}_2$, $\bar{z}_3$, $\bar{z}_4$, may also be used to obtain values of z_1, z_2, z_3, z_4, of a modified form. In the results, let the values of ϵ, η, ι, ω, as obtained in § 288, be inserted, so that

$$-\frac{1}{\Omega}\bar{z}_1=\frac{1}{\Omega^{\frac{1}{2}}}\frac{\sigma}{\kappa}\begin{vmatrix}u_2, & u_3, & u_4\\ v_2, & v_3, & v_4\\ w_2, & w_3, & w_4\end{vmatrix}+\alpha\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\lambda\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-p'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

with similar expressions for $\bar{z}_2$, $\bar{z}_3$, $\bar{z}_4$. We have (§§ 284, 286)

$$\frac{1}{\sigma}(A\lambda+H\mu+G\nu+L\varpi)=\frac{1}{\rho}u_1-v_1,$$

$$\frac{1}{\tau}(A\alpha+H\beta+G\gamma+L\delta)=u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_1\sigma'-w_1\sigma;$$

hence, multiplying the equations for the quantities $\bar{z}_i$ by A, H, G, L, respectively, and adding, we obtain the result

$$-z_1=\frac{1}{\Omega^{\frac{1}{2}}}\frac{\sigma}{\kappa}\begin{vmatrix} A, & H, & G, & L \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}+\sigma\tau\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)\left\{u_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_1\sigma'-w_1\sigma\right\}$$

$$+\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}\left(\frac{\sigma}{\rho}u_1-\sigma v_1\right)-u_1\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)$$

$$=\frac{1}{\Omega^{\frac{1}{2}}}\frac{\sigma}{\kappa}\begin{vmatrix} A, & H, & G, & L \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}-Uu_1+Vv_1-Ww_1,$$

where

$$U=\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\frac{\sigma}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)+\frac{1}{\rho\tau^2}-\sigma\tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)$$

$$=\frac{1}{\rho\tau^2}+\frac{1}{\sigma\tau}\frac{d}{ds}\left\{\tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\},$$

$$V=\frac{1}{\tau^2}+\frac{1}{\sigma\tau}\frac{d}{ds}\left(\tau\frac{d\sigma}{ds}\right),$$

$$W=\sigma^2\tau\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right).$$

The general result is, for $i=1, 2, 3, 4$,

$$-z_i=\frac{1}{\Omega^{\frac{1}{2}}}\frac{\sigma}{\kappa}\begin{vmatrix} A_{1i}, & A_{2i}, & A_{3i}, & A_{4i} \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}-Uu_i+Vv_i-Ww_i.$$

From these values, two inferences can be made. In the first place, let the four relations, giving ϵ, η, ι, ω, be multiplied by y_1, y_2, y_3, y_4, respectively, and the products be added; then, when regard is paid to the postulated forms of l_3, l_4, l_5, we find

$$\frac{l_5}{\sigma\tau\kappa}+l_4\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+l_3\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-y'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)$$

$$=-\frac{1}{\Omega}(y_1\bar{z}_1+y_2\bar{z}_2+y_3\bar{z}_3+y_4\bar{z}_4).$$

Next, let this relation be compared with the later form of relation connecting l_5 and Y'''; and then it appears that we have the equation

$$Y''' + \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) Y' - 3Y\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right) = -\frac{1}{\Omega}(y_1\bar{z}_1 + y_2\bar{z}_2 + y_3\bar{z}_3 + y_4\bar{z}_4).$$

We have already had the relations

$$Y' = -\frac{1}{\Omega}(y_1\bar{v}_1 + y_2\bar{v}_2 + y_3\bar{v}_3 + y_4\bar{v}_4),$$

$$Y'' + \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right) Y = -\frac{1}{\Omega}(y_1\bar{w}_1 + y_2\bar{w}_2 + y_3\bar{w}_3 + y_4\bar{w}_4).$$

Of these three relations, the expression for Y'' can be deduced from the expression for Y' by differentiation, and the expression for Y''' can be deduced from the expression for Y'' also by differentiation; but this actual verification of these statements requires the use of later results connected (§ 297) with the partial differential equations of the second order satisfied by the point-coordinates of position in the domain.

Some covariants and their geometrical significance.

290. Various inferences, concerning concomitants of the whole system of quantics, can be derived from the last form of expression for l_5.

(i) Multiply the equation by y', and add the products for all the dimensions of the plenary space: then, as

$$\sum y_i y' = u_i,$$

we have

$$\begin{aligned}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) &= \frac{1}{\Omega}\sum_i u_i\bar{z}_i \\ &= \frac{1}{\Omega}\sum (A_{i1}p' + A_{i2}q' + A_{i3}r' + A_{i4}t')(a_{1i}z_1 + a_{2i}z_2 + a_{3i}z_3 + a_{4i}z_4) \\ &= z_1p' + z_2q' + z_3r' + z_4t',\end{aligned}$$

a result already established.

(ii) Multiply the equation by l_3, and add the products as before: then

$$\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma\tau^2} = -\frac{1}{\Omega}\sum_i \{z_i(\sum y_i l_3)\}.$$

Now (§ 284)

$$\sum y_i l_3 = \frac{\sigma}{\rho}u_i - \sigma v_i,$$

so that the right-hand side

$$\begin{aligned}&= \frac{\sigma}{\Omega}\sum_i (v_i\bar{z}_i) - \frac{\sigma}{\rho}\frac{1}{\Omega}\sum_i u_i\bar{z}_i \\ &= \frac{\sigma}{\Omega}\sum_i v_i\bar{z}_i - \frac{\sigma}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right);\end{aligned}$$

and therefore

$$\frac{1}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{1}{\sigma}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma^2\tau^2}=\frac{1}{\Omega}\sum_i v_i\bar{z}_i$$

$$=\frac{1}{\Omega}\sum av_1z_1,$$

a covariant of the system.

(iii) Multiply the equation by l_4, and add the products as before: then

$$\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)=-\frac{1}{\Omega}\sum_i\{\bar{z}_i(\sum y_il_4)\}.$$

It was proved (§ 286) that

$$\frac{1}{\tau}\sum y_il_4=u_i\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)-v_i\sigma'-w_i\sigma;$$

and therefore

$$\frac{\sigma}{\tau}\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)=-\frac{1}{\Omega}\left[\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\left(\sum_i u_i\bar{z}_i\right)-\sigma'\left(\sum_i v_i\bar{z}_i\right)-\sigma\left(\sum_i w_i\bar{z}_i\right)\right]$$

$$=-\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)$$

$$+\sigma'\left[\frac{1}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{1}{\sigma}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma^2\tau^2}\right]$$

$$+\frac{\sigma}{\Omega}\sum_i w_i\bar{z}_i.$$

Consequently, after a slight transformation,

$$\frac{1}{\Omega}\sum aw_1z_1=\frac{d}{ds}\left(\frac{1}{\rho}\right)\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{d}{ds}\left(\frac{1}{\sigma}\right)\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)+\frac{1}{\sigma\tau}\frac{d}{ds}\left(\frac{1}{\sigma\tau}\right),$$

a result which can also be derived by differentiation from the earlier equation (§ 286)

$$\frac{1}{\Omega}\sum aw_1^2=\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\}^2+\left\{\frac{d}{ds}\left(\frac{1}{\sigma}\right)\right\}^2+\frac{1}{\sigma^2\tau^2}.$$

(iv) Multiply the equation by l_5, and add the products as before: then

$$\frac{1}{\sigma\tau\kappa}=-\frac{1}{\Omega}\sum_i\{\bar{z}_i(\sum y_il_5)\}.$$

Now

$$\sum y_il_5=A_{1i}\epsilon+A_{2i}\eta+A_{3i}\iota+A_{4i}\omega,$$

and therefore

$$\sum_i\bar{z}_i(\sum y_il_5)=(A_{1i}\epsilon+A_{2i}\eta+A_{3i}\iota+A_{4i}\omega)(a_{1i}z_1+a_{2i}z_2+a_{3i}z_3+a_{4i}z_4)$$

$$=\Omega(\epsilon z_1+\eta z_2+\iota z_3+\omega z_4).$$

From this result, we can proceed in two ways.

In the first way, we substitute for ϵ the value (§ 288)

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau}\epsilon=\begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ w_2, & w_3, & w_4 \end{vmatrix},$$

and the corresponding values for η, ι, ω; then, after a slight reduction, we find

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^3\tau^2\kappa}=\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix},$$

in accordance with the equation already obtained (p. 316).

In the second way, we substitute for ϵ the value (p. 317)

$$-\frac{\epsilon}{\sigma\tau\kappa}=\frac{\bar{z}_1}{\Omega}+a\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+\lambda\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-p'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

and the corresponding values for η, ι, ω; and, again after some reduction, we find

$$\frac{1}{\Omega}\sum az_1^2=\frac{1}{\sigma^2\tau^2\kappa^2}+\sigma^2\left\{\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)\right\}^2+\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}^2+\left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)\right\}^2.$$

This last expression for the concomitant $\sum az_1^2$ of the system can be obtained by squaring the equation

$$-\frac{1}{\Omega}\sum y_i\bar{z}_i=\frac{l_5}{\sigma\tau\kappa}+l_4\sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)+l_3\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma\tau^2}\right\}-y'\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

and adding for all the dimensions of the plenary space, using the relations affecting the sums of the squares, and the sums of the products in pairs, of the typical direction-cosines y', l_3, l_4, l_5, of principal lines of the domainal geodesic.

We thus have the geometrical values of the concomitants

$$\sum au_1z_1,\quad \sum av_1z_1,\quad \sum aw_1z_1,\quad \sum az_1^2,$$

similar to the earlier expressions (p. 314) for

$$\begin{matrix} \sum au_1^2, & \sum au_1v_1, & \sum au_1w_1, \\ & \sum av_1^2\;, & \sum av_1w_1, \\ & & \sum aw_1^2\;. \end{matrix}$$

291. Corresponding results may be deduced similarly from the equation in Y''' which is

$$Y'''+\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)Y'-3Y\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)=-\frac{1}{\Omega}(y_1\bar{z}_1+y_2\bar{z}_2+y_3\bar{z}_3+y_4\bar{z}_4).$$

(i) Let the equation be multiplied by y_i, and the results be added for all the dimensions of the plenary space; then we have

$$\sum y_i Y''' = -z_i + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) v_i,$$

in accordance with an earlier result.

(ii) Let the equation be multiplied by Y, and the results be added similarly; then

$$\sum Y Y''' = 3\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right),$$

which can be derived solely from the Frenet equations adapted to a geodesic in any amplitude.

(iii) Let the equation be multiplied by Y', and the results be added similarly. Now

$$\sum y_i Y' = -v_i,$$

for all the values of i; and we therefore have

$$\sum Y' Y''' + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) \sum Y'^2 = \frac{1}{\Omega}(v_1 z_1 + v_2 z_2 + v_3 z_3 + v_4 z_4)$$

$$= \frac{1}{\Omega} \sum a v_1 z_1.$$

But

$$\sum Y'^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2},$$

so that

$$\sum Y' Y'' = -\frac{\rho'}{\rho^3} - \frac{\sigma'}{\sigma^3};$$

consequently

$$\sum Y' Y''' + \sum Y''^2 = -\frac{\rho''}{\rho^3} + 3\frac{\rho'^2}{\rho^4} - \frac{\sigma''}{\sigma^3} + 3\frac{\sigma'^2}{\sigma^4},$$

or, because (§ 286, *Ex.* 2)

$$\sum Y''^2 = \frac{1}{\sigma^2 \tau^2} + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4} + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2,$$

we have

$$\sum Y' Y''' = -\frac{\rho''}{\rho^3} + 2\frac{\rho'^2}{\rho^4} - \frac{\sigma''}{\sigma^3} + 2\frac{\sigma'^2}{\sigma^4} - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2 - \frac{1}{\sigma^2 \tau^2}.$$

Substituting these values, we find

$$\frac{1}{\Omega} \sum a v_1 z_1 = \frac{1}{\rho} \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) + \frac{1}{\sigma} \frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma^2 \tau^2},$$

in accordance with the result already (p. 320) obtained.

(iv) Let the equation be multiplied by Y'', and the results be added similarly. Now

$$\sum y_i Y'' = -w_i,$$

for all the values of i; and

$$\sum YY'' = -\sum Y'^2 = -\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right),$$

$$\sum Y'Y'' = -\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)$$

$$\sum Y'''Y'' = \frac{1}{\sigma\tau}\frac{d}{ds}\left(\frac{1}{\sigma\tau}\right)+\frac{d}{ds}\left(\frac{1}{\rho}\right)\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{d}{ds}\left(\frac{1}{\sigma}\right)\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)$$

$$-\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right).$$

When these values are substituted, we find

$$\frac{1}{\Omega}\sum aw_1z_1 = \frac{1}{\sigma\tau}\frac{d}{ds}\left(\frac{1}{\sigma\tau}\right)+\frac{d}{ds}\left(\frac{1}{\rho}\right)\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)+\frac{d}{ds}\left(\frac{1}{\sigma}\right)\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right),$$

in accordance with an earlier result.

(v) Let the equation be multiplied by Y''' and the results be added similarly: or let the equation be squared and the results be added. By either process, we find (after the necessary respective reductions, and substituting the value already obtained for $\sum az_1^2$)

$$\sum Y'''^2 = \frac{1}{\sigma^2\tau^2\kappa^2}+\sigma^2\left\{\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)\right\}^2+\left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)\right\}^2$$

$$+9\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)^2+\left\{\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)-\frac{1}{\rho}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)^2\right\},$$

which also can be deduced from the Frenet equations adapted to a geodesic.

(vi) Let the equation be multiplied by y_1' and the products be added similarly for all the dimensions of the plenary space. Now we have had (§ 285, *Ex.* 3)

$$\sum Y'y_1' = -(\gamma_1 v_1+\delta_1 v_2+\theta_1 v_3+\phi_1 v_4),$$

$$\sum y_1 y_1' = \gamma_1 A+\delta_1 H+\theta_1 G+\phi_1 L,$$

$$\sum y_2 y_1' = \gamma_1 H+\delta_1 B+\theta_1 F+\phi_1 M,$$

among others; hence

$$\sum y_1' Y''' - \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)(\gamma_1 v_1+\delta_1 v_2+\theta_1 v_3+\phi_1 v_4) - 3v_1\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)$$

$$= -\frac{1}{\Omega}\sum(\gamma_1 A+\delta_1 H+\theta_1 G+\phi_1 L)(az_1+hz_2+gz_3+lz_4)$$

$$= -(\gamma_1 z_1+\delta_1 z_2+\theta_1 z_3+\phi_1 z_4).$$

Likewise for multiplication by y_2', by y_3', by y_4'; the general result, for $i=1, 2, 3, 4$, is

$$\sum y_i' Y''' = 3v_i\left(\frac{\rho'}{\rho^3}+\frac{\sigma'}{\sigma^3}\right)$$
$$+(\gamma_i v_1+\delta_i v_2+\theta_i v_3+\phi_i v_4)\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)$$
$$-(\gamma_i z_1+\delta_i z_2+\theta_i z_3+\phi_i z_4).$$

Ex. Establish the relations:

$$\text{(i)}\ \sum l_3 Y''' = \frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma}\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right);$$

$$\text{(ii)}\ \sum l_4 Y''' = \sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right);$$

$$\text{(iii)}\ \sum l_5 Y''' = \frac{1}{\sigma\tau\kappa};$$

and shew that, for any value of m,

$$\sum l_m \frac{d^{m-2}Y}{ds^{m-2}} = \frac{1}{\rho_2\rho_3\cdots\rho_{m-1}},$$

where $1/\rho_2$, $1/\rho_3$, $1/\rho_4$, ..., are the successive curvatures in the Frenet equation next in rank after the circular curvature.

(All these relations hold for a geodesic in a general amplitude.)

Parametric curves of the domain and gremial lines of a domainal geodesic.

292. There are two sets of leading lines in the tangent block of the domain. One of the sets is composed of the tangents to the four parametric curves; the typical direction-cosines of the four lines are respectively $y_1A^{-\frac{1}{2}}$, $y_2B^{-\frac{1}{2}}$, $y_3C^{-\frac{1}{2}}$, $y_4D^{-\frac{1}{2}}$. The other set is constituted by four of the principal lines of a domainal geodesic, being the tangent, the binormal, the trinormal, and the quartinormal; the typical direction-cosines of these four lines are respectively y', l_3, l_4, l_5.

Each set of four cosines is expressible linearly in terms of the other set. The linear expressions for y', l_3, l_4, l_5, in terms of y_1, y_2, y_3, y_4, have been given in preceding sections. The corresponding linear expressions for y_1, y_2, y_3, y_4, in terms of y', l_3, l_4, l_5, are given by the comprehensive formula (for $i=1, 2, 3, 4$)

$$y_i = y'u_i$$
$$+l_3\sigma\left(\frac{lu_i}{\rho}-v_i\right)$$
$$+l_4\tau\left\{\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)u_i-\sigma'v_i-\sigma w_i\right\}$$
$$+l_5\frac{\sigma^2\tau}{\Omega^{\frac{1}{2}}}\begin{vmatrix} A_{1i}, & A_{2i}, & A_{3i}, & A_{4i} \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix},$$

which admits of easy verification, by taking the respective sums

$$\sum y_i y', \quad \sum y_i l_3, \quad \sum y_i l_4, \quad \sum y_i l_5,$$

and inserting the established values of these sums.

Principal values and principal directions of the circular curvature of geodesics.

293. As the results will be required later, consider the principal values of the circular curvature of domainal geodesics, whether the complete aggregate be taken, or the aggregate touching a domainal region be taken, or the aggregate touching a domainal surface be taken.

We have (§ 281), for all geodesics, the relations

$$\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right)=\frac{2v_1}{\rho}, \quad \frac{\partial}{\partial q'}\left(\frac{1}{\rho}\right)=\frac{2v_2}{\rho}, \quad \frac{\partial}{\partial r'}\left(\frac{1}{\rho}\right)=\frac{2v_3}{\rho}, \quad \frac{\partial}{\partial t'}\left(\frac{1}{\rho}\right)=\frac{2v_4}{\rho}.$$

(i) When we take the complete aggregate of all domainal geodesics, the only condition imposed on the direction-variables is

$$\sum Ap'^2=1.$$

Accordingly, the principal values are to be obtained by making $1/\rho$ a maximum or a minimum, for all sets of variables p', q', r', t', subject to this single condition. The critical equations are

$$\frac{v_i}{\rho}=Tu_i, \quad (i=1, 2, 3, 4),$$

the value of the multiplier T being undetermined in the formation of the critical equations. Multiplying the four equations by p', q', r', t', respectively, and adding, we have

$$T=\frac{1}{\rho^2};$$

and therefore the critical equations are

$$v_1=\frac{u_1}{\rho}, \quad v_2=\frac{u_2}{\rho}, \quad v_3=\frac{u_3}{\rho}, \quad v_4=\frac{u_4}{\rho}.$$

It follows that, at any point on a domainal geodesic which there touches one of the principal directions (that is, directions of principal circular curvature), the torsion of the domainal geodesic vanishes (§ 284). To the general result, we shall return later.

(ii) When we take the aggregate of the domainal geodesics which touch a parametric region $\epsilon(p, q, r, t)=0$, two conditions are imposed on the direction-variables, being

$$\sum Ap'^2=1, \quad \epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t'=0.$$

Accordingly, the critical equations for the principal values of the circular curvature of domainal geodesics touching the region are

$$\frac{v_i}{\rho}=Tu_i+W\epsilon_i,$$

for $i=1, 2, 3, 4$, the multipliers T and W being left undetermined in the formation of the critical equations.

First, multiply the four equations by p', q', r', t', and add: then, when account is taken of the limiting conditions affecting the direction-variables, the result becomes

$$\frac{1}{\rho^2}=T,$$

so that the equations are

$$v_i-\frac{u_i}{\rho}=W\rho\epsilon_i, \quad (i=1, 2, 3, 4).$$

Next, proceeding from this modified form, we have

$$\sum a\left(v_1-\frac{u_1}{\rho}\right)^2=W^2\rho^2\sum a\epsilon_1^2.$$

Now

$$\sum au_1^2=\Omega, \quad \sum au_1v_1=\frac{\Omega}{\rho}, \quad \sum av_1^2=\Omega\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right), \quad \sum a\epsilon_1^2=\Omega\epsilon_n^2;$$

and therefore

$$\frac{1}{\sigma^2}=W^2\rho^2\epsilon_n^2,$$

giving a value of W; and thus the four equations are

$$v_i=\frac{1}{\rho}u_i+\frac{1}{\sigma}\frac{\epsilon_i}{\epsilon_n},$$

for $i=1, 2, 3, 4$.

One inference is immediate. We have

$$\sum a\epsilon_1v_1=\frac{1}{\rho}\sum a\epsilon_1u_1+\frac{1}{\sigma\epsilon_n}\sum a\epsilon_1^2;$$

but $\sum a\epsilon_1^2=\Omega\epsilon_n^2$, and

$$\sum a\epsilon_1u_1=\Omega(\epsilon_1p'+\epsilon_2q'+\epsilon_3r'+\epsilon_4t')=0,$$

and therefore

$$\sum a\epsilon_1v_1=\Omega\frac{\epsilon_n}{\sigma},$$

it being remembered that the directions now are the principal directions of circular curvature of the domainal geodesics which originate in the region $\epsilon(p, q, r, t)=0$.

(iii) When we take the aggregate of domainal geodesics, which originate in an assigned superficial orientation in the domain, we represent this orientation as the intersection of two given parametric regions

$$\epsilon(p, q, r, t)=0, \quad \iota(p, q, r, t)=0.$$

Let I denote the inclination of the regions; and let dn and $d\nu$ denote domainal small arcs along the domainal normals to $\epsilon=0$ and $\iota=0$ respectively; then

$$\cos I=\frac{1}{\Omega\epsilon_n\epsilon_\nu}\sum a\epsilon_1\iota_1.$$

There now are three conditions imposed on the direction-variables p', q', r', t'; and thus the critical equations for principal values of circular curvature among the domainal geodesics in the specified superficial orientation are found to be

$$v_k=\frac{1}{\rho}u_k+\lambda\epsilon_k+\mu\iota_k, \quad (k=1, 2, 3, 4),$$

where the multipliers λ and μ still have to be found.

From these equations, we have

$$\sum av_1\epsilon_1=\frac{1}{\rho}\sum au_1\epsilon_1+\lambda\sum a\epsilon_1^2+\mu\sum a\epsilon_1\iota_1,$$

or, as $\sum au_1\epsilon_1=\Omega(\epsilon_1p'+\epsilon_2q'+\epsilon_3r'+\epsilon_4t')=0$, the equation is

$$\frac{1}{\Omega\epsilon_n}\sum av_1\epsilon_1=\lambda\epsilon_n+\mu\iota_\nu\cos I.$$

Similarly, we find

$$\frac{1}{\Omega\iota_\nu}\sum av_1\iota_1=\lambda\epsilon_n\cos I+\mu\iota_\nu.$$

The covariants $\sum av_1\epsilon_1$ and $\sum av_1\iota_1$ are connected with the torsion of geodesics in a domainal region (Chap. XXVIII); thus the multipliers λ and μ may be regarded as definite. The directions, occurring in these equations, are the principal directions of circular curvature for the selected domainal geodesics; and they are distinct from the principal directions in the earlier aggregate.

Ex. 1. Verify the relation

$$\lambda^2\epsilon_n^2+2\lambda\mu\epsilon_n\iota_\nu\cos I+\mu^2\iota_\nu^2=\frac{\Omega}{\sigma^2}.$$

Ex. 2. In the result as obtained, the orientation at O is assigned as the superficial intersection of parametric regions $\epsilon=0$ and $\iota=0$ passing through O. The orientation may also be assigned by a set of orientation-variables s_{23}, s_{31}, s_{12}, s_{14}, s_{24}, s_{34}, so that a direction p', q', r', t', in the orientation satisfies two independent equations

$$p's_{23}+q's_{31}+r's_{12}=0, \quad p's_{24}+q's_{41}+t's_{12}=0,$$

and two similar (but not independent) relations. Let U_i and V_i be defined by the expressions

$$V_i = v_1 s_{1i} + v_2 s_{2i} + v_3 s_{3i} + v_4 s_{4i},$$

$$U_i = u_1 s_{1i} + u_2 s_{2i} + u_3 s_{3i} + u_4 s_{4i};$$

then it can be proved that the principal directions of the circular curvature of domainal geodesics, which originate in the orientation specified by the variables s_{ij}, are given by the equations

$$V_i = \frac{1}{\rho} U_i,$$

for $i = 1, 2, 3, 4$, only two of the four equations constituting an independent pair of equations; and they are to be combined with the two independent relations between the direction-variables and the orientation-variables.

CHAPTER XXV

Geodesics in a Free Domain: Non-Gremial Properties

Fifth normal (quintinormal) of a domainal geodesic: the fifth curvature.

294. Thus far, the curvatures of a domainal geodesic, which have been considered in addition to the (prime) circular curvature, are the gremial curvatures associated with principal lines lying within the tangent block of the domain. We now proceed to initiate the consideration of the non-gremial curvatures of the domainal geodesic; the principal lines associated with these have direction-cosines typically denoted by l_μ, for values $\mu=6, 7, \ldots, N$, in succession, N denoting the number of dimensions of the plenary space.

The typical direction-cosine l_5 of the quartinormal has been obtained in the form

$$\frac{l_5}{\sigma^2\tau}=\Omega^{-\frac{1}{2}}\begin{vmatrix} y, & y_2, & y_3, & y_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}=\Omega^{-\frac{1}{2}}\,|\,y_1,\ u_1,\ v_1,\ w_1\,|,$$

with an abbreviated notation for the determinant. Differentiating along the geodesic, and using the Frenet equation, we have

$$\frac{1}{\sigma^2\tau}\left(\frac{l_6}{\rho_5}-\frac{l_4}{\kappa}\right)+l_5\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)$$
$$=\Omega^{-\frac{1}{2}}\,|\,y_1',\ u_1,\ v_1,\ w_1\,|+\sum y_1\frac{d}{ds}\{\Omega^{-\frac{1}{2}}\,|\,u_2,\ v_2,\ w_2\,|\}.$$

Now

$$\begin{aligned}\frac{d}{ds}\begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ w_2, & w_3, & w_4 \end{vmatrix} = {}& S\{(v_3w_4-v_4w_3)(\gamma_2u_1+\delta_2u_2+\theta_2u_3+\phi_2u_4)\} \\ &+S\{(w_3u_4-w_4u_3)(w_2+\gamma_2v_1+\delta_2v_2+\theta_2v_3+\phi_2v_4)\} \\ &+S\{(u_3v_4-u_4v_3)(z_2+\gamma_2w_1+\delta_2w_2+\theta_2w_3+\phi_2w_4)\},\end{aligned}$$

where the summation-symbol S implies summation for the three terms which arise from the determinant by the cyclical interchanges of the suffixes 2, 3, 4; and the right-hand side can be expressed in the form

$$\begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ z_2, & z_3, & z_4 \end{vmatrix}+(\gamma_1+\delta_2+\theta_3+\phi_4)\begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ w_2, & w_3, & w_4 \end{vmatrix}-\begin{vmatrix} \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

Also, we have

$$\frac{\Omega'}{2\Omega}=\frac{1}{2\Omega}(\Omega_1 p'+\Omega_2 q'+\Omega_3 r'+\Omega_4 t')=\gamma_1+\delta_2+\theta_3+\phi_4\,;$$

and therefore

$$\Omega^{\frac{1}{2}}\frac{d}{ds}\{\Omega^{-\frac{1}{2}}\,|\,u_2,\ v_2,\ w_2\,|\}=|\,u_2,\ v_2,\ z_2\,|-|\,\gamma_1,\ u_1,\ v_1,\ w_1\,|\,.$$

Similarly for the other like terms in the summation in the equation for l_6; and therefore

$$\frac{l_6}{\sigma^2\tau\rho_5}+l_5\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)-\frac{l_4}{\sigma^2\tau\kappa}$$

$$=\ \Omega^{-\frac{1}{2}}\,|\,y_1',\ u_1,\ v_1,\ w_1\,|+\Omega^{-\frac{1}{2}}\,|\,y_1,\ u_1,\ v_1,\ z_1\,|$$

$$-\,\Omega^{-\frac{1}{2}}\begin{vmatrix} y_1\gamma_1+y_2\delta_1+y_3\theta_1+y_4\phi_1, & u_1, & v_1, & w_1 \\ y_1\gamma_2+y_2\delta_2+y_3\theta_2+y_4\phi_2, & u_2, & v_2, & w_2 \\ y_1\gamma_3+y_2\delta_3+y_3\theta_3+y_4\phi_3, & u_3, & v_3, & w_3 \\ y_1\gamma_4+y_2\delta_4+y_3\theta_4+y_4\phi_4, & u_4, & v_4, & w_4 \end{vmatrix}.$$

Ex. Verify the relations

$$\frac{d}{ds}\left(\frac{\epsilon}{\sigma^2\tau}\right)=\Omega^{-\frac{1}{2}}\begin{vmatrix} u_2, & u_3, & u_4 \\ v_2, & v_3, & v_4 \\ z_2, & z_3, & z_4 \end{vmatrix}-\frac{1}{\sigma^2\tau}(\gamma_1\epsilon+\gamma_2\eta+\gamma_3\iota+\gamma_4\omega),$$

$$-\frac{d}{ds}\left(\frac{\eta}{\sigma^2\tau}\right)=\Omega^{-\frac{1}{2}}\begin{vmatrix} u_3, & u_4, & u_1 \\ v_3, & v_4, & v_1 \\ z_3, & z_4, & z_1 \end{vmatrix}-\frac{1}{\sigma^2\tau}(\delta_1\epsilon+\delta_2\eta+\delta_3\iota+\delta_4\omega),$$

$$\frac{d}{ds}\left(\frac{\iota}{\sigma^2\tau}\right)=\Omega^{-\frac{1}{2}}\begin{vmatrix} u_4, & u_1, & u_2 \\ v_4, & v_1, & v_2 \\ z_4, & z_1, & z_2 \end{vmatrix}-\frac{1}{\sigma^2\tau}(\theta_1\epsilon+\theta_2\eta+\theta_3\iota+\theta_4\omega),$$

$$-\frac{d}{ds}\left(\frac{\omega}{\sigma^2\tau}\right)=\Omega^{-\frac{1}{2}}\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ z_1, & z_2, & z_3 \end{vmatrix}-\frac{1}{\sigma^2\tau}(\phi_1\epsilon+\phi_2\eta+\phi_3\iota+\phi_4\omega).$$

Next, with the notation

$$\eta_{i1}p'+\eta_{i2}q'+\eta_{i3}r'+\eta_{i4}t'=\eta_i,$$

for $i=1,\,2,\,3,\,4$, we have

$$\begin{aligned} y_1' &= y_{11}p'+y_{12}q'+y_{13}r'+y_{14}t' \\ &= \eta_1+y_1\gamma_1+y_2\delta_1+y_3\theta_1+y_4\phi_1\,; \end{aligned}$$

and, generally,

$$y_i'=\eta_i+y_1\gamma_i+y_2\delta_i+y_3\theta_i+y_4\phi_i.$$

Thus the first term and the third term in the foregoing equation combine; and we have

$$\frac{l_6}{\sigma^2\tau\rho_5}+l_5\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)-\frac{l_4}{\sigma^2\tau\kappa}=\Omega^{-\frac{1}{2}}\,|\,\eta_1,\,u_1,\,v_1,\,w_1\,|+\Omega^{-\frac{1}{2}}\,|\,y_1,\,u_1,\,v_1,\,z_1\,|.$$

The second term on the right-hand side can be evaluated. By the results in § 292, there are four relations of the form

$$y_1=y'u_1+l_3\sigma\left(\frac{u_1}{\rho}-v_1\right)+l_4\tau\left\{\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)u_1-\sigma'v_1-\sigma w_1\right\}+l_5\sum l_5y_1.$$

In this second term, let these values of y_1, y_2, y_3, y_4, be substituted, and the result be arranged as a linear combination in y', l_3, l_4, l_5. The total coefficient of y' in $|\,y_1,\,u_1,\,v_1,\,z_1\,|$ becomes

$$|\,u_1,\,u_1,\,v_1,\,z_1\,|=0.$$

The total coefficient of l_3

$$=\sigma\left|\,\frac{u_1}{\rho}-v_1,\,u_1,\,v_1,\,z_1\,\right|=0.$$

The total coefficient of l_4, similarly,

$$=-\sigma\tau\,|\,w_1,\,u_1,\,v_1,\,z_1\,|$$

$$=-\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau\kappa},$$

by the expression for the coil found in § 288. The total coefficient of l_5 in the determinant $|\,y_1,\,u_1,\,v_1,\,z_1\,|$, after the substitutions,

$$=|\,\textstyle\sum l_5y_1,\,u_1,\,v_1,\,z_1\,|.$$

In § 288, the relation

$$\Omega^{\frac{1}{2}}l_5=\sigma^2\tau\begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}$$

was obtained; and therefore

$$\sum l_5y_1=\frac{\sigma^2\tau}{\Omega^{\frac{1}{2}}}\begin{vmatrix} A, & H, & G, & L \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}$$

$$=\sigma\tau\kappa(Uu_1-Vv_1+Ww_1-z_1),$$

by the result on p. 318, with the determinate values of U, V, W, there given. Generally, we have (for $i=1, 2, 3, 4$)

$$\sum l_5y_i=\sigma\tau\kappa(Uu_i-Vv_i+Ww_i-z_i),$$

the value of W being given by

$$W=\sigma^2\tau\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right).$$

Hence

$$\left|\sum l_5y_1, u_1, v_1, z_1\right|=W\sigma\tau\kappa\left|w_1, u_1, v_1, z_1\right|;$$

and the value of the determinant on the right-hand side has been obtained in § 288 as $\Omega^{\frac{1}{2}}/\sigma^3\tau^2\kappa$; therefore

$$\left|\sum l_5y_1, u_1, v_1, z_1\right|=\Omega^{\frac{1}{2}}\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right).$$

Consequently, we have

$$\Omega^{-\frac{1}{2}}\left|y_1, u_1, v_1, z_1\right|=l_5\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)-\frac{l_4}{\sigma^2\tau\kappa}.$$

The final form of the expression for the typical direction-cosine l_6 thus becomes

$$\frac{l_6}{\sigma^2\tau\rho_5}=\Omega^{-\frac{1}{2}}\left|\eta_1, u_1, v_1, w_1\right|=\Omega^{-\frac{1}{2}}\begin{vmatrix} \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

Once more using the values of ϵ, η, ι, ω, we can state the result also in the form

$$\frac{l_6}{\rho_5}=\eta_1\epsilon+\eta_2\eta+\eta_3\iota+\eta_4\omega.$$

In connection with these quantities $\eta_1, \eta_2, \eta_3, \eta_4$, we recall the equation for the circular curvature and the direction of the prime normal in the form

$$\frac{Y}{\rho}=\eta_1p'+\eta_2q'+\eta_3r'+\eta_4t'.$$

Magnitudes of rank five.

295. At this stage, we introduce magnitudes X_{ij}, connected with the direction-cosines of the fifth normal in the same manner as the secondary magnitudes $\bar{A}_{ij}$ are connected with the direction-cosines of the prime normal. They are defined by the equation

$$X_{ij}=\sum l_6\eta_{ij},$$

for all the combinations $i, j, =1, 2, 3, 4$; and we write

$$X_{i1}p'+X_{i2}q'+X_{i3}r'+X_{i4}t'=\xi_i,$$

for $i=1, 2, 3, 4$. Then we have

$$\frac{1}{\rho_5}X_{ij}=\epsilon\sum\eta_1\eta_{ij}+\eta\sum\eta_2\eta_{ij}+\iota\sum\eta_3\eta_{ij}+\omega\sum\eta_4\eta_{ij};$$

and the quantities ϵ, η, ι, ω, are expressible in terms of earlier magnitudes already used, while their coefficients, being of the type

$$\sum \eta_k \eta_{ij} = p' \sum \eta_{1k}\eta_{ij} + q' \sum \eta_{2k}\eta_{ij} + r' \sum \eta_{3k}\eta_{ij} + t' \sum \eta_{4k}\eta_{ij},$$

also are expressible in terms of magnitudes already used.

Moreover, an expression for the fifth curvature $1/\rho_5$ is obtainable in the form

$$\frac{1}{{\rho_5}^2} = \sum (\eta_1\epsilon + \eta_2\eta + \eta_3\iota + \eta_4\omega)^2,$$

also expressible in terms of the same magnitudes.

Evidently

$$\xi_i = \sum l_6(\eta_{i1}p' + \eta_{i2}q' + \eta_{i3}r' + \eta_{i4}t') = \sum l_6\eta_i,$$

for all the values of i. Further

$$\begin{aligned}\xi_1 p' + \xi_2 q' + \xi_3 r' + \xi_4 t' &= \sum l_6(\eta_1 p' + \eta_2 q' + \eta_3 r' + \eta_4 t')\\ &= \frac{1}{\rho}\sum l_6 Y = 0,\end{aligned}$$

by the properties of the geodesic. This relation, among the quantities ξ, is of persistent recurrence; an equivalent form is

$$\sum_i \sum_j X_{ij} x_i' x_j' = 0.$$

Again, multiplying the equation

$$\frac{l_6}{\rho_5} = \eta_1\epsilon + \eta_2\eta + \eta_3\iota + \eta_4\omega$$

throughout by l_6 and adding the results, we have

$$\frac{1}{\rho_5} = \xi_1\epsilon + \xi_2\eta + \xi_3\iota + \xi_4\omega\,;$$

and an equivalent form, obtained by substituting the values of ϵ, η, ι, ω, from § 288, is

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau\rho_5} = \begin{vmatrix} \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

The foregoing value for $1/\rho_5$ can also be derived simply as follows. As the principal line, connected with the curvature $1/\rho_5$ of the domainal geodesic and having l_6 for its typical direction-cosine, is orthogonal to the first five principal lines of the geodesic which have y', Y, l_3, l_4, l_5, for their typical direction-cosines, it is orthogonal to the tangent block of the domain: hence the relations

$$\sum y_i l_6 = 0$$

are satisfied, for $i=1, 2, 3, 4$. Differentiating the relation for $i=1$ along the geodesic arc, we have

$$\sum y_1' l_6 + \sum y_1 \left(\frac{l_7}{\rho_6} - \frac{l_5}{\rho_5}\right) = 0.$$

Now

$$y_1' = \eta_1 + y_1\gamma_1 + y_2\delta_1 + y_3\theta_1 + y_4\phi_1,$$

and therefore

$$\sum y_1' l_6 = \sum l_6(\eta_1 + y_1\gamma_1 + y_2\delta_1 + y_3\theta_1 + y_4\phi_1)$$
$$= \sum l_6\eta_1 = \xi_1.$$

Also, as the principal line of the domainal geodesic, with l_7 as its typical direction-cosine, is orthogonal to all the principal lines of the geodesic earlier in rank, it is orthogonal to the tangent block of the domain, and so is at right angles to every direction in that block; hence

$$\sum y_j l_7 = 0,$$

for $j=1, 2, 3, 4$. Thus the equation becomes

$$\xi_1 = \frac{1}{\rho_5}\sum y_1 l_5$$
$$= \frac{1}{\rho_5}(A\epsilon + H\eta + G\iota + L\omega);$$

and similarly

$$\xi_2 = \frac{1}{\rho_5}(H\epsilon + B\eta + F\iota + M\omega),$$

$$\xi_3 = \frac{1}{\rho_5}(G\epsilon + F\eta + C\iota + N\omega),$$

$$\xi_4 = \frac{1}{\rho_5}(L\epsilon + M\eta + N\iota + D\omega).$$

Multiply these equations by ϵ, η, ι, ω, respectively, and add the four results; then, as $\sum A\epsilon^2 = \sum l_5^2 = 1$, we have

$$\frac{1}{\rho_5} = \xi_1\epsilon + \xi_2\eta + \xi_3\iota + \xi_4\omega,$$

the result already established.

Moreover, resolving the equations for the four quantities ϵ, η, ι, ω, we have the relations

$$\frac{\Omega}{\rho_5}\epsilon = a\xi_1 + h\xi_2 + g\xi_3 + l\xi_4,$$

$$\frac{\Omega}{\rho_5}\eta = h\xi_1 + b\xi_2 + f\xi_3 + m\xi_4,$$

$$\frac{\Omega}{\rho_5}\iota = g\xi_1 + f\xi_2 + c\xi_3 + n\xi_4,$$

$$\frac{\Omega}{\rho_5}\omega = l\xi_1 + m\xi_2 + n\xi_3 + d\xi_4;$$

and we therefore also have

$$\frac{\Omega}{\rho_5^2}=\frac{\Omega}{\rho_5}(\epsilon\xi_1+\eta\xi_2+\iota\xi_3+\omega\xi_4)=\sum a\xi_1^2,$$

thus providing another covariantive expression for the fifth curvature.

Magnitudes of rank greater than five.

296. Again, we have the relations

$$\sum y_i l_m=0,$$

for $i=1, 2, 3, 4$, where $m=6, 7, \ldots, N$, and l_m is the typical direction-cosine of the mth principal line.

For all values of $m>6$, we have

$$\sum y_i' l_m+\sum y_i\left(\frac{l_{m+1}}{\rho_m}-\frac{l_{m-1}}{\rho_{m-1}}\right)=0,$$

while, for such values,

$$\sum y_i l_{m+1}=0, \quad \sum y_i l_{m-1}=0;$$

consequently

$$\sum y_i' l_m=0.$$

Now

$$y_i'=\eta_i+y_1\gamma_i+y_2\delta_i+y_3\theta_i+y_4\phi_i:$$

and therefore

$$\sum \eta_i l_m=0,$$

for all the values of $i=1, 2, 3, 4$, and for values of m greater than 6, while, for those values of i,

$$\sum \eta_i l_6=\xi_i,$$

the non-vanishing quantities ξ being subject to the relation

$$\xi_1 p'+\xi_2 q'+\xi_3 r'+\xi_4 t'=0.$$

The relation

$$p'\sum \eta_1 l_m+q'\sum \eta_2 l_m+r'\sum \eta_3 l_m+t'\sum \eta_4 l_m=0$$

is satisfied, being merely the equivalent of $\sum Yl_m=0$.

We note that the relation

$$\sum \eta_i l_m=0,$$

for $i=1, 2, 3, 4$, is satisfied for all values of m except $m=2$ and $m=6$; while

$$\sum \eta_i l_2=\sum \eta_i Y=v_i, \quad \sum \eta_i l_6=\xi_i,$$

being two exceptions which are of fundamental importance.

Partial differential equations of the second order satisfied by space-coordinates.

297. We can now construct some partial differential equations which formally appear to be of the second order and are satisfied by the space-variables of any point in the domain.

For the general frame of coordinate axes in the plenary homaloidal space of the domain, we can substitute the orthogonal frame of any particular geodesic drawn in a domainal direction p', q', r', t'. Also within this orthogonal frame, we can substitute, for the tangent, the binormal, the trinormal, and the quartinormal of the geodesic, the directions of the four parametric curves in the tangent block of the domain; and in this modified frame, all the axes other than those four directions remain orthogonal to one another, while each of these axes is at right angles to each of the four directions.

To this modified frame, any directed quantity can be referred; and an expression for the quantity can be obtained linearly, in terms of typical direction-cosines and appropriate coefficients. Thus for any magnitude connected with a typical variable y, the appropriate typical direction-cosines now are y_1, y_2, y_3, y_4, together with Y, l_6, l_7, ..., l_N, the latter belonging to the axes in the modified frame which have remained principal lines of the geodesic. Consider, for instance, a quantity such as y_{11}, connected with the variable y typical of a space-position; for all the variables of this position, there is a relation

$$y_{11}=y_1\alpha_1+y_2\alpha_2+y_3\alpha_3+y_4\alpha_4+Y\beta_2+l_6\beta_6+l_7\beta_7+\ldots+l_N\beta_N,$$

where the coefficients α, β, are the same through the set of quantities y_{11}, and have to be determined for the set. There are equations

$$\sum Yy_i=0, \quad \sum l_m y_i=0,$$

the summations being taken over all the space-dimensions, and the equations holding for all values of $m \geqslant 6$ and for $i=1, 2, 3, 4$. By earlier results,

$$\sum y_i y_{11}=A_{1i}\Gamma_{11}+A_{2i}\Delta_{11}+A_{3i}\Theta_{11}+A_{4i}\Phi_{11},$$

for the same values of i, so that

$$\alpha_1=\Gamma_{11}, \quad \alpha_2=\Delta_{11}, \quad \alpha_3=\Theta_{11}, \quad \alpha_4=\Phi_{11}.$$

Similarly, owing also to the further relations

$$\sum Yl_m=0,$$

we have

$$\beta_2=\sum Yy_{11}=\bar{A}_{11}.$$

The magnitude

$$y_{ij}-y_1\Gamma_{ij}-y_2\Delta_{ij}-y_3\Theta_{ij}-y_4\Phi_{ij}$$

has been denoted by η_{ij}; so that, for the instance considered,

$$\eta_{11}=Y\bar{A}_{11}+l_6\beta_6+l_7\beta_7+\ldots+l_N\beta_N.$$

In the same way, proceeding for all the magnitudes η_{ij}, we have equations

$$\eta_{ij}=Y\bar{A}_{ij}+l_6(\beta_{ij})_6+l_7(\beta_{ij})_7+\ldots+l_N(\beta_{ij})_N.$$

Further, we have had quantities X_{ij} defined by the equation

$$X_{ij}=\sum l_6\eta_{ij},$$

so that we can take, as a preliminary form of partial differential equation satisfied by y,

$$\eta_{ij} = Y\bar{A}_{ij} + l_6 X_{ij} + l_7 (\beta_{ij})_7 + \ldots + l_N (\beta_{ij})_N .$$

Obviously

$$(\beta_{ij})_m = \sum \eta_{ij} l_m,$$

for all the values $m = 7, \ldots, N$. As

$$\eta_i = \eta_{i1} p' + \eta_{i2} q' + \eta_{i3} r' + \eta_{i4} t',$$

we have

$$p'(\beta_{i1})_m + q'(\beta_{i2})_m + r'(\beta_{i3})_m + t'(\beta_{i4})_m = \sum \eta_i l_m = 0,$$

for all these values of m. Accordingly, take the foregoing equation for $j = 1, 2, 3, 4$; multiply by p', q', r', t', for the respective values of j, and add: then, as

$$\bar{A}_{1i} p' + \bar{A}_{2i} q' + \bar{A}_{3i} r' + \bar{A}_{4i} t' = v_i,$$

$$X_{1i} p' + X_{2i} q' + X_{3i} r' + X_{4i} t' = \xi_1,$$

with the earlier notation, it follows that

$$\eta_i = Y v_i + l_6 \xi_i,$$

holding for all the values $i = 1, 2, 3, 4$.

These equations can be regarded as the definite form of partial differential equations satisfied by the space-coordinates of a point on the domainal geodesic in the direction p', q', r', t'.

Before discussing certain analytical combinations of these quantities η_i, we note that, as

$$\sum Y y_k = 0, \quad \sum l_6 y_k = 0,$$

for all the values of k, $= 1, 2, 3, 4$, we have

$$\sum \eta_i y_k = 0,$$

for all the values of i and k: or the quantities η_i, for the different quantities η and any the same value of i, are the spatial components of a magnitude E_i, which is orthogonal to the tangent block of the domain and which therefore lies in a homaloid orthogonal to this block. The form of the partial equations shews that the direction of E_i lies in a plane through the prime normal and the fifth normal of the domainal geodesic; and therefore all the four magnitudes E_i, for $i = 1, 2, 3, 4$, lie in this same plane which, connected with a domainal geodesic, is orthogonal to the domain. We shall return later to this result; meanwhile, we note the property

$$\sum a\eta_1^2 = \sum a v_1^2 + \sum a \xi_1^2$$

$$= \Omega \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\rho_5^2} \right).$$

Quadratic combinations of the non-gremial magnitudes η.

298. The sums of the quadratic combinations of the four typical quantities η_1, η_2, η_3, η_4, the sums being taken over the dimensions of the plenary space, satisfy certain relations ultimately based upon the aggregate of equations

$$\eta_i = Yv_i + l_6\xi_i.$$

To express these sums of quadratic combinations, we write

$$c_{ij} = \sum \eta_i^{(m)}\eta_j^{(m)}$$

for all the values of i and j, independently of one another, with the space-summation for $m=1, \ldots, N$. The quantity $c_{ii}c_{jj} - c_{ij}^2$, being equal to

$$\sum\sum \{\eta_i^{(m)}\eta_j^{(n)} - \eta_i^{(n)}\eta_j^{(m)}\}^2$$

taken for the space-summations m, n, $=1, \ldots, N$, is essentially a positive quantity, when not zero. We therefore introduce magnitudes m_{ij} such that

$$m_{ij} = (c_{ii}c_{jj} - c_{ij}^2)^{\frac{1}{2}},$$

assigning a positive value to the radical, and assuming $j > i$ for the quantities m_{ij}, because the value of the radical is here unaltered by the interchange of i and j, while $m_{kk}=0$. Manifestly

$$c_{ii} = v_i^2 + \xi_i^2, \quad c_{ij} = v_iv_j + \xi_i\xi_j;$$

and therefore every determinant of the form

$$\begin{vmatrix} c_{i\alpha}, & c_{i\beta}, & c_{i\gamma} \\ c_{j\alpha}, & c_{j\beta}, & c_{j\gamma} \\ c_{k\alpha}, & c_{k\beta}, & c_{k\gamma} \end{vmatrix},$$

for all values of i, j, k, and of α, β, γ, from the range 1, 2, 3, 4, must vanish: while the first minors of such a determinant do not vanish. Accordingly, for the determinant

$$\begin{vmatrix} c_{11}, & c_{12}, & c_{13}, & c_{14} \\ c_{21}, & c_{22}, & c_{23}, & c_{24} \\ c_{31}, & c_{32}, & c_{33}, & c_{34} \\ c_{41}, & c_{42}, & c_{43}, & c_{44} \end{vmatrix},$$

every first minor vanishes, while the second minors do not vanish; and there are algebraical identities among these second minors. To obtain these identities, we introduce angles θ_1, θ_2, θ_3, θ_4, with the definitions

$$v_i = c_{ii}^{\frac{1}{2}}\cos\theta_i, \quad \xi_i = c_{ii}^{\frac{1}{2}}\sin\theta_i;$$

and for the canonical case of reference, we shall assume

$$\theta_1 < \theta_2 < \theta_3 < \theta_4.$$

Then

$$c_{ij} = v_iv_j + \xi_i\xi_j = c_{ii}^{\frac{1}{2}}c_{jj}^{\frac{1}{2}}\cos(\theta_i - \theta_j),$$

and therefore, with all these assumptions,

$$-m_{ij}=(c_{ii}c_{jj})^{\frac{1}{2}}\sin(\theta_i-\theta_j).$$

Further,

$$v_i\xi_l-v_l\xi_i=(c_{ii}c_{ll})^{\frac{1}{2}}\sin(\theta_l-\theta_i)=m_{il},$$

when $l>i$.

Now let numbers i, j, k, l, be selected * from the range 1, 2, 3, 4, such that

$$i\leqslant j,\quad i<l,\quad j<k.$$

Then we have

$$\begin{aligned} c_{ij}c_{kl}-c_{ik}c_{jl}&=(v_iv_j+\xi_i\xi_j)(v_kv_l+\xi_k\xi_l)\\ &\qquad-(v_iv_k+\xi_i\xi_k)(v_jv_l+\xi_j\xi_l)\\ &=(v_i\xi_l-v_l\xi_i)(v_j\xi_k-v_k\xi_j)\\ &=m_{il}m_{jk}.\end{aligned}$$

It follows that all the second minors of the vanishing determinant of four rows can be expressed in terms of the six diagonal minors $m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}$; and, because of the manifest identity

$$(c_{12}c_{34}-c_{13}c_{24})+(c_{13}c_{24}-c_{14}c_{23})+(c_{14}c_{23}-c_{12}c_{34})=0,$$

these six minors are themselves subject to the condition

$$m_{12}m_{34}-m_{13}m_{24}+m_{14}m_{23}=0.$$

The full expression of all the second minors of the determinant of four rows in terms of these diagonal minors is

$$\left.\begin{aligned} c_{22}c_{33}-c_{23}^2&=m_{23}^2\\ c_{33}c_{11}-c_{13}^2&=m_{13}^2\\ c_{11}c_{22}-c_{12}^2&=m_{12}^2\end{aligned}\right\},\qquad \left.\begin{aligned} c_{11}c_{44}-c_{14}^2&=m_{14}^2\\ c_{22}c_{44}-c_{24}^2&=m_{24}^2\\ c_{33}c_{44}-c_{34}^2&=m_{34}^2\end{aligned}\right\},$$

$$\left.\begin{aligned} c_{11}c_{23}-c_{12}c_{13}&=m_{12}m_{13}\\ c_{11}c_{24}-c_{12}c_{14}&=m_{12}m_{14}\\ c_{11}c_{34}-c_{13}c_{14}&=m_{13}m_{14}\end{aligned}\right\},\qquad \left.\begin{aligned} c_{22}c_{13}-c_{12}c_{23}&=-m_{12}m_{23}\\ c_{22}c_{14}-c_{12}c_{24}&=-m_{12}m_{24}\\ c_{22}c_{34}-c_{23}c_{24}&=m_{23}m_{24}\end{aligned}\right\},$$

$$\left.\begin{aligned} c_{33}c_{12}-c_{13}c_{23}&=m_{13}m_{23}\\ c_{33}c_{14}-c_{13}c_{34}&=-m_{13}m_{34}\\ c_{33}c_{24}-c_{23}c_{34}&=-m_{23}m_{34}\end{aligned}\right\},\qquad \left.\begin{aligned} c_{44}c_{12}-c_{14}c_{24}&=m_{14}m_{24}\\ c_{44}c_{13}-c_{14}c_{34}&=m_{14}m_{34}\\ c_{44}c_{23}-c_{24}c_{34}&=m_{24}m_{34}\end{aligned}\right\},$$

$$\left.\begin{aligned} c_{12}c_{34}-c_{23}c_{14}&=m_{13}m_{24}\\ c_{23}c_{24}-c_{12}c_{34}&=-m_{14}m_{23}\\ c_{13}c_{24}-c_{14}c_{23}&=m_{12}m_{34}\end{aligned}\right\}.$$

* The selection is the same as is adopted in connection with the Riemann four-index symbols in § 16.

The formal comparison of these twenty-one minors, with the twenty-one Riemann four-index symbols of § 16, is obvious from the formulæ

$$(\alpha\beta,\ \gamma\delta)=\sum(\eta_{\alpha\gamma}\eta_{\beta\delta}-\eta_{\alpha\delta}\eta_{\beta\gamma}),$$
$$m_{il}m_{jk}=c_{ij}c_{kl}-c_{ik}c_{jl},$$

where there are no limitations of inequality among α, β, γ, δ, while the integers i, j, k, l, are subject to the inequalities $i \leqslant j$, $i<l$, $j<k$. In fact, the Riemann symbols are connected with quantities $\sum \eta_{\alpha\beta}\eta_{\gamma\delta}$; the m-combinations are similarly connected with the quantities $(\sum \eta_\alpha\eta_\beta)(\sum \eta_\gamma\eta_\delta)$.

Linear relations among the magnitudes η.

299. Again, owing to the typical relations

$$\eta_\iota=Yv_\iota+l_6\xi_\iota$$

for $i=2, 3, 4$, there are the conditions

$$\|\ \eta_2,\ \eta_3,\ \eta_4\ \|=0,\quad \|\ \eta_3,\ \eta_4,\ \eta_1\ \|=0,\quad \|\ \eta_4,\ \eta_1,\ \eta_2\ \|=0,\quad \|\ \eta_1,\ \eta_2,\ \eta_3\ \|=0,$$

holding among the magnitudes η. Accordingly, there are equations of the type

$$E\eta_3=P\eta_1+Q\eta_2,$$

there being one such equation for all the magnitudes η_1, η_2, η_3, corresponding to the different space-variables with coefficients E, P, Q, the same throughout. To determine these coefficients E, P, Q, we multiply by η_1 and add for all the space-dimensions: then multiply by η_2 and add again for all the space-dimensions. Thus

$$Ec_{13}=Pc_{11}+Qc_{12},\quad Ec_{23}=Pc_{12}+Qc_{22},$$

so that

$$P(c_{11}c_{22}-c_{12}^2)=E(c_{13}c_{22}-c_{12}c_{23}),$$
$$Q(c_{11}c_{22}-c_{12}^2)=E(c_{11}c_{23}-c_{12}c_{13}),$$

that is, by the preceding relations,

$$Pm_{12}^2=-Em_{12}m_{23},\quad Qm_{12}^2=Em_{12}m_{13};$$

and thus the foregoing typical equation is

$$m_{23}\eta_1-m_{13}\eta_2+m_{12}\eta_3=0.$$

Similarly for typical equations connecting each group of three from the four quantities; the complete set is

$$\left.\begin{array}{l} m_{23}\eta_1-m_{13}\eta_2+m_{12}\eta_3 \qquad\qquad =0 \\ m_{24}\eta_1-m_{14}\eta_2 \qquad\qquad +m_{12}\eta_4=0 \\ m_{34}\eta_1 \qquad\qquad -m_{14}\eta_3+m_{13}\eta_4=0 \\ \qquad\qquad m_{34}\eta_2-m_{24}\eta_3+m_{23}\eta_4=0 \end{array}\right\}$$

it being remembered that, under the adopted convention, we use positive quantities m_{ij} such that $i < j$. Also, these four equations are equivalent to only two linearly independent equations, because

$$m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23} = 0.$$

Further, in an equation such as the first of them, let the values $\eta_i = Yv_i + l_6\xi_i$ be substituted; then we have

$$Y(m_{23}v_1 - m_{13}v_2 + m_{12}v_3) + l_6(m_{23}\xi_1 - m_{13}\xi_2 + m_{12}\xi_3) = 0,$$

holding for each of the space-dimensions. Multiply by Y and add the results: then

$$m_{23}v_1 - m_{13}v_2 + m_{12}v_3 = 0,$$

and so

$$m_{23}\xi_1 - m_{13}\xi_2 + m_{12}\xi_3 = 0.$$

The complete sets of formal relations, thus derived from the foregoing four equations, are

$$\left.\begin{array}{llll} m_{23}v_1 - m_{13}v_2 + m_{12}v_3 & & & = 0 \\ m_{24}v_1 - m_{14}v_2 & & + m_{12}v_4 & = 0 \\ m_{34}v_1 & - m_{14}v_3 & + m_{13}v_4 & = 0 \\ \quad m_{34}v_2 - m_{24}v_3 & & + m_{23}v_4 & = 0 \end{array}\right\},$$

$$\left.\begin{array}{llll} m_{23}\xi_1 - m_{13}\xi_2 + m_{12}\xi_3 & & & = 0 \\ m_{24}\xi_1 - m_{14}\xi_2 & & + m_{12}\xi_4 & = 0 \\ m_{34}\xi_1 & - m_{14}\xi_3 & + m_{13}\xi_4 & = 0 \\ \quad m_{34}\xi_2 - m_{24}\xi_3 & & + m_{23}\xi_4 & = 0 \end{array}\right\},$$

each set containing only two linearly independent relations because

$$m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23} = 0.$$

Expression for the fifth curvature.

300. Next, the relation

$$\xi_1 p' + \xi_2 q' + \xi_3 r' + \xi_4 t' = 0$$

has been established; in it, let the values

$$\xi_i = c_{ii}^{\frac{1}{2}} \sin\theta_i$$

(for $i = 1, 2, 3, 4$) be substituted, so that we have

$$\begin{aligned} c_{11}^{\frac{1}{2}}p' \sin\theta_1 + c_{22}^{\frac{1}{2}}q' \{\sin\theta_1 \cos(\theta_2 - \theta_1) + \cos\theta_1 \sin(\theta_2 - \theta_1)\} & \\ + c_{33}^{\frac{1}{2}}r' \{\sin\theta_1 \cos(\theta_3 - \theta_1) + \cos\theta_1 \sin(\theta_3 - \theta_1)\} & \\ + c_{44}^{\frac{1}{2}}t' \{\sin\theta_1 \cos(\theta_4 - \theta_1) + \cos\theta_1 \sin(\theta_4 - \theta_1)\} &= 0. \end{aligned}$$

On the left-hand side, the coefficient of $\sin\theta_1$

$$=c_{11}^{\frac{1}{2}}p'+c_{22}^{\frac{1}{2}}q'\frac{c_{12}}{(c_{11}c_{22})^{\frac{1}{2}}}+c_{33}^{\frac{1}{2}}r'\frac{c_{13}}{(c_{11}c_{33})^{\frac{1}{2}}}+c_{44}^{\frac{1}{2}}t'\frac{c_{14}}{(c_{11}c_{44})^{\frac{1}{2}}}$$

$$=c_{11}^{-\frac{1}{2}}(c_{11}p'+c_{12}q'+c_{13}r'+c_{14}t');$$

but, from the definitions of c_{ij},

$$c_{11}p'+c_{12}q'+c_{13}r'+c_{14}t'=\sum\{\eta_1(\eta_1p'+\eta_2q'+\eta_3r'+\eta_4t')\}$$

$$=\sum\eta_1\frac{Y}{\rho}$$

$$=\frac{v_1}{\rho}=\frac{1}{\rho}c_{11}^{\frac{1}{2}}\cos\theta_1;$$

and thus the selected aggregate of terms involving $\sin\theta_1$

$$=\frac{1}{\rho}\cos\theta_1\sin\theta_1.$$

On the same left-hand side, the coefficient of $\cos\theta_1$

$$=c_{22}^{\frac{1}{2}}q'\frac{m_{12}}{(c_{11}c_{22})^{\frac{1}{2}}}+c_{33}^{\frac{1}{2}}r'\frac{m_{13}}{(c_{11}c_{33})^{\frac{1}{2}}}+c_{44}^{\frac{1}{2}}t'\frac{m_{14}}{(c_{11}c_{44})^{\frac{1}{2}}}$$

$$=c_{11}^{-\frac{1}{2}}(m_{12}q'+m_{13}r'+m_{14}t').$$

Thus the equation becomes

$$\frac{1}{\rho}\cos\theta_1\sin\theta_1+c_{11}^{-\frac{1}{2}}(m_{12}q'+m_{13}r'+m_{14}t')\cos\theta_1=0.$$

Now, in general, $\cos\theta_1$ does not vanish; hence, as

$$\xi_1=c_{11}^{\frac{1}{2}}\sin\theta_1,$$

the equation is

$$\frac{\xi_1}{\rho}=-m_{12}q'-m_{13}r'-m_{14}t'.$$

The value of ξ_2 is obtained similarly by making θ_2 (instead of θ_1) the dominating angle in the analysis: likewise for ξ_3 and ξ_4. The full set of values is

$$\left.\begin{aligned}
\frac{\xi_1}{\rho}&= \qquad\qquad -m_{12}q'-m_{13}r'-m_{14}t'\\
\frac{\xi_2}{\rho}&=m_{12}p' \qquad\qquad -m_{23}r'-m_{24}t'\\
\frac{\xi_3}{\rho}&=m_{13}p'+m_{23}q' \qquad\qquad -m_{34}t'\\
\frac{\xi_4}{\rho}&=m_{14}p'+m_{24}q'+m_{34}r'
\end{aligned}\right\}$$

We can verify at once that, in virtue of the relation

$$m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23} = 0,$$

these values satisfy the set of four equations, linear in $\xi_1, \xi_2, \xi_3, \xi_4$, obtained in § 299.

By means of these values of $\xi_1, \xi_2, \xi_3, \xi_4$, we can modify the expression of the fifth curvature of the domainal geodesic in the equation (§ 295)

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau\rho_5} = \begin{vmatrix} \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

Let the determinant be expanded in terms of second minors, so that it will consist of six terms each of which is the product of two second minors. In a term, such as

$$-\begin{vmatrix} u_3, & u_4 \\ w_3, & w_4 \end{vmatrix} \begin{vmatrix} \xi_1, & \xi_2 \\ v_1, & v_2 \end{vmatrix},$$

the minor in the second factor is

$$\begin{vmatrix} \xi_1, & \xi_2 \\ v_1, & v_2 \end{vmatrix} = -\rho\{(m_{12}q' + m_{13}r' + m_{14}t')v_2 + (m_{12}p' - m_{23}r' - m_{24}t')v_1\};$$

and we have established (§ 299) the equations

$$m_{23}v_1 - m_{13}v_2 + m_{12}v_3 = 0,$$

$$m_{24}v_1 - m_{14}v_2 + m_{12}v_4 = 0,$$

so that

$$\begin{vmatrix} \xi_1, & \xi_2 \\ v_1, & v_2 \end{vmatrix} = -\rho m_{12}(v_1p' + v_2q' + v_3r' + v_4t') = -m_{12},$$

because

$$v_1p' + v_2q' + v_3r' + v_4t' = \frac{1}{\rho}.$$

Hence the selected term becomes

$$m_{12}\begin{vmatrix} u_3, & u_4 \\ w_3, & w_4 \end{vmatrix}.$$

Similarly for the other five terms: the final result is

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau\rho_5} = m_{23}\begin{vmatrix} u_1, & u_4 \\ w_1, & w_4 \end{vmatrix} - m_{13}\begin{vmatrix} u_2, & u_4 \\ w_2, & w_4 \end{vmatrix} + m_{12}\begin{vmatrix} u_3, & u_4 \\ w_3, & w_4 \end{vmatrix}$$

$$+ m_{14}\begin{vmatrix} u_2, & u_3 \\ w_2, & w_3 \end{vmatrix} - m_{24}\begin{vmatrix} u_1, & u_3 \\ w_1, & w_3 \end{vmatrix} + m_{34}\begin{vmatrix} u_1, & u_2 \\ w_1, & w_2 \end{vmatrix}.$$

Ex. As illustrations of the analysis, it is convenient to adduce the values of certain concomitants of the system involving the magnitudes ξ_1, ξ_2, ξ_3, ξ_4.

We have had the relations (§§ 289, 291)

$$\xi_1 = \frac{1}{\rho_5}(A\epsilon + H\eta + G\iota + L\omega),$$

$$\frac{1}{\sigma\tau\kappa}(A\epsilon + H\eta + G\iota + L\omega) - Uu_1 + Vv_1 - Ww_1 = -z_1,$$

where

$$U = \frac{1}{\rho\tau^2} + \frac{1}{\sigma\tau}\frac{d}{ds}\left\{\tau\frac{d}{ds}\left(\frac{\sigma}{\rho}\right)\right\}, \quad V = \frac{\sigma''}{\sigma} + \frac{\sigma'\tau'}{\sigma\tau} + \frac{1}{\tau^2}, \quad -W = \frac{2\sigma'}{\tau} + \frac{\tau'}{\tau};$$

and therefore

$$\frac{\rho_5}{\sigma\tau\kappa}\xi_1 = Uu_1 - Vv_1 + Ww_1 - z_1,$$

or, generally for $i = 1, 2, 3, 4$,

$$\frac{\rho_5}{\sigma\tau\kappa}\xi_i = Uu_i - Vv_i + Ww_i - z_i.$$

As

$$\sum au_1^2 = \Omega, \quad \sum au_1v_1 = \Omega\frac{1}{\rho}, \quad \sum au_1w_1 = \Omega\frac{d}{ds}\left(\frac{1}{\rho}\right), \quad \sum au_1z_1 = \Omega\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right),$$

we have

$$\begin{aligned}\frac{\rho_5}{\Omega\sigma\tau\kappa}\sum au_1\xi_1 &= U - \frac{1}{\rho}V + W\frac{d}{ds}\left(\frac{1}{\rho}\right) - \frac{d^2}{ds^2}\left(\frac{1}{\rho}\right)\\ &= 0,\end{aligned}$$

when the values of U, V, W are substituted. The result can be obtained more immediately; for

$$au_1 + hu_2 + gu_3 + lu_4 = \Omega p',$$

and so for the other combinations: hence

$$\sum au_1\xi_1 = \Omega\sum \xi_1 p' = 0,$$

by the condition attaching to the quantities ξ_1, ξ_2, ξ_3, ξ_4.

Next, we have had the results (§§ 288, 290)

$$\begin{aligned}\sum av_1^2 &= \Omega\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right),\\ \sum av_1w_1 &= -\Omega\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right),\\ \sum av_1z_1 &= \Omega\left\{\frac{1}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) + \frac{1}{\sigma}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma^2\tau}\right\};\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\rho_5}{\Omega\sigma\tau\kappa}\sum av_1\xi_1 = \frac{1}{\rho}U - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)V - \left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right)W\\ - \left\{\frac{1}{\rho}\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) + \frac{1}{\sigma}\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma^2\tau}\right\}\\ = 0,\end{aligned}$$

on substitution and reduction : that is,

$$\sum a v_1 \xi_1 = 0.$$

Proceeding in the same way, and using results already established, we find

$$\sum a w_1 \xi_1 = 0,$$

$$\frac{1}{\Omega} \sum a z_1 \xi_1 = -\frac{1}{\sigma \tau \kappa \rho_5}.$$

From these forms, in combination with the forms of the similar concomitants obtained earlier, many concomitants can be constructed and their geometrical values can be deduced.

Locus of centres of circular curvature of concurrent geodesics.

301. One of the configurations, non-gremial to the domain, is the locus of the centres of circular curvature of concurrent geodesics. As each centre lies on the prime normal of its geodesic, this line being orthogonal to the tangent block, the locus in question must lie in the homaloid orthogonal to the tangent block : and one form of the equations of the orthogonal homaloid is

$$\sum (\bar{y} - y) y_1 = 0, \quad \sum (\bar{y} - y) y_2 = 0, \quad \sum (\bar{y} - y) y_3 = 0, \quad \sum (\bar{y} - y) y_4 = 0.$$

But the character of the locus is affected by the dimensional extent of the homaloidal plenary space of the domain.

When the domain is primary, so that the plenary space is quintuple, the orthogonal homaloid is simply the unique line which is normal to the domain at a point and is the direction of the prime normals of all the domainal geodesics through the point. Definite parts of this normal constitute the range of the centres of circular curvature ; but it is not a locus in the customary sense, as a single point of the range can be the centre of circular curvature for an unlimited number of domainal geodesics.

When a domain is not primary, the direction of a prime normal of a domainal geodesic depends partly upon the direction of the geodesic. Let the direction-variables of the geodesic be p', q', r', t', as usual ; and let y_c be the typical space-variable of the centre of circular curvature of the geodesic, so that we have

$$y_c - y = Y \rho,$$

and therefore

$$\frac{1}{\rho^2}(y_c - y) = \frac{Y}{\rho} = \sum \eta_{11} p'^2,$$

there being as many such equations as there are dimensions in the plenary space. The character of the locus of the centre of circular curvature depends upon the number of these dimensions. The locus always lies in the orthogonal homaloid of the region. All the directions, typified by the quantities η_{ij}, lie in the homaloid, because the equations

$$\sum y_k \eta_{ij} = 0$$

are satisfied for all values of i, j, k; and there are ten such quantities η_{ij}. When the plenary space is sextuple, the orthogonal homaloid is a plane; thus only two of the ten directions, typified by η_{ij}, can be taken as independent, and the required locus is a curve in this orthogonal plane. When the plenary space is septuple, the orthogonal homaloid of the domain is a flat: then only three of the ten directions, typified by η_{ij}, can be taken as independent, and the required locus lies in this orthogonal flat. When the plenary space is octuple, the orthogonal homaloid is a block; four of the directions, typified by η_{ij}, can be taken as independent, and the required locus lies in this orthogonal block. And so on: the three specified cases will be considered in turn.

Moreover, it will appear that there is not a proper locus of the centre of circular curvature when the plenary space is sextuple or septuple. When the plenary space is sextuple, a proper locus is provided only for a set of geodesics originating in any superficial orientation in the domain. When the plenary space is septuple, a proper locus is provided only for a set of geodesics originating either in any superficial orientation or in any regional orientation in the domain.

The centre-locus when the plenary space is sextuple.

302. When the plenary homaloidal space of the domain is sextuple, the orthogonal homaloid of the domain is a plane. As a plane can contain only two organically independent directions, we take the directions, typified by η_{11} and η_{12} respectively, to be axes of reference in the plane. Let $\bar{x}$, $\bar{z}$, be coordinates of any centre of circular curvature, referred to those axes; then, writing

$$\mathsf{a}=\sum \eta_{11}{}^2, \quad \mathsf{b}=\sum \eta_{12}{}^2, \quad \mathsf{h}=\sum \eta_{11}\eta_{12}=(\mathsf{ab})^{\frac{1}{2}} \cos \omega,$$

where ω denotes the inclination of the axes, we have

$$\mathsf{a}^{\frac{1}{2}}(\bar{x}+\bar{z}\cos\omega)=\sum \eta_{11}(y_c-y),$$

$$\mathsf{b}^{\frac{1}{2}}(\bar{x}\cos\omega+\bar{z})=\sum \eta_{12}(y_c-y),$$

with the relation

$$\bar{x}^2+2\bar{x}\bar{z}\cos\omega+\bar{z}^2=\rho^2.$$

Let

$$\bar{X}=\frac{\mathsf{a}^{\frac{1}{2}}}{\rho^2}(\bar{x}+\bar{z}\cos\omega), \quad \bar{Z}=\frac{\mathsf{b}^{\frac{1}{2}}}{\rho^2}(\bar{x}\cos\omega+\bar{z});$$

then

$$\bar{X}=\sum \eta_{11}\frac{y_c-y}{\rho^2}=\sum \eta_{11}\frac{Y}{\rho}=\sum \{\eta_{11}(\sum \eta_{11}p'^2)\},$$

$$\bar{Z}=\sum \eta_{12}\frac{y_c-y}{\rho^2}=\sum \eta_{12}\frac{Y}{\rho}=\sum \{\eta_{12}(\sum \eta_{11}p'^2)\},$$

where the inner summation on the right-hand side is over the four parameters and the outer summation is over the six space-variables. Also there is the permanent arc-relation. Thus there are three equations involving the four quantities p', q', r', t'. It follows that, if no distinction is made between real and complex

quantities, every point in the plane is a centre for an infinitude of concurrent geodesics whose tangents at O constitute a conical surface: in real geometry, there might be ranges in the plane to the points of which no real geodesics correspond. But, in the absence of relations reducing the number of independent magnitudes among p', q', r', t', there is no proper locus. If only one condition is imposed, still there is no locus: every point of the plane is potentially a centre for a finite number of geodesics concurrent in O.

In order that elimination of the direction-variables may become possible, there must be only a couple of independent quantities in the elimination; accordingly, two conditions must be imposed upon the magnitudes p', q', r', t'. The simplest instance occurs when the directions, of the geodesics considered, originate in a superficial orientation which itself may be arbitrary. Such an orientation is provided by the assignment of two different arbitrary directions, represented by direction-variables $\mathbf{p}'$, $\mathbf{q}'$, $\mathbf{r}'$, $\mathbf{t}'$, and P', Q', R', T'; and then any direction p', q', r', t', lying in this orientation, has its variables represented by four equations of the form

$$p' = \lambda \mathbf{p}' + \mu \mathbf{P}',$$

where λ and μ are parameters in the four equations such that

$$\lambda^2 + 2\lambda\mu \cos\epsilon + \mu^2 = 1,$$

ϵ denoting the angle between the assumed directions. We now have

$$\frac{Y}{\rho} = \lambda^2 \sum \eta_{11} \mathbf{p}'^2 + 2\lambda\mu \sum \eta_{11} \mathbf{p}' \mathbf{P}' + \mu^2 \sum \eta_{11} \mathbf{P}'^2;$$

and the equations, for the determination of the locus, become

$$\overline{X} = \sum \left(\eta_{11} \frac{Y}{\rho} \right) = \lambda^2 E_0 + 2\lambda\mu F_0 + \mu^2 G_0,$$

$$\overline{Z} = \sum \left(\eta_{12} \frac{Y}{\rho} \right) = \lambda^2 e_0 + 2\lambda\mu f_0 + \mu^2 g_0,$$

where E_0, F_0, G_0, e_0, f_0, g_0, are quantities belonging solely to the quantities determining the assumed orientation, and are unaffected by the parameters λ and μ determining the direction-variables of any geodesic.

The elimination is simple. Let

$$U = \begin{vmatrix} F_0 - \overline{X}\cos\epsilon, & G_0 - \overline{X} \\ f_0 - \overline{Z}\cos\epsilon, & g_0 - \overline{Z} \end{vmatrix}, \quad W = \begin{vmatrix} E_0 - \overline{X}, & F_0 - \overline{X}\cos\epsilon \\ e_0 - \overline{Z}, & f_0 - \overline{Z}\cos\epsilon \end{vmatrix},$$

$$V = \begin{vmatrix} E_0 - \overline{X}, & G_0 - \overline{X} \\ e_0 - \overline{Z}, & g_0 - \overline{Z} \end{vmatrix},$$

where U, V, W, manifestly are linear non-homogeneous functions of $\overline{X}$ and $\overline{Z}$; then the required eliminant is

$$V^2 - 4UW = 0,$$

an equation of the form

$$k_0 \overline{X}^2 + 2k_1 \overline{X}\,\overline{Y} + k_2 \overline{Y}^2 + 2k_3 \overline{X} + 2k_4 \overline{Y} + k_5 = 0.$$

But

$$\overline{X}=\mathrm{a}^{\frac{1}{2}}\frac{\bar{x}+\bar{z}\cos\omega}{\bar{x}^2+2\bar{x}\bar{z}\cos\omega+\bar{z}^2},\qquad \overline{Z}=\mathrm{b}^{\frac{1}{2}}\frac{\bar{x}\cos\omega+\bar{z}}{\bar{x}^2+2\bar{x}\bar{z}\cos\omega+\bar{z}^2};$$

and therefore the equation represents a lemniscate curve in the orthogonal plane of the domain, the plenary space being sextuple.

Ex. Investigate the possibility of a conic-locus arising (i), as the intersection of two orthogonal planes at consecutive points of the domainal geodesics, originating in an assigned orientation, the plenary space being sextuple: and (ii), as the inverse pedal of the foregoing lemniscate curve.

The centre-locus when the plenary space is septuple.

303. When the plenary homaloidal space of the domain is septuple, so that any centre-locus must lie in a flat (being the orthogonal homaloid of the domain), there are three organically independent directions in that flat. We assume three directions, typified by η_{11}, η_{12}, η_{13}, as constituting three directions of reference in that flat; and we denote by $\bar{u}$, $\bar{v}$, $\bar{w}$, the coordinates of any centre of circular curvature of a domainal geodesic, referred to these axes; so that, if we take

$$\begin{aligned}\mathrm{a}&=\sum\eta_{11}^2, & \mathrm{f}&=\sum\eta_{12}\eta_{13}=(\mathrm{bc})^{\frac{1}{2}}\cos l\ ,\\ \mathrm{b}&=\sum\eta_{12}^2, & \mathrm{g}&=\sum\eta_{11}\eta_{13}=(\mathrm{ca})^{\frac{1}{2}}\cos m,\\ \mathrm{c}&=\sum\eta_{13}^2, & \mathrm{h}&=\sum\eta_{11}\eta_{12}=(\mathrm{ab})^{\frac{1}{2}}\cos n\ ,\end{aligned}$$

l, m, n, being the angles between the axes in pairs, we have

$$\begin{aligned}\overline{U}\rho^2&=\mathrm{a}^{\frac{1}{2}}(\bar{u}+\bar{v}\cos n+\bar{w}\cos m)=\sum\eta_{11}(y_c-y),\\ \overline{V}\rho^2&=\mathrm{b}^{\frac{1}{2}}(\bar{u}\cos n+\bar{v}+\bar{w}\cos l)\ =\sum\eta_{12}(y_c-y),\\ \overline{W}\rho^2&=c^{\frac{1}{2}}(\bar{u}\cos m+\bar{v}\cos l+\bar{w})\ =\sum\eta_{13}(y_c-y),\end{aligned}$$

while

$$\rho^2=\bar{u}^2+\bar{v}^2+\bar{w}^2+2\bar{v}\bar{w}\cos l+2\bar{w}\bar{u}\cos m+2\bar{u}\bar{v}\cos n.$$

We thus have

$$\begin{aligned}\overline{U}&=\frac{1}{\rho^2}\sum\eta_{11}(y_c-y)=\sum\eta_{11}\frac{Y}{\rho}=\sum\{\eta_{11}(\sum\eta_{11}p'^2)\},\\ \overline{V}&=\frac{1}{\rho^2}\sum\eta_{12}(y_c-y)=\sum\eta_{12}\frac{Y}{\rho}=\sum\{\eta_{12}(\sum\eta_{11}p'^2)\},\\ \overline{W}&=\frac{1}{\rho^2}\sum\eta_{13}(y_c-y)=\sum\eta_{13}\frac{Y}{\rho}=\sum\{\eta_{13}(\sum\eta_{11}p'^2)\},\end{aligned}$$

where the inner summation on the right-hand side is taken over the quadratic combinations of the direction-variables p', q', r', t', and the outer summation is

taken over the seven dimensions of the plenary homaloidal space. Also there is the permanent arc-relation

$$\sum Ap'^2 = 1.$$

Thus there are four equations, in all, involving four quantities p', q', r', t', not homogeneously. Unless at least one other relation should subsist among the four quantities, elimination of all four is impossible: in that event, there is no proper locus of the centre of circular curvature of all the domainal geodesics through a point of the domain, because every point of the flat is potentially a centre.

But elimination becomes possible if either two relations, or only one relation, be imposed upon p', q', r', t'. When two such relations are imposed, the eliminant consists of two equations involving $\bar{u}$, $\bar{v}$, $\bar{w}$, and positional magnitudes of the domain; hence the locus consists of a skew curve (the intersection of two surfaces) in the orthogonal flat of the domain. When only a single relation is imposed, the eliminant consists of a single equation involving $\bar{u}$, $\bar{v}$, $\bar{w}$, and positional magnitudes of the domain; the locus then is a surface in the orthogonal flat.

(i) When two restricting relations exist, consider the instance which arises when the aggregate of retained geodesics have directions originating in a superficial orientation in the domain. Such an orientation in arbitrary choice can be regarded as determinate by the assignment of two arbitrary directions in the domain. When the direction-variables of these two directions are p', q', r', t', and P', Q', R', T', respectively, then any direction in the superficial orientation thus determined can be represented by direction-variables

$$p' = \lambda \mathrm{p}' + \mu P',$$

with like values for q', r', t'. When these values are inserted, the foregoing equations for the centre of circular curvature can be expressed in the forms

$$\begin{aligned}
\bar{U} &= \lambda^2 A_1 + 2\lambda\mu H_1 + \mu^2 B_1, \\
\bar{V} &= \lambda^2 A_{12} + 2\lambda\mu H_{12} + \mu^2 B_{12}, \\
\bar{W} &= \lambda^2 A_2 + 2\lambda\mu H_2 + \mu^2 B_2,
\end{aligned}$$

while the form of the permanent arc-relation now is

$$1 = \lambda^2 A_0 + 2\lambda\mu H_0 + \mu^2 B_0,$$

with evident significance for the various symbols A, H, B.

To obtain the locus, it now suffices to eliminate the two parametric quantities λ and μ between the four equations. One eliminant equation is obtainable after resolving the first three of the equations for λ^2, $\lambda\mu$, μ^2; it has the form

$$4 \begin{vmatrix} \bar{U}, & \bar{V}, & \bar{W} \\ H_1, & H_{12}, & H_2 \\ B_1, & B_{12}, & B_2 \end{vmatrix} \begin{vmatrix} \bar{U}, & \bar{V}, & \bar{W} \\ A_1, & A_{12}, & A_2 \\ H_1, & H_{12}, & H_2 \end{vmatrix} = \begin{vmatrix} \bar{U}, & \bar{V}, & \bar{W} \\ A_1, & A_{12}, & A_2 \\ B_1, & B_{12}, & B_2 \end{vmatrix}^2.$$

Another eliminant equation manifestly is

$$\begin{vmatrix} \bar{U}, & A_1, & H_1, & B_1 \\ \bar{V}, & A_{12}, & H_{12}, & B_{12} \\ \bar{W}, & A_2, & H_2, & B_2 \\ 1, & A_0, & H_0, & B_0 \end{vmatrix} = 0.$$

Owing to the homogeneity of the first eliminant equation in the quantities $\bar{U}$, $\bar{V}$, $\bar{W}$, with a common denominator factor ρ^2, it becomes a homogeneous equation of the second degree in the coordinates $\bar{u}$, $\bar{v}$, $\bar{w}$: that is, it represents a quadric cone with its vertex at the originating point O of the domain. The second equation represents a sphere through O. Both the cone and the sphere lie in the orthogonal flat of the domain. Hence the locus is a skew quartic curve lying in the flat; and it is the locus of the centres of circular curvature of domainal geodesics the directions of which originate in a superficial orientation lying in the domain.

(ii) As a simple instance in which a single restricting condition is imposed upon the range of variation of the variables p', q', r', t', consider the effect of the requirement that every domainal geodesic through O shall touch a parametric region $\theta(p, q, r, t)=0$, so that

$$\theta_1 p' + \theta_2 q' + \theta_3 r' + \theta_4 t' = 0.$$

Effectively, this can be stated as a condition that the directions of the domainal geodesics shall originate in a regional orientation with orientation-variables θ_1, θ_2, θ_3, θ_4.

The variables p', q', r', t', have now to be eliminated between this condition, the three equations expressing $\bar{U}$, $\bar{V}$, $\bar{W}$, and the permanent arc-equation. We can use the earlier analysis of § 257, after a preliminary modification of removing the quantity t' by means of the orientation-condition. When we substitute

$$t' = -\frac{1}{\theta_4}(\theta_1 p' + \theta_2 q' + \theta_3 r'),$$

we have

$$\bar{U} = Q_1, \quad \bar{V} = Q_2, \quad \bar{W} = Q_3,$$

while the permanent arc-relation becomes

$$Q_0 = 1,$$

where Q_1, Q_2, Q_3, Q_0, are homogeneous quadratic forms in p', q', r'; and therefore we have to eliminate p', q', r', between the three equations

$$(A_1 - A_0\bar{U},\ B_1 - B_0\bar{U},\ C_1 - C_0\bar{U},\ F_1 - F_0\bar{U},\ G_1 - G_0\bar{U},\ H_1 - H_0\bar{U} \between p', q', r')^2 = 0,$$

$$(A_2 - A_0\bar{V},\ B_2 - B_0\bar{V},\ C_2 - C_0\bar{V},\ F_2 - F_0\bar{V},\ G_2 - G_0\bar{V},\ H_2 - H_0\bar{V} \between p', q', r')^2 = 0,$$

$$(A_3 - A_0\bar{W},\ B_3 - B_0\bar{W},\ C_3 - C_0\bar{W},\ F_3 - F_0\bar{W},\ G_3 - G_0\bar{W},\ H_3 - H_0\bar{W} \between p', q', r')^2 = 0,$$

which are homogeneous and quadratic in p', q', r'.

The detailed process of elimination is the same as that used (*l.c.*) for the determination of the locus of the centre of circular curvature of regional geodesics when the plenary space of the region is sextuple. The eliminant required is of the type

$$E_0+\frac{E_1}{\rho^2}+\frac{E_2}{\rho^4}+\frac{E_3}{\rho^6}+\frac{E_4}{\rho^8}=0,$$

where

$$\rho^2=\bar{u}^2+\bar{v}^2+\bar{w}^2+2\bar{v}\bar{w}\cos l+2\bar{w}\bar{u}\cos m+2\bar{u}\bar{v}\cos n,$$

and for $i=0, 1, 2, 3, 4$, the symbol E_i denotes a function of $\bar{u}$, $\bar{v}$, $\bar{w}$, homogeneous and of degree i in the coordinates $\bar{u}$, $\bar{v}$, $\bar{w}$.

Hence the locus is a surface, lying in the orthogonal flat of the domain; the surface is of degree eight, and it has a conical point (real or imaginary) of the fourth order at O. It is the locus of the centres of circular curvature of domainal geodesics through O, the directions of which either are tangential to a region containing the point O or originate in the regional orientation at that initial point.

The centre-locus when the plenary space is octuple.

304. Similarly, when the plenary homaloidal space of the domain is octuple, there is a locus of centres of circular curvatures of domainal geodesics through a point O. The locus arises through the complete aggregate of these geodesics in all the possible directions through O; it lies in the block, which is orthogonal to the domain; and, being represented by a single equation, it represents a region in that orthogonal block, that is, a primary region.

Four directions, organically independent of one another, appertain to the orientation of the block. We assume that four such directions can be taken along lines typified by four of the ten magnitudes η_{ij}; and we assume them to belong to the magnitudes η_{11}, η_{22}, η_{33}, η_{44}. We write

$$\bar{U}\rho^2=(\sum\eta_{11}{}^2)^{\frac{1}{2}}(\bar{u}+\bar{v}\cos\widehat{12}+\bar{w}\cos\widehat{13}+\bar{z}\cos\widehat{14})=\sum\eta_{11}(y_c-y),$$

$$\bar{V}\rho^2=(\sum\eta_{22}{}^2)^{\frac{1}{2}}(\bar{u}\cos\widehat{12}+\bar{v}+\bar{w}\cos\widehat{23}+\bar{z}\cos\widehat{24})=\sum\eta_{22}(y_c-y),$$

$$\bar{W}\rho^2=(\sum\eta_{33}{}^2)^{\frac{1}{2}}(\bar{u}\cos\widehat{13}+\bar{v}\cos\widehat{23}+\bar{w}+\bar{z}\cos\widehat{34})=\sum\eta_{33}(y_c-y),$$

$$\bar{Z}\rho^2=(\sum\eta_{44}{}^2)^{\frac{1}{2}}(\bar{u}\cos\widehat{14}+\bar{v}\cos\widehat{24}+\bar{w}\cos\widehat{34}+\bar{z})=\sum\eta_{44}(y_c-y),$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{z}$, denote the coordinates of a centre of circular curvature referred to the chosen axes in the orthogonal block, where $\cos\widehat{ij}$ denotes the inclination of the directions typified by η_{ii} and η_{jj}, and where

$$\begin{aligned}\rho^2=\bar{u}^2+\bar{v}^2+\bar{w}^2+\bar{z}^2&\\ +2\bar{v}\bar{w}\cos\widehat{23}+2\bar{w}\bar{u}\cos\widehat{13}+2\bar{u}\bar{v}\cos\widehat{12}&\\ +2\bar{u}\bar{z}\cos\widehat{14}+2\bar{v}\bar{z}\cos\widehat{24}+2\bar{w}\bar{z}\cos\widehat{34}&.\end{aligned}$$

Moreover, the permanent arc-relation is

$$\sum Ap'^2=1.$$

To obtain the equation representing the centre-locus, we have to eliminate p', q', r', t', among these five equations. It will consist of a single relation between $\bar{u}, \bar{v}, \bar{w}, \bar{z}$, the coordinates relative to the axes in the block orthogonal to the domain, the other quantities occurring in the relation being solely positional magnitudes of the domain. Hence the centre-locus is a region in that block; and it is the locus of the centres of circular curvature of all the domainal geodesics passing through the initial point O.

The eliminant obviously can be taken as the eliminant of the equations

$$Q_1 = \sum_i \sum_j \{(\sum \eta_{11}\eta_{ij}) - \bar{U}A_{ij}\} x_i' x_j' = 0,$$

$$Q_2 = \sum_i \sum_j \{(\sum \eta_{22}\eta_{ij}) - \bar{V}A_{ij}\} x_i' x_j' = 0,$$

$$Q_3 = \sum_i \sum_j \{(\sum \eta_{33}\eta_{ij}) - \bar{W}A_{ij}\} x_i' x_j' = 0,$$

$$Q_4 = \sum_i \sum_j \{(\sum \eta_{44}\eta_{ij}) - \bar{Z}A_{ij}\} x_i' x_j' = 0,$$

which are homogeneous in the four magnitudes p', q', r', t', with $i, j = 1, 2, 3, 4$. There appears to be no reasonably simple process for the formation of the eliminant in specific explicit terms. The general theory of elimination indicates that the ultimate equation is of degree sixteen in the coordinates $\bar{u}, \bar{v}, \bar{w}, \bar{z}$, and that the region thus represented has a conical point of degree eight at the initial point O of the region.

As a corollary, it can be inferred that, when the plenary space of the domain has more than eight dimensions, the locus of the centres of circular curvature of the concurrent domainal geodesics still is a primary region in a block determined by leading lines, the directions of which are typified by $\eta_{11}, \eta_{22}, \eta_{33}, \eta_{44}$; its degree, and its nature at O, are the same as when the plenary space of the domain is octuple.

Ex. Determine, for any plenary space of more than seven dimensions, the centre-locus of concurrent domainal geodesics which originate either in a superficial orientation or in a regional orientation.

CHAPTER XXVI

GEODESIC TRIANGLES: SPHERICITY OF A DOMAIN IN A SUPERFICIAL ORIENTATION.

Sphericity of a domain in a superficial orientation.

305. The expression for the Riemann measure of the superficial curvature in any orientation is obtained by means of geodesic surfaces in the configuration. We therefore, as in § 270, take two directions represented by the variables x_1', x_2', x_3', x_4', and z_1', z_2', z_3', z_4', in the domain; if p', q', r', t', be the variables for any other direction which lies in the orientation defined by the two directions, we have *

$$\begin{vmatrix} p', & q', & r' \\ x_1', & x_2', & x_3' \\ z_1', & z_2', & z_3' \end{vmatrix} = 0, \qquad \begin{vmatrix} p', & q', & t' \\ x_1', & x_2', & x_4' \\ z_1', & z_2', & z_4' \end{vmatrix} = 0,$$

that is,

$$r's_{12} = -(p's_{23} + q's_{31}), \quad t's_{12} = -(p's_{24} + q's_{41}).$$

Consequently, for the domainal arcs on the geodesic surface at O in the domain, the permanent arc-relation $\sum Ap'^2 = 1$, on the elimination of r' and t', becomes

$$Ep'^2 + 2Fp'q' + Gq'^2 = 1,$$

where

$$E = A_{11} - \frac{2}{s_{12}}(A_{13}s_{23} + A_{14}s_{24}) + \frac{1}{s_{12}^2}(A_{33}s_{23}^2 + 2A_{34}s_{23}s_{24} + A_{44}s_{24}^2),$$

$$F = A_{12} - \frac{1}{s_{12}}(A_{13}s_{31} + A_{23}s_{23} + A_{14}s_{41} + A_{24}s_{24})$$
$$\qquad + \frac{1}{s_{12}^2}\{A_{33}s_{23}s_{31} + A_{34}(s_{23}s_{41} + s_{31}s_{24}) + A_{44}s_{24}s_{41}\},$$

$$G = A_{22} - \frac{2}{s_{12}}(A_{23}s_{31} + A_{24}s_{41}) + \frac{1}{s_{12}^2}(A_{33}s_{31}^2 + 2A_{34}s_{31}s_{41} + A_{44}s_{41}^2).$$

When these values are substituted in the magnitude $EG - F^2$, and the expression is reduced by means of the relation

$$s_{12}s_{34} + s_{13}s_{42} + s_{14}s_{23} = 0,$$

we find

$$s_{12}^2(EG - F^2) = \sum (A_{ik}A_{jl} - A_{il}A_{jk})s_{ij}s_{kl},$$

summed for values of i, j, k, l, from the range of these values that are admissible.

Again, the geodesics at O on the geodesic surface are actually the domainal

* A different process, applicable to a general amplitude, has been given in § 117.

geodesics in the respective possible directions; so that the magnitude of the circular curvature and the direction-cosines of the prime normal for a superficial geodesic are the same, respectively, as for the domainal geodesic in the assigned direction. For the latter, we have

$$\frac{Y}{\rho} = \sum \eta_{11} p'^2 ;$$

and therefore, when we take account of the values of r' and t' which require the domainal geodesic to lie in the surface, we have

$$\frac{Y}{\rho} = \bar{\eta}_{11} p'^2 + 2\bar{\eta}_{12} p'q' + \bar{\eta}_{22} q'^2,$$

the right-hand side being derived from $\sum \eta_{11} p'^2$ by the substitutions

$$r's_{12} = -(p's_{23} + q's_{31}), \quad t's_{12} = -(p's_{24} + q's_{41}).$$

The analysis for the construction of E, F, G, manifestly leads to the values of $\bar{\eta}_{11}$, $\bar{\eta}_{12}$, $\bar{\eta}_{22}$, by the substitution of $\eta_{\alpha\beta}$ for $A_{\alpha\beta}$, for all the values of α, β, $=1, 2, 3, 4$; and therefore

$$s_{12}^2 (\bar{\eta}_{11}\bar{\eta}_{22} - \bar{\eta}_{12}^2) = \sum (\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}) s_{ij} s_{kl}.$$

When this result is summed for all the dimensions of the plenary space, we have

$$\begin{aligned} s_{12}^2 \sum (\bar{\eta}_{11}\bar{\eta}_{22} - \bar{\eta}_{12}^2) &= \sum [\{\sum (\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk})\} s_{ij} s_{kl}] \\ &= \sum (ij, kl) s_{ij} s_{kl}, \end{aligned}$$

on introducing the four-index symbols.

But for the surface, the Riemann measure of curvature, being its sphericity K, is given by (§ 113)

$$K(EG - F^2) = \sum (\bar{\eta}_{11}\bar{\eta}_{22} - \bar{\eta}_{12}^2);$$

and this measure for the geodesic surface is defined as the measure for the domain in the orientation of the surface. Accordingly, the sphericity of the domain, in the orientation defined by the variables s_{12}, s_{23}, s_{31}, s_{14}, s_{24}, s_{34}, is equal to

$$\frac{\sum (ij, kl) s_{ij} s_{kl}}{\sum (A_{ik}A_{jl} - A_{il}A_{jk}) s_{ij} s_{kl}}.$$

The quantities of the type $\sum_i \sum_j \frac{d\Gamma_{ij}}{ds_k} p_i' p_j'$.

306. In the discussion, alike of small geodesic triangles in a domain and of parallel geodesics, certain combinations of quantities such as

$$\sum_i \sum_j (\xi_i'\zeta_j' + \xi_j'\zeta_i') \frac{d\Gamma_{ij}}{ds_k}$$

continually recur, the variables ξ' and ζ' being similar to the variables x'.

In § 23, we obtained the parametric derivatives of the Christoffel symbols in the form

$$\frac{\partial}{\partial x_a}\{\beta\gamma, \mu\} = \{a\beta\gamma, \mu\} + \sum_{\omega}[\{a\beta, \omega\}\{\gamma\omega, \mu\} + \{a\gamma, \omega\}\{\beta\omega, \mu\}]$$
$$+\frac{1}{3\Omega}\sum_{\delta} a_{\mu\delta}[(\beta\delta, \gamma a) + (\gamma\delta, \beta a)],$$

valid for all values of a, β, γ, μ. Denoting the geodesic direction-variables by x_θ', we therefore have

$$\frac{d}{ds}\{\beta\gamma, \mu\} = \sum_{\theta}[\{\beta\gamma\theta, \mu\}x_\theta']$$
$$+\sum_{\omega}\sum_{\theta} x_\theta'[\{\beta\theta, \omega\}\{\gamma\omega, \mu\} + \{\gamma\theta, \omega\}\{\beta\omega, \mu\}]$$
$$+\frac{1}{3\Omega}\sum_{\delta}\sum_{\theta} a_{\mu\delta}x_\theta'[(\beta\delta, \gamma\theta) + \gamma\delta, \beta\theta)].$$

The result holds for a general amplitude. When it is used for a domain, we denote by p_k', q_k', r_k', t_k', the direction-variables of the domainal geodesic along which the arc-differentiation ds_k is effected: further, by p_i', q_i', r_i', t_i', and p_j', q_j', r_j', t_j', the direction-variables of any two directions, which may be the same as one another or be distinct. Also, we denote the quantity

$$\left(\frac{d\Gamma_{11}}{ds_k}, \ldots, \frac{d\Gamma_{44}}{ds_k} \between p_i', q_i', r_i', t_i' \between p_j', q_j', r_j', t_j'\right)$$

by

$$\left(\frac{d\Gamma}{ds_k} \between i \between j\right),$$

with a corresponding notation for the like quantities involving derivatives of Δ, Θ, Φ. Then

$$\left(\frac{d\Gamma}{ds_k} \between i \between j\right) = (\Gamma_{300}p_i'p_j'p_k')$$
$$+(\sum\Gamma_{11}p_i')(\sum\sum\Gamma_{11}p_j'p_k') + (\sum\Gamma_{11}p_j')(\sum\sum\Gamma_{11}p_i'p_k')$$
$$+(\sum\Gamma_{12}p_i')(\sum\sum\Delta_{11}p_j'p_k') + (\sum\Gamma_{12}p_j')(\sum\sum\Delta_{11}p_i'p_k')$$
$$+(\sum\Gamma_{13}p_i')(\sum\sum\Theta_{11}p_j'p_k') + (\sum\Gamma_{13}p_j')(\sum\sum\Theta_{11}p_i'p_k')$$
$$+(\sum\Gamma_{14}p_i')(\sum\sum\Phi_{11}p_j'p_k') + (\sum\Gamma_{14}p_j')(\sum\sum\Phi_{11}p_i'p_k')$$
$$+\frac{1}{3\Omega}\sum_{\delta}\sum_{\theta} a_{1\delta}x_\theta'\sum_{\lambda}\sum_{\mu}[(\lambda\delta, \mu\theta) + (\mu\delta, \lambda\theta)]z_\lambda'u_\mu',$$

where the variables x' are p_k', q_k', r_k', t_k', the variables z' are the variables p_i', q_i', r_i', t_i', and the variables u' are the variables p_j', q_j', r_j', t_j', while the symbols $(\sum\Gamma_{1\phi}p_m')$ bear the significance

$$(\sum\Gamma_{1\phi}p_m') = \Gamma_{1\phi}p_m' + \Gamma_{2\phi}q_m' + \Gamma_{3\phi}r_m' + \Gamma_{4\phi}t_m'$$

for all values of ϕ and m, and

$$(\Gamma_{300}p_i'p_j'p_k')=\sum_\theta\sum_\lambda\sum_\mu\{\lambda\mu\theta,1\}x_\theta'z_\lambda'u_\mu'.$$

In the expression for

$$\left(\frac{d\Delta}{ds_k}\bigg)\,i\,\bigg)\,j\right),$$

the first term is $(\Delta_{300}p_i'p_j'p_k')$, the coefficients in the next four lines involving single summations are $\sum\Delta_{1\phi}p_m'$, and $a_{2\delta}$ takes the place of $a_{1\delta}$ in the sixth line. Similarly we obtain the expressions for

$$\left(\frac{d\Theta}{ds_k}\bigg)\,i\,\bigg)\,j\right),\quad \left(\frac{d\Phi}{ds_k}\bigg)\,i\,\bigg)\,j\right).$$

It will be found convenient to have the whole quantity in the sixth line, involving quadruple summation, set out in somewhat longer form. For this purpose, we write

$$s_{\alpha\beta}=u_\alpha'x_\beta'-x_\alpha'u_\beta',\quad t_{\alpha\beta}=u_\alpha'z_\beta'-z_\alpha'u_\beta',$$

for all values of α and β, where

$$\begin{aligned} z_1',\ z_2',\ z_3',\ z_4',\ &=p_i',\ q_i',\ r_i',\ t_i',\\ u_1',\ u_2',\ u_3',\ u_4',\ &=p_j',\ q_j',\ r_j',\ t_j',\\ x_1',\ x_2',\ x_3',\ x_4',\ &=p_k',\ q_k',\ r_k',\ t_k',\end{aligned}$$

$s_{\alpha\beta}$ and $t_{\alpha\beta}$ being orientation-variables of different orientations. Also we write

$$(\gamma\delta)_{ki}=\sum_\alpha\sum_\beta(\gamma\delta,\alpha\beta)s_{\alpha\beta},\quad (\gamma\delta)_{kj}=\sum_\alpha\sum_\beta(\gamma\delta,\alpha\beta)t_{\alpha\beta},$$

$$K_\theta(i,kj)=p_i'(\theta 1)_{kj}+q_i'(\theta 2)_{kj}+r_i'(\theta 3)_{kj}+t_i'(\theta 4)_{kj}',$$

for all values of i, j, k, where manifestly $(\gamma\delta)_{ik}=-(\gamma\delta)_{ki}$, and $(\gamma\delta)_{jk}=-(\gamma\delta)_{kj}$.

Then, if the whole of the sixth line in the expression for $\left(\frac{d\Gamma}{ds_k}\bigg)\,i\,\bigg)\,j\right)$ be denoted by $\frac{1}{3\Omega}U_1$, we have

$$U_1=\sum_\mu a_{1\mu}[K_\mu(i,kj)+K_\mu(j,ki)].$$

The corresponding portions in the expressions for

$$\left(\frac{d\Delta}{ds_k}\bigg)\,i\,\bigg)\,j\right),\quad \left(\frac{d\Theta}{ds_k}\bigg)\,i\,\bigg)\,j\right),\quad \left(\frac{d\Phi}{ds_k}\bigg)\,i\,\bigg)\,j\right)$$

respectively are denoted by

$$\frac{1}{3\Omega}U_2,\quad \frac{1}{3\Omega}U_3,\quad \frac{1}{3\Omega}U_4,$$

where

$$U_\lambda=\sum_\mu a_{\lambda\mu}[K_\mu(i,kj)+K_\mu(j,ki)],$$

for all the values $\lambda=1, 2, 3, 4$, and $a_{\lambda\mu}$ is the minor of $A_{\lambda\mu}$ in Ω.

Further, as for the similar single summations and double summations which arose in connection with a region (§§ 172, 212), we introduce symbols under the definitions

$$\left.\begin{aligned}\alpha_i&=\Gamma_{11}p_i'+\Gamma_{12}q_i'+\Gamma_{13}r_i'+\Gamma_{14}t_i'\\ \beta_i&=\Gamma_{12}p_i'+\Gamma_{22}q_i'+\Gamma_{23}r_i'+\Gamma_{24}t_i'\\ \gamma_i&=\Gamma_{13}p_i'+\Gamma_{23}q_i'+\Gamma_{33}r_i'+\Gamma_{34}t_i'\\ \delta_i&=\Gamma_{14}p_i'+\Gamma_{24}q_i'+\Gamma_{34}r_i'+\Gamma_{44}t_i'\end{aligned}\right\},$$

$$\left.\begin{aligned}\xi_i&=\sum\Delta_{11}p_i'\\ \eta_i&=\sum\Delta_{12}p_i'\\ \zeta_i&=\sum\Delta_{13}p_i'\\ \varpi_i&=\sum\Delta_{14}p_i'\end{aligned}\right\},\quad \left.\begin{aligned}\phi_i&=\sum\Theta_{11}p_i'\\ \chi_i&=\sum\Theta_{12}p_i'\\ \psi_i&=\sum\Theta_{13}p_i'\\ \omega_i&=\sum\Theta_{14}p_i'\end{aligned}\right\},\quad \left.\begin{aligned}\kappa_i&=\sum\Phi_{11}p_i'\\ \lambda_i&=\sum\Phi_{12}p_i'\\ \mu_i&=\sum\Phi_{13}p_i'\\ \nu_i&=\sum\Phi_{14}p_i'\end{aligned}\right\},$$

where the summations in the last three groups of magnitudes are similar to those in α_i, β_i, γ_i, δ_i; and, for the double summations,

$$\left.\begin{aligned}\alpha_i p_j'+\beta_i q_j'+\gamma_i r_j'+\delta_i t_j'&=\sum\sum\Gamma_{11}p_i'p_j'=\bar{\gamma}_{ij}\\ \xi_i p_j'+\eta_i q_j'+\zeta_i r_j'+\varpi_i t_j'&=\sum\sum\Delta_{11}p_i'p_j'=\bar{\delta}_{ij}\\ \phi_i p_j'+\chi_i q_j'+\psi_i r_j'+\omega_i t_j'&=\sum\sum\Theta_{11}p_i'p_j'=\bar{\theta}_{ij}\\ \kappa_i p_j'+\lambda_i q_j'+\mu_i r_j'+\nu_i t_j'&=\sum\sum\Phi_{11}p_i'p_j'=\bar{\phi}_{ij}\end{aligned}\right\}.$$

Manifestly

$$\bar{\gamma}_{ii}=-p_i'',\quad \bar{\delta}_{ii}=-q_i'',\quad \bar{\theta}_{ii}=-r_i'',\quad \bar{\phi}_{ii}=-t_i''.$$

Then the results become

$$\left.\begin{aligned}\left(\frac{d\Gamma}{ds_k}\middle)\, i\,\middle(\,\right)j\right)&=(\Gamma_{300}p_i'p_j'p_k')+\alpha_i\bar{\gamma}_{jk}+\beta_i\bar{\delta}_{jk}+\gamma_i\bar{\theta}_{jk}+\delta_i\bar{\phi}_{jk}\\ &\qquad+\frac{1}{3\Omega}U_1+\alpha_j\bar{\gamma}_{ik}+\beta_j\bar{\delta}_{ik}+\gamma_j\bar{\theta}_{ik}+\delta_j\bar{\phi}_{ik}\\ \left(\frac{d\Delta}{ds_k}\middle)\, i\,\middle(\,\right)j\right)&=(\Delta_{300}p_i'p_j'p_k')+\xi_i\bar{\gamma}_{jk}+\eta_i\bar{\delta}_{jk}+\zeta_i\bar{\theta}_{jk}+\varpi_i\bar{\phi}_{jk}\\ &\qquad+\frac{1}{3\Omega}U_2+\xi_j\bar{\gamma}_{ik}+\eta_j\bar{\delta}_{ik}+\zeta_j\bar{\theta}_{ik}+\varpi_j\bar{\phi}_{ik}\\ \left(\frac{d\Theta}{ds_k}\middle)\, i\,\middle(\,\right)j\right)&=(\Theta_{300}p_i'p_j'p_k')+\phi_i\bar{\gamma}_{jk}+\chi_i\bar{\delta}_{jk}+\psi_i\bar{\theta}_{jk}+\omega_i\bar{\phi}_{jk}\\ &\qquad+\frac{1}{3\Omega}U_3+\phi_j\bar{\gamma}_{ik}+\chi_j\bar{\delta}_{ik}+\psi_j\bar{\theta}_{ik}+\omega_j\bar{\phi}_{ik}\\ \left(\frac{d\Phi}{ds_k}\middle)\, i\,\middle(\,\right)j\right)&=(\Phi_{300}p_i'p_j'p_k')+\kappa_i\bar{\gamma}_{jk}+\lambda_i\bar{\delta}_{jk}+\mu_i\bar{\theta}_{jk}+\nu_i\bar{\phi}_{jk}\\ &\qquad+\frac{1}{3\Omega}U_4+\kappa_j\bar{\gamma}_{ik}+\lambda_j\bar{\delta}_{ik}+\mu_j\bar{\theta}_{ik}+\nu_j\bar{\phi}_{ik}\end{aligned}\right\}.$$

The numbers i, j, k, may be different from one another, or equalities may subsist among them; in the latter event, simplifications occur in the expressions of the quantities U_1, U_2, U_3, U_4.

Quantities $\sum_i \frac{dA_{ij}}{ds_k} p_i'$, $\sum\sum \frac{d^2A_{ij}}{ds_k^2} p_i'p_j'$.

307. In connection with the same investigations concerning small geodesic triangles and adjacent parallel geodesics, we require similar combinations, of the first and the second arc-derivatives of the primary magnitudes A_{ij}, with sets of direction-variables for directions not along the geodesic.

In addition to the abbreviating symbols already used, we use (for the domain) the definitions (§ 31) of u_1, u_2, u_3, u_4, so that

$$\left.\begin{aligned} u_1^{(\iota)} &= Ap_\iota' + Hq_\iota' + Gr_\iota' + Lt_\iota' \\ u_2^{(\iota)} &= Hp_\iota' + Bq_\iota' + Fr_\iota' + Mt_\iota' \\ u_3^{(\iota)} &= Gp_\iota' + Fq_\iota' + Cr_\iota' + Nt_\iota' \\ u_4^{(\iota)} &= Lp_\iota' + Mq_\iota' + Nr_\iota' + Dt_\iota' \end{aligned}\right\}.$$

Also, with the convention

$$x_1' = p', \quad x_2' = q', \quad x_3' = r', \quad x_4' = t',$$

we define quantities P_μ, Q_μ, R_μ, T_μ, by the relations, for $a = 1, 2, 3, 4$,

$$P_a = \tfrac{1}{3} \frac{\partial}{\partial x_a'} \left(\sum\sum\sum \Gamma_{a\beta\gamma} x_a' x_\beta' x_\gamma'\right),$$

$$Q_a = \tfrac{1}{3} \frac{\partial}{\partial x_a'} \left(\sum\sum\sum \Delta_{a\beta\gamma} x_a' x_\beta' x_\gamma'\right),$$

$$R_a = \tfrac{1}{3} \frac{\partial}{\partial x_a'} \left(\sum\sum\sum \Theta_{a\beta\gamma} x_a' x_\beta' x_\gamma'\right),$$

$$S_a = \tfrac{1}{3} \frac{\partial}{\partial x_a'} \left(\sum\sum\sum \Phi_{a\beta\gamma} x_a' x_\beta' x_\gamma'\right),$$

with the implied assumption that, in P_a, Q_a, R_a, S_a, the direction-variables x' are p', q', r', t'.

Now (§ 268)

$$\begin{aligned} \frac{\partial A_{\lambda\mu}}{\partial x_e} = {} & \Gamma_{e\lambda} A_{\mu 1} + \Delta_{e\lambda} A_{\mu 2} + \Theta_{e\lambda} A_{\mu 3} + \Phi_{e\lambda} A_{\mu 4} \\ & + \Gamma_{e\mu} A_{\lambda 1} + \Delta_{e\mu} A_{\lambda 2} + \Theta_{e\mu} A_{\lambda 3} + \Phi_{e\mu} A_{\lambda 4}, \end{aligned}$$

for all values of λ, μ, e. Hence as

$$\frac{d}{ds} = \sum_e x_e' \frac{\partial}{\partial x_e},$$

and as we shall take the quantities x_e' (for the values of e) to be p', q', r', t', we have

$$\frac{dA}{ds} = \frac{dA_{11}}{ds} = 2(aA + \xi H + \phi G + \kappa L),$$

$$\frac{dB}{ds} = 2(\beta H + \eta B + \chi F + \lambda M),$$

$$\frac{dC}{ds}=2(\gamma G+\zeta F+\psi C+\mu N),$$

$$\frac{dD}{ds}=2(\delta L+\varpi M+\omega N+\nu D),$$

$$\frac{dF}{ds}=(\beta G+\eta F+\chi C+\lambda N)+(\gamma H+\zeta B+\psi F+\mu M),$$

$$\frac{dG}{ds}=(\gamma A+\zeta H+\psi G+\mu L)+(\alpha G+\xi F+\phi C+\kappa N),$$

$$\frac{dH}{ds}=(\alpha H+\xi B+\phi F+\kappa M)+(\beta A+\eta H+\chi G+\lambda L),$$

$$\frac{dL}{ds}=(\delta A+\varpi H+\omega G+\nu L)+(\alpha L+\xi M+\phi N+\kappa D),$$

$$\frac{dM}{ds}=(\delta H+\varpi B+\omega F+\nu M)+(\beta L+\eta M+\chi N+\lambda D),$$

$$\frac{dN}{ds}=(\delta G+\varpi F+\omega C+\nu N)+(\gamma L+\zeta M+\psi N+\mu D).$$

Consequently

$$p_i'\frac{dA}{ds_j}+q_i'\frac{dH}{ds_j}+r_i'\frac{dG}{ds_j}+t_i'\frac{dL}{ds_j}$$
$$=(A\bar\gamma_{ij}+H\bar\delta_{ij}+G\bar\theta_{ij}+L\bar\phi_{ij})+\{\alpha_j u_1^{(i)}+\xi_j u_2^{(i)}+\phi_j u_3^{(i)}+\kappa_j u_4^{(i)}\},$$

$$p_i'\frac{dH}{ds_j}+q_i'\frac{dB}{ds_j}+r_i'\frac{dF}{ds_j}+t_i'\frac{dM}{ds_j}$$
$$=(H\bar\gamma_{ij}+B\bar\delta_{ij}+F\bar\theta_{ij}+M\bar\phi_{ij})+\{\beta_j u_1^{(i)}+\eta_j u_2^{(i)}+\chi_j u_3^{(i)}+\lambda_j u_4^{(i)}\},$$

$$p_i'\frac{dG}{ds_j}+q_i'\frac{dF}{ds_j}+r_i'\frac{dC}{ds_j}+t_i'\frac{dN}{ds_j}$$
$$=(G\bar\gamma_{ij}+F\bar\delta_{ij}+C\bar\theta_{ij}+N\bar\phi_{ij})+\{\gamma_j u_1^{(i)}+\zeta_j u_2^{(i)}+\psi_j u_3^{(i)}+\mu_j u_4^{(i)}\},$$

$$p_i'\frac{dL}{ds_j}+q_i'\frac{dM}{ds_j}+r_i'\frac{dN}{ds_j}+t_i'\frac{dD}{ds_j}$$
$$=(L\bar\gamma_{ij}+M\bar\delta_{ij}+N\bar\theta_{ij}+D\bar\phi_{ij})+\{\delta_j u_1^{(i)}+\varpi_j u_2^{(i)}+\omega_j u_3^{(i)}+\nu_j u_4^{(i)}\};$$

and also

$$\left.\sum\sum\frac{dA}{ds_j}p_e'p_i'=\begin{array}{l} u_1^{(e)}\bar\gamma_{ij}+u_2^{(e)}\bar\delta_{ij}+u_3^{(e)}\bar\theta_{ij}+u_4^{(e)}\bar\phi_{ij} \\ +u_1^{(i)}\bar\gamma_{ej}+u_2^{(i)}\bar\delta_{ej}+u_3^{(i)}\bar\theta_{ej}+u_4^{(i)}\bar\phi_{ej}\end{array}\right\}.$$

308. We shall also require quantities of the type

$$\sum\sum\frac{d^2A}{ds_k^2}p_i'p_j',$$

both when p_i', q_i', r_i', t_i', and p_j', q_j', r_j', t_j', denote different directions and when they denote the same direction. Now

$$\frac{d^2A_{\lambda\mu}}{ds^2}=\sum_e\sum_f\frac{\partial^2A_{\lambda\mu}}{\partial x_e\partial x_f}x_e'x_f'+\sum_e\frac{\partial A_{\lambda\mu}}{\partial x_e}x_e'',$$

where x_1', x_2', x_3', x_4', as before denote the direction-variables of the geodesic. We had

$$\frac{\partial A_{\lambda\mu}}{\partial x_e}=\sum_a\left[A_{\mu a}\{e\lambda, a\}+A_{\lambda a}\{e\mu, a\}\right];$$

and therefore

$$\begin{aligned}\frac{\partial^2 A_{\lambda\mu}}{\partial x_e\partial x_f}= &\sum_a\left[A_{\mu a}\frac{\partial}{\partial x_f}\{e\lambda, a\}+A_{\lambda a}\frac{\partial}{\partial x_f}\{e\mu, a\}\right]\\ &+\sum_a\sum_h\{e\lambda, a\}[A_{\mu h}\{fa, h\}+A_{ha}\{f\mu, h\}]\\ &+\sum_a\sum_h\{e\mu, a\}[A_{\lambda h}\{fa, h\}+A_{ha}\{f\lambda, h\}].\end{aligned}$$

The parametric derivatives of $\{e\lambda, a\}$ and $\{e\mu, a\}$ are known (as found in § 306), and are to be substituted on the right-hand side.

With these values, the second arc-derivatives of the primary magnitudes assume the forms

$$\begin{aligned}\frac{d^2A_{11}}{ds^2}=\frac{d^2A}{ds^2}= &-\tfrac{2}{3}\sum_e\sum_f(1e, 1f)x_e'x_f'\\ &+\ 2(AP_1+HQ_1+GR_1+LS_1)+2(A, \dots, D\between\alpha, \xi, \phi, \kappa)^2\\ &+4\alpha(A\alpha\ +H\xi\ +G\phi\ +L\kappa)\ +4\xi(A\beta+H\eta+G\chi+L\lambda)\\ &+4\eta(A\gamma\ +H\zeta\ +G\psi\ +L\mu)\ +4\kappa(A\delta+H\varpi+G\omega+L\nu);\end{aligned}$$

$$\begin{aligned}\frac{d^2A_{22}}{ds^2}=\frac{d^2B}{ds^2}= &-\tfrac{2}{3}\sum_e\sum_f(2e, 2f)x_e'x_f'\\ &+\ 2(HP_2+BQ_2+FR_2+MS_2)+2(A, \dots, D\between\beta, \eta, \chi, \lambda)^2\\ &+4\beta(H\alpha\ +B\xi\ +F\phi\ +M\kappa)\ +4\eta(H\beta+B\eta+F\chi+M\lambda)\\ &+4\chi(H\gamma\ +B\zeta\ +F\psi\ +M\mu)\ +4\lambda(H\delta+B\varpi+F\omega+M\nu);\end{aligned}$$

$$\begin{aligned}\frac{d^2A_{33}}{ds^2}=\frac{d^2C}{ds^2}= &-\tfrac{2}{3}\sum_e\sum_f(3e, 3f)\,x_e'x_f'\\ &+\ 2(GP_3+FQ_3+CR_3+NS_3)+2(A, \dots, D\between\gamma, \zeta, \psi, \mu)^2\\ &+4\gamma(G\alpha\ +F\xi\ +C\phi\ +N\kappa)\ +4\zeta(G\beta+F\eta+C\chi+N\lambda)\\ &+4\psi(G\gamma\ +F\zeta\ +C\psi\ +N\mu)\ +4\mu(G\delta+F\varpi+C\omega+N\nu);\end{aligned}$$

$$\begin{aligned}\frac{d^2A_{44}}{ds^2}=\frac{d^2D}{ds^2}= &-\tfrac{2}{3}\sum_e\sum_f(4e, 4f)\,x_e'x_f'\\ &+\ 2(LP_4+MQ_4+NR_4+DS_4)+2(A, \dots, D\between\delta, \varpi, \omega, \nu)^2\\ &+4\delta(L\alpha\ +M\xi\ +N\phi\ +D\kappa)\ +4\varpi(L\beta+M\eta+N\chi+D\lambda)\\ &+4\omega(L\gamma\ +M\zeta\ +N\psi\ +D\mu)\ +4\nu(L\delta\ +M\varpi+N\omega+D\nu);\end{aligned}$$

$$\frac{d^2A_{23}}{ds^2}=\frac{d^2F}{ds^2}=-\tfrac{1}{3}\sum_e\sum_f[(2e,3f)+(3e,2f)]\,x_e'x_f'$$

$$\begin{aligned}
&(HP_3+BQ_3+FR_3+MS_3)+\ (GP_2+FQ_2+CR_2+NR_3)\\
&\qquad+2(A,\ldots,D\between\beta,\eta,\chi,\lambda\between\gamma,\zeta,\psi,\mu)\\
&+2\gamma(H\alpha\ +B\xi\ +F\phi\ +M\kappa)\ +2\beta(G\alpha+F\xi\ +C\phi\ +N\kappa)\\
&+2\zeta(H\beta\ +B\eta\ +F\chi\ +M\lambda)\ +2\eta(G\beta+F\eta\ +C\chi\ +N\lambda)\\
&+2\psi(H\gamma\ +B\zeta\ +F\psi\ +M\mu)\ +2\chi(G\gamma+F\zeta\ +C\psi\ +N\mu)\\
&+2\mu(H\delta\ +B\varpi\ +F\omega\ +M\nu)\ +2\lambda(G\delta+F\varpi\ +C\omega\ +N\nu),
\end{aligned}$$

with similar expressions for $\frac{d^2G}{ds^2}\left(=\frac{d^2A_{13}}{ds^2}\right)$ and $\frac{d^2H}{ds^2}\left(=\frac{d^2A_{12}}{ds^2}\right)$; and

$$\frac{d^2A_{14}}{ds^2}=\frac{d^2L}{ds^2}=-\tfrac{1}{3}\sum_e\sum_f[(1e,4f)+(1f,4e)]\,x_e'x_f'$$

$$\begin{aligned}
&+\ \ (AP_4+HQ_4+GR_4+LS_4)+\ (LP_1+MQ_1+NR_1+DS_1)\\
&\qquad+2(A,\ldots,D\between\alpha,\xi,\phi,\kappa\between\delta,\varpi,\omega,\nu)\\
&+2\delta\,(A\alpha\ +H\xi\ +G\phi\ +L\kappa)\ +2\alpha(L\alpha+M\xi\ +N\phi\ +D\kappa)\\
&+2\varpi(A\beta\ +H\eta\ +G\chi\ +L\lambda)\ +2\xi(L\beta+M\eta\ +N\chi\ +D\lambda)\\
&+2\omega(A\gamma\ +H\zeta\ +G\psi\ +L\mu)\ +2\phi(L\gamma+M\zeta\ +N\psi\ +D\mu)\\
&+2\nu\ (A\delta\ +H\varpi\ +G\omega\ +L\nu)\ +2\kappa(L\delta+M\varpi\ +N\omega\ +D\nu),
\end{aligned}$$

with similar expressions for $\frac{d^2M}{ds^2}\left(=\frac{d^2A_{24}}{ds^2}\right)$ and $\frac{d^2N}{ds^2}\left(=\frac{d^2A_{34}}{ds^2}\right)$.

Then, with the notation of § 306 for the significance of the symbols $s_{\alpha\beta}$ and $t_{\alpha\beta}$ as the orientation-variables, framed by the combination of the direction p_k', q_k', r_k', t_k', with the directions p_i', q_i', r_i', t_i', and p_j', q_j', r_j', t_j', respectively, we have

$$\begin{aligned}
&\sum\sum\frac{d^2A}{ds_k^2}p_i'p_j'\\
&\quad=-\tfrac{2}{3}\sum(\gamma\delta,\alpha\beta)\,s_{\alpha\beta}t_{\gamma\delta}\\
&\qquad+u_1^{(i)}(\Gamma_{300}p_k'^2p_j')+u_2^{(i)}(\Delta_{300}p_k'^2p_j')+u_3^{(i)}(\Theta_{300}p_k'^2p_j')+u_4^{(i)}(\Phi_{300}p_k'^2p_j')\\
&\qquad+u_1^{(j)}(\Gamma_{300}p_k'^2p_i')+u_2^{(j)}(\Delta_{300}p_k'^2p_i')+u_3^{(j)}(\Theta_{300}p_k'^2p_i')+u_4^{(j)}(\Phi_{300}p_k'^2p_i')\\
&\qquad\qquad+2(A,\ldots,D\between\bar\gamma_{ki},\bar\delta_{ki},\bar\theta_{ki},\bar\phi_{ki}\between\bar\gamma_{kj},\bar\delta_{kj},\bar\theta_{kj},\bar\phi_{kj})\\
&\qquad+2u_1^{(i)}\{\alpha_k\bar\gamma_{kj}+\beta_k\bar\delta_{kj}+\gamma_k\bar\theta_{kj}+\ \delta_k\bar\phi_{kj}\}+2u_1^{(j)}\{\alpha_k\bar\gamma_{ki}+\beta_k\bar\delta_{ki}+\gamma_k\bar\theta_{ki}+\ \delta_k\bar\phi_{ki}\}\\
&\qquad+2u_2^{(i)}\{\xi_k\bar\gamma_{kj}+\eta_k\bar\delta_{kj}+\zeta_k\bar\theta_{kj}+\varpi_k\bar\phi_{kj}\}+2u_2^{(j)}\{\xi_k\bar\gamma_{ki}+\eta_k\bar\delta_{ki}+\zeta_k\bar\theta_{ki}+\varpi_k\bar\phi_{ki}\}\\
&\qquad+2u_3^{(i)}\{\phi_k\bar\gamma_{kj}+\chi_k\bar\delta_{kj}+\psi_k\bar\theta_{kj}+\omega_k\bar\phi_{kj}\}+2u_3^{(j)}\{\phi_k\bar\gamma_{ki}+\chi_k\bar\delta_{ki}+\psi_k\bar\theta_{ki}+\omega_k\bar\phi_{ki}\}\\
&\qquad+2u_4^{(i)}\{\kappa_k\bar\gamma_{kj}+\lambda_k\bar\delta_{kj}+\mu_k\bar\theta_{kj}+\ \nu_k\bar\phi_{kj}\}+2u_4^{(j)}\{\kappa_k\bar\gamma_{ki}+\lambda_k\bar\delta_{ki}+\mu_k\bar\theta_{ki}+\ \nu_k\bar\phi_{ki}\},
\end{aligned}$$

holding for values of i, j, k, that may be different from one another or may be the same in any combinations.

When i and j are the same, the orientation-variables s and t are the same; and then the first line becomes

$$-\tfrac{2}{3}\sum(\gamma\delta, \alpha\beta)s_{\alpha\beta}s_{\gamma\delta},$$

which (except as to a factor) is the Riemann measure of curvature of the domain (§ 305) in the superficial orientation defined by the variables $s_{\alpha\beta}$. If $k=i$, the variables $s_{\alpha\beta}$ vanish; if $k=j$, the variables $t_{\alpha\beta}$ vanish; in either of these events, the first line vanishes.

309. In the immediately preceding formula, the second arc-derivatives of the primary magnitudes (as, initially, the second arc-derivatives of the parameters) are effected along the same domainal geodesic in each instance. Another formula of the same type is required for the evaluation of certain expressions connected with the parallelism of geodesics: it involves second arc-derivatives, such as $\dfrac{d^2A}{ds_1 ds_2}$, taken concurrently along different geodesics. So far as immediate applications are concerned, the fundamental necessity is a knowledge of the changes in the direction-variables of a set of domainal geodesics, drawn parallel to one geodesic G_2 at various points along a different geodesic G_1, which serves as a basic line of reference. The complete expression of those direction-variables depends on the law of parallelism adopted. But all the diverse laws suggested have one rudimentary property in common which, for any direction-variables at a point along G_1 for a geodesic parallel to G_2, provides the same first two terms, being the finite part and the part which is of the first order in the length of the arc-distance along G_1.

Thus if ds_1 denote an element of arc along G_1, the direction-variables of which at O are p_1', q_1', r_1', t_1'; if G_2 be a different domainal geodesic through O with direction-variables p_2', q_2', r_2', t_2'; and if through a point P on G_1 near O, the small geodesic arc OP being denoted by x, a geodesic be drawn parallel to G_2 under any of the suggested laws of parallelism, a direction variable P_2' of this new geodesic is given by

$$P_2' = p_2' - x\sum\Gamma_{11}p_1'p_2' + O(P_2),$$

where $O(P_2)$ is an aggregate of terms of the second and higher orders, the form of $O(P_2)$ depending upon the particular law adopted. If then we define $\dfrac{dp_2'}{ds_1}$ as given by

$$\frac{dp_2'}{ds_1} = \operatorname*{Lim}_{x\to 0}\left\{(P_2' - p_2')\frac{1}{x}\right\},$$

we have

$$\frac{dp_2'}{ds} = -\sum\Gamma_{11}p_1'p_2' = -\bar{\gamma}_{12},$$

with the preceding notation; and similarly, under this convention,

$$\frac{dq_2'}{ds_1} = -\bar{\delta}_{12}, \quad \frac{dr_2'}{ds_1} = -\bar{\theta}_{12}, \quad \frac{dt_2'}{ds_1} = -\bar{\phi}_{12}.$$

We likewise can take G_2 as a basic geodesic and can draw, through successive points Q along G_2, domainal geodesics parallel to G_1. Let the direction-variables of a geodesic, thus drawn through Q, be denoted by P_1', Q_1', R_1', T_1', and let the small arc OQ be denoted by y; then we have

$$P_1' = p_1' - y \sum \Gamma_{11} p_2' p_1' + O(P_1),$$

where $O(P_1)$, like $O(P_2)$, is an aggregate of terms of the second and higher order in small quantities, its precise form depending on the law of parallelism adopted. If, as before, we define a quantity $\dfrac{dp_1'}{ds_2}$ as given by

$$\frac{dp_1'}{ds_2} = \operatorname*{Lim}_{y\to 0} \left\{ (P_1' - p_1') \frac{1}{y} \right\},$$

we have

$$\frac{dp_1'}{ds_2} = -\sum \Gamma_{11} p_2' p_1' = -\bar{\gamma}_{12},$$

and similarly

$$\frac{dq_1'}{ds_2} = -\bar{\delta}_{12}, \quad \frac{dr_1'}{ds_2} = -\bar{\theta}_{12}, \quad \frac{dt_1'}{ds_2} = -\bar{\phi}_{12}.$$

Hence, with this definition of successive differentiations, the geodesics G_1 and G_2 being organically independent of one another, we have

$$\frac{d^2 z}{ds_1 ds_2} = \frac{d^2 z}{ds_2 ds_1},$$

where z is any one of the domainal parameters. It follows that, if ψ denote any function solely of position in the domain, we have

$$\frac{d^2 \psi}{ds_1 ds_2} = \frac{d^2 \psi}{ds_2 ds_1}.$$

But this commutative quality of operations is restricted solely to two such differentiations, each of the first order: it does not extend to higher orders. Thus, with the foregoing formulæ, we can establish a result

$$\frac{d}{ds_2}\left(\frac{d^2 p}{ds_1^2}\right) - \frac{d^2}{ds_1^2}\left(\frac{dp}{ds_2}\right) = \frac{1}{\Omega} \sum_{\mu} \{a_{1\mu} K_{\mu}(1, 12)\},$$

and so for other instances.

After these explanations, and using the earlier analysis of § 308, it will be sufficient to state the result. We denote four arc-derivatives by direction-variables z_i', z_j', z_k', z_l', for $z = p, q, r, t$, and the elements of arcs by ds_i, ds_j, ds_k, ds_l; we denote orientation-variables, $s_{\alpha\beta}$ from directions ds_k and ds_l, $\bar{s}_{\alpha\beta}$ from directions

ds_l and ds_ι, $t_{\alpha\beta}$ from directions ds_k and ds_j, and $\bar{t}_{\alpha\beta}$ from directions ds_l and ds_j. Then we have

$$\begin{aligned}\sum \frac{d^2 A}{ds_k ds_l} p_i' p_j' = &-\tfrac{1}{3} \sum (\gamma\delta, \alpha\beta)[s_{\alpha\beta}\bar{t}_{\gamma\delta} + \bar{s}_{\alpha\beta} t_{\gamma\delta}] \\ &+ \sum [u_1^{(i)}(\Gamma_{300} p_k' p_l' p_j')] + \sum [u_1^{(j)}(\Gamma_{300} p_k' p_l' p_i')] \\ &+ (A \between \bar{\gamma}_{ki} \between \bar{\gamma}_{lj}) + (A \between \bar{\gamma}_{kj} \between \bar{\gamma}_{li}) \\ &+ \sum [u_1^{(i)}(\alpha_k \bar{\gamma}_{lj} + \beta_k \bar{\delta}_{lj} + \gamma_k \bar{\theta}_{lj} + \delta_k \bar{\phi}_{lj})] \\ &+ \sum [u_1^{(i)}(\alpha_l \bar{\gamma}_{kj} + \beta_l \bar{\delta}_{kj} + \gamma_l \bar{\theta}_{kj} + \delta_l \bar{\phi}_{kj})] \\ &+ \sum [u_1^{(j)}(\alpha_k \bar{\gamma}_{li} + \beta_k \bar{\delta}_{li} + \gamma_k \bar{\theta}_{li} + \delta_k \bar{\phi}_{li})] \\ &+ \sum [u_1^{(j)}(\alpha_l \bar{\gamma}_{ki} + \beta_l \bar{\delta}_{ki} + \gamma_l \bar{\theta}_{ki} + \delta_l \bar{\phi}_{ki})],\end{aligned}$$

the summations in the last four lines being over the four quantities $u_1^{(\lambda)}$, $u_2^{(\lambda)}$, $u_3^{(\lambda)}$, $u_4^{(\lambda)}$, each with its appropriate factor as in the formula at the end of § 308.

Small geodesic triangles in a domain.

310. To consider the form of a domain in the vicinity of a point O, we draw two geodesics, OA in a direction p_1', q_1', r_1', t_1', and OB in a direction p_2', q_2', r_2', t_2'; along OA we take a small arc OU, $=x$, and along OB we take a small arc OV, $=y$, where x and y are small quantities of the same order; and we join UV by a domainal geodesic $SUVT$. The direction-variables of the geodesic UV, at U in the direction UV, are denoted by p', q', r', t', and we shall find it convenient to take

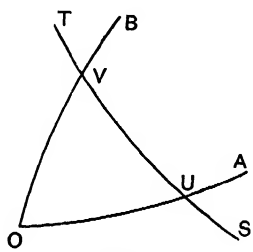

FIG. 31.

$$p' = p_0' + P, \quad q' = q_0' + Q, \quad r' = r_0' + R, \quad t' = t_0' + T,$$

where p_0', q_0', r_0', t_0', are finite, while P, Q, R, T, are small quantities. The arc-length of the portion UV of the geodesic is denoted by $\bar{w}$, and we shall find it convenient to take

$$\bar{w} = w + W,$$

where w definitely is of the same order of small quantities as x and y, while W will be proved to be of the third order of small quantities.

To investigate the nature of the domain near O, we use the obvious property that a point V can be reached, either by a direct geodesic path OV or by a broken geodesic path made up of geodesic arcs OU and UV; and the values of the parameters at V must be the same, by the two paths. Earlier investigations, for a surface and a region respectively, have indicated that quantities of the third order must be retained and are sufficient in the initial stages. Thus the value of the p-parameter at V, when that position is attained by the path OV,

$$= p + y p_2' + \tfrac{1}{2} y^2 p_2'' + \tfrac{1}{6} y^3 p_2''' + \dots,$$

and, when the position is attained by the path OV and UV, its value is

$$=p+xp_1'+\tfrac{1}{2}x^2p_1''+\tfrac{1}{6}x^3p_1'''+\dots$$
$$+\bar{w}p'+\tfrac{1}{2}\bar{w}^2p''+\tfrac{1}{6}\bar{w}^3p'''+\dots,$$

where, in estimating p'' and p''', it must be remembered that these magnitudes belong to the geodesic UV at U. The equality at V of the two values of p therefore gives an equation

$$\bar{w}p'+\tfrac{1}{2}\bar{w}^2p''+\tfrac{1}{6}\bar{w}^3p'''=yp_2'-xp_1'+\tfrac{1}{2}(y^2p_2''-x^2p_1'')+\tfrac{1}{6}(y^3p_2'''-x^3p_1'''),$$

up to the third order of small quantities inclusive; and, similarly, the equality at V of the respective values of q, r, t, gives the similar equations

$$\bar{w}q'+\tfrac{1}{2}\bar{w}^2q''+\tfrac{1}{6}\bar{w}^3q'''=yq_2'-xq_1'+\tfrac{1}{2}(y^2q_2''-x^2q_1'')+\tfrac{1}{6}(y^3q_2'''-x^3q_1'''),$$
$$\bar{w}r'+\tfrac{1}{2}\bar{w}^2r''+\tfrac{1}{6}\bar{w}^3r'''=yr_2'-xr_1'+\tfrac{1}{2}(y^2r_2''-x^2r_1'')+\tfrac{1}{6}(y^3r_2'''-x^3r_1'''),$$
$$\bar{w}t'+\tfrac{1}{2}\bar{w}^2t''+\tfrac{1}{6}\bar{w}^3t'''=yt_2'-xt_1'+\tfrac{1}{2}(y^2t_2''-x^2t_1'')+\tfrac{1}{6}(y^3t_2'''-x^3t_1''').$$

We require to make approximations, up to the third order of small quantities inclusive, to the values of all the quantities on the left-hand sides of these equations.

(i) Let the foregoing values be substituted for p', q', r', t', and $\bar{w}$, with the further limitation

$$\sum Ap_0'^2=1,$$

which implies that p_0', q_0', r_0', t_0', are the variables of a direction * through O. All the magnitudes involving x^2, y^2, x^3, y^3, $\bar{w}^2$, $\bar{w}^3$, are of the second order or higher; as also, in fact, is part of $\bar{w}$; hence, equating terms of the first order in the four equations, we have

$$wp_0'=yp_2'-xp_1',\quad wq_0'=yq_2'-xq_1',\quad wr_0'=yr_2'-xr_1',\quad wt_0'=yt_2'-xt_1'.$$

We denote by $\widehat{12}$ the angle UOV measured positively from OU to OV; then $\widehat{10}$ similarly will be an approximation to the angle AUV, and $\widehat{20}$ similarly will be an approximation to the angle BVT, both measured positively. Thus

$$\cos\widehat{12}=\sum Ap_1'p_2',\quad \cos\widehat{10}=\sum Ap_1'p_0',\quad \cos\widehat{20}=\sum Ap_2'p_0';$$

the angles $\pi-\widehat{10}$, $\widehat{20}$, are approximations to the internal angles U, V, of the geodesic triangle OUV, the angle O being $\widehat{12}$. We easily find

$$w^2=x^2+y^2-2xy\cos\widehat{12},$$

$$w=y\cos\widehat{20}-x\cos\widehat{10},\quad x=y\cos\widehat{12}-w\cos\widehat{10},\quad y=x\cos\widehat{12}+w\cos\widehat{20},$$

$$\frac{x}{\sin\widehat{20}}=\frac{y}{\sin\widehat{10}}=\frac{w}{\sin\widehat{12}},$$

being relations of a plane triangle with linear sides x, y, w, and angles $\widehat{12}$, $\pi-\widehat{10}$, $\widehat{20}$.

* It will appear that a domainal geodesic in this direction is parallel to the geodesic UV at U, the parallelism being estimated along OUA.

(ii) For the next approximation, we retain terms of the second order of small quantities; and therefore, as p', q', r', t', are multiplied by $\bar{w}$ on the left-hand side, we can regard P, Q, R, T, in the values

$$p'=p_0'+P,\quad q'=q_0'+Q,\quad r'=r_0'+R,\quad t'=t_0'+T,$$

as being of the first order for this approximation; also, W can be regarded as of the second order.

The values p', q', r', t', are the accurate values of the direction-variables of the geodesic UV at U: thus, in the expression

$$\bar{w}^2 p''=-\bar{w}^2[\sum(\Gamma_{11})_U p'^2],$$

we must take the values of the magnitudes Γ_{ij} at U. These differ from their values at O by small quantities of the first and higher orders; here, they are multiplied by $\bar{w}^2$, itself already of the second and higher orders; consequently, in the second-order approximation, we can take

$$\bar{w}=w,\quad (\Gamma_{ij})_U=\Gamma_{ij},\quad p'=p_0',\quad q'=q_0',\quad r'=r_0',\quad t'=t_0',$$

in the right-hand side, and therefore, for the present purpose,

$$\bar{w}^2 p''=w^2 p_0''.$$

The term involving $\bar{w}^3$ can be neglected, here; and

$$\bar{w}p'=(w+W)(p_0'+P)=wp_0'+Wp_0'+wP,$$

the term WP being of the third order and consequently also negligible. Thus the second-order approximations give the relations

$$\begin{aligned}
Wp_0'+wP+\tfrac{1}{2}w^2p_0''&=\tfrac{1}{2}(y^2p_2''-x^2p_1''),\\
Wq_0'+wQ+\tfrac{1}{2}w^2q_0''&=\tfrac{1}{2}(y^2q_2''-x^2q_1''),\\
Wr_0'+wR+\tfrac{1}{2}w^2r_0''&=\tfrac{1}{2}(y^2r_2''-x^2r_1''),\\
Wt_0'+wT+\tfrac{1}{2}w^2t_0''&=\tfrac{1}{2}(y^2t_2''-x^2t_1'').
\end{aligned}$$

But

$$\begin{aligned}
w^2p_0''&=-w^2\sum\Gamma_{11}p_0'^2\\
&=-\sum\Gamma_{11}(yp_2'-xp_1')^2\\
&=y^2p_2''+x^2p_1''+2xy(\sum\Gamma_{11}p_2'p_1')\\
&=y^2p_2''-x^2p_1''+2x\{\sum\Gamma_{11}p_1'(yp_2'-xp_1')\}\\
&=y^2p_2''-x^2p_1''+2xw(\sum\Gamma_{11}p_1'p_0')\\
&=y^2p_2''-x^2p_1''+2xw\bar{\gamma}_{10},
\end{aligned}$$

with the former notation; and so the first equation becomes

$$Wp_0'+w(P+x\bar{\gamma}_{10})=0.$$

Similarly the other three equations become

$$Wq_0' + w(Q + x\delta_{10}) = 0,$$
$$Wr_0' + w(R + x\bar{\theta}_{10}) = 0,$$
$$Wt_0' + w(T + x\bar{\phi}_{10}) = 0.$$

Also, account has to be taken of the fact that p', q', r', t', are direction-variables at the point U in the domain, so that the permanent arc-relation

$$\sum A_U p'^2 = 1$$

must be satisfied ; and here, for the present approximation, it is sufficient to take account of small quantities of the first order. Now

$$A_U = A + x\frac{dA}{ds_1},$$

up to the first order : while, up to the same order,

$$p'^2 = p_0'^2 + 2p_0'P, \quad p'q' = p_0'q_0' + p_0'Q + q_0'P,$$

and so on. The arc-relation thus becomes, to the first order,

$$\sum A_0 p_0'^2 + x\left(\sum \frac{dA}{ds_1} p_0'^2\right) + 2\{u_1^{(0)}P + u_2^{(0)}Q + u_3^{(0)}R + u_4^{(0)}T\} = 1.$$

But $\sum A_0 p_0'^2 = 1$; and taking $e=0$, $i=0$, $j=1$, in the value obtained (§ 307) for $\sum \frac{dA}{ds_j} p_e' p_i'$, we have

$$\sum \frac{dA}{ds} p_0'^2 = 2\{u_1^{(0)}\bar{\gamma}_{01} + u_2^{(0)}\delta_{01} + u_3^{(0)}\bar{\theta}_{01} + u_4^{(0)}\bar{\phi}_{01}\} ;$$

the arc-relation therefore requires a condition

$$u_1^{(0)}\{P + \bar{\gamma}_{01}\} + u_2^{(0)}\{Q + \delta_{01}\} + u_3^{(0)}\{R + \bar{\theta}_{01}\} + u_4^{(0)}\{T + \bar{\phi}_{01}\} = 0.$$

Let the four earlier equations be multiplied by $u_1^{(0)}$, $u_2^{(0)}$, $u_3^{(0)}$, $u_4^{(0)}$, respectively, and the results be added ; when the last condition is used, as well as the relation

$$u_1^{(0)}p_0' + u_2^{(0)}q_0' + u_3^{(0)}r_0' + u_4^{(0)}t_0' = \sum Ap_0'^2 = 1,$$

we find

$$W = 0,$$

that is, up to the second order of small quantities.

Hence W is of the third order of small quantities at least : we shall use the symbol to denote the part of the accurate value which actually is of the third order. Also, when this inference is used in connection with the four equations, we have

$$P + x\bar{\gamma}_{10} = 0, \quad Q + x\delta_{10} = 0, \quad R + x\bar{\theta}_{10} = 0, \quad T + x\bar{\phi}_{10} = 0,$$

accurately up to the first order inclusive. Accordingly, we may write

$$\begin{aligned} p' &= p_0' - x\bar{\gamma}_{10} + P_0, \\ q' &= q_0' - x\bar{\delta}_{10} + Q_0, \\ r' &= r_0' - x\bar{\theta}_{10} + R_0, \\ t' &= t_0' - x\bar{\phi}_{10} + T_0, \\ \bar{w} &= w + W, \end{aligned}$$

where W is of the third order, and P_0, Q_0, R_0, T_0, are of the second order.

It may be noted, in passing, that the relations

$$p' - p_0' = -x\bar{\gamma}_{10}, \quad q' - q_0' = -x\bar{\delta}_{10}, \quad r' - r_0' = -x\bar{\theta}_{10}, \quad t' - t_0' = -x\bar{\phi}_{10},$$

satisfy the primary conditions (§ 380) that the geodesic at U, with direction-cosines p', q', r', t', should be parallel to the geodesic at O, with direction-cosines p_0', q_0', r_0', t_0', the parallelism being estimated relative to the geodesic OUA.

(iii) For approximations up to the third order of small quantities inclusive, we neglect all quantities in the four parameter-equations which are of order higher than three, and we neglect all quantities in the permanent arc-relation at U which are of order higher than two. Moreover, as the terms of the first order in the equations have balanced, and likewise the terms of the second order, we need now retain only terms of the third order in the parameter-equations; and, similarly, we retain only terms of the second order in the arc-relation.

We have

$$\bar{w}p' = (w + W)(p_0' - x\bar{\gamma}_{10} + P_0),$$

and therefore the terms in the p-equation to be retained from the quantity $\bar{w}p'$

$$= Wp_0' + wP_0.$$

In the term $\frac{1}{2}\bar{w}^2 p''$, the quantity $\bar{w}^2$, $=(w+W)^2$, is equal to w^2, up to the third order inclusive. The quantity p'' is to be taken at U; and therefore

$$-p'' = \sum (\Gamma_{11})_U p'^2$$

accurately. But p'' is multiplied by w^2, so that we need take only its terms of the first order of small quantities, in order to obtain the third-order approximation. Now, to this first order,

$$(\Gamma_{11})_U = \Gamma_{11} + x\frac{d\Gamma_{11}}{ds_1}, \quad p'^2 = p_0'^2 - 2xp_0'\bar{\gamma}_{10},$$

and similarly for the other combinations; hence the terms to be retained in $-p''$

$$= x\sum \frac{d\Gamma_{11}}{ds_1} p_0'^2 - 2x(\alpha_0\bar{\gamma}_{10} + \beta_0\bar{\delta}_{10} + \gamma_0\bar{\theta}_{10} + \kappa_0\bar{\phi}_{10}).$$

By taking $i=j=0$, $k=1$, in the result of § 306, we have

$$\sum \frac{d\Gamma_{11}}{ds_1} p_0'^2 = (\Gamma_{300} p_0'^2 p_1') + 2(\alpha_0 \bar{\gamma}_{10} + \beta_0 \bar{\delta}_{10} + \gamma_0 \bar{\theta}_{10} + \kappa_0 \bar{\phi}_{10}) + \frac{1}{3\Omega} U_1,$$

where, in the present instance,

$$U_1 = 2 \sum_\mu a_{1\mu} K_\mu (0, 10).$$

Consequently, the terms of the third order in the p-equation, to be retained out of the term $\frac{1}{2}\bar{w}^2 p''$,

$$= -\tfrac{1}{2} w^2 x \left[(\Gamma_{300} p_0'^2 p_1') + \frac{2}{3\Omega} \sum_\mu a_{1\mu} K_\mu (0, 10) \right].$$

In the term $\frac{1}{6}\bar{w}^3 p'''$ the quantity $\bar{w}^3$, $=(w+W)^3$, is equal to w^3, up to the third order inclusive. The quantity p''' is to be taken at U, and therefore

$$-p''' = \sum (\Gamma_{300})_U p'^3,$$

accurately. But p''' is multiplied by w^3, so that we need retain only the finite terms in order to obtain the third-order approximation. Thus

$$(\Gamma_{300})_U = \Gamma_{300}, \quad p' = p_0',$$

for this approximation; and the whole contribution of the term

$$= -\tfrac{1}{6} w^3 (\Gamma_{300} p_0'^3).$$

Now

$$\begin{aligned} w^3 p_0'^3 &= (y p_2' - x p_1')^3 \\ &= y^3 p_2'^3 - x^3 p_1'^3 - 3xy p_2' p_1' (y p_2' - x p_1') \\ &= y^3 p_2'^3 - x^3 p_1'^3 - 3xyw p_0' p_1' p_2'; \end{aligned}$$

and therefore the terms, to be retained out of $\frac{1}{6}\bar{w}^3 p'''$,

$$= \tfrac{1}{6}(y^3 p_2''' - x^3 p_1''') + \tfrac{1}{2} xyw (\Gamma_{300} p_0' p_1' p_2').$$

When these values are inserted in the p-equation, to obtain the third-order approximation, it becomes

$$W p_0' + w P_0 - \tfrac{1}{2} w^2 x \left[(\Gamma_{300} p_0'^2 p_1') + \frac{2}{3\Omega} \left\{ \sum_\mu a_{1\mu} K_\mu (0, 10) \right\} \right] + \tfrac{1}{2} xyw (\Gamma_{300} p_0' p_1' p_2') = 0.$$

But

$$-w p_0' + y p_2' = x p_1';$$

and therefore the equation is

$$W p_0' + w [P_0 + \tfrac{1}{2} x^2 (\Gamma_{300} p_0' p_1'^2)] = \frac{w^2 x}{3\Omega} \sum_\mu \{ a_{1\mu} K_\mu (0, 10) \}.$$

Similarly, the equations connected with the other three parameters of the domain become

$$Wq_0' + w[Q_0 + \tfrac{1}{2}x^2(\varDelta_{300}p_0p_1'^2)] = \frac{w^2x}{3\Omega}\sum_\mu \{a_{2\mu}K_\mu(0, 10)\},$$

$$Wr_0' + w[R_0 + \tfrac{1}{2}x^2(\varTheta_{300}p_0p_1'^2)] = \frac{w^2x}{3\Omega}\sum_\mu \{a_{3\mu}K_\mu(0, 10)\},$$

$$Wt_0' + w[T_0 + \tfrac{1}{2}x^2(\varPhi_{300}p_0p_1'^2)] = \frac{w^2x}{3\Omega}\sum_\mu \{a_{4\mu}K_\mu(0, 10)\}.$$

It remains to take account of the corresponding approximation to be derived from the arc-relation

$$\sum A_U p'^2 = 1$$

at U; and, here, we require the second-order terms. The finite terms have balanced; the first-order terms likewise have balanced; and we therefore need retain only the actual terms of the second order on the left-hand side. Now, up to this order inclusive,

$$A_U = A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2},$$

$$p'^2 = p_0'^2 - 2xp_0'\bar{\gamma}_{10} + (2p_0'P_0 + x^2\bar{\gamma}_{10}^2);$$

and therefore the terms to be retained provide the condition

$$\begin{aligned}\sum\{A(2p_0'P_0 + x^2\bar{\gamma}_{10}^2)\}& \\ -2x^2\sum\left(\frac{dA}{ds_1}p_0'\bar{\gamma}_{10}\right)& \\ +\tfrac{1}{2}x^2\sum\left(\frac{d^2A}{ds_1^2}p_0'^2\right)&=0.\end{aligned}$$

In this equation, the terms arising out of the first line

$$= 2\{u_1^{(0)}P_0 + u_2^{(0)}Q_0 + u_3^{(0)}R_0 + u_4^{(0)}S_0\} + x^2(\sum A\bar{\gamma}_{10}^2).$$

For the terms in the second line, the coefficient of $-2x^2$

$$\begin{aligned}= \quad &\bar{\gamma}_{01}\left(p_0'\frac{dA}{ds_1} + q_0'\frac{dH}{ds_1} + r_0'\frac{dG}{ds_1} + t_0'\frac{dL}{ds_1}\right)\\ +&\bar{\delta}_{01}\left(p_0'\frac{dH}{ds_1} + q_0'\frac{dB}{ds_1} + r_0'\frac{dF}{ds_1} + t_0'\frac{dM}{ds_1}\right)\\ +&\bar{\theta}_{01}\left(p_0'\frac{dG}{ds_1} + q_0'\frac{dF}{ds_1} + r_0'\frac{dC}{ds_1} + t_0'\frac{dN}{ds_1}\right)\\ +&\bar{\phi}_{01}\left(p_0'\frac{dL}{ds_1} + q_0'\frac{dM}{ds_1} + r_0'\frac{dN}{ds_1} + t_0'\frac{dD}{ds_1}\right).\end{aligned}$$

The values of the coefficients of $\bar{\gamma}_{01}$, $\bar{\delta}_{01}$, $\bar{\theta}_{01}$, $\bar{\phi}_{01}$, are derived from the results in

§ 307 by taking $i=0, j=1$; when these values are substituted, the aggregate of the second line in the equation

$$\begin{aligned}= &-2x^2\sum A\bar{\gamma}_{01}{}^2\\ &-2x^2[\quad u_1{}^{(0)}\{\alpha_1\bar{\gamma}_{01}+\beta_1\bar{\delta}_{01}+\gamma_1\bar{\theta}_{01}+\delta_1\bar{\phi}_{01}\}\\ &\qquad+u_2{}^{(0)}\{\xi_1\bar{\gamma}_{01}+\eta_1\bar{\delta}_{01}+\zeta_1\bar{\theta}_{01}+\varpi_1\bar{\phi}_{01}\}\\ &\qquad+u_3{}^{(0)}\{\phi_1\bar{\gamma}_{01}+\chi_1\bar{\delta}_{01}+\psi_1\bar{\theta}_{01}+\omega_1\bar{\phi}_{01}\}\\ &\qquad+u_4{}^{(0)}\{\kappa_1\bar{\gamma}_{01}+\lambda_1\bar{\delta}_{01}+\mu_1\bar{\theta}_{01}+\nu_1\bar{\phi}_{01}\}].\end{aligned}$$

The coefficient of $\frac{1}{2}x^2$ in the third line is obtained from the value of

$$\sum\sum\frac{d^2A}{ds_k{}^2}p_i'p_j'$$

in § 308 by taking $i=j=0$, $k=1$; the variables s and t in that expression are the same; and we use the variables $s_{\alpha\beta}$, with the directions p_0', q_0', r_0', t_0', and p_1', q_1', r_1', t_1', in the sense

$$s_{12}=\begin{vmatrix}p_0', & q_0'\\ p_1', & q_1'\end{vmatrix},\quad s_{34}=\begin{vmatrix}r_0', & t_0'\\ r_1', & t_1'\end{vmatrix},$$

and like expressions for s_{23}, s_{31}, s_{24}, s_{14}. The actual value of the coefficient in question

$$\begin{aligned}&=-\tfrac{2}{3}\sum(\alpha\beta,\gamma\delta)s_{\alpha\beta}s_{\gamma\delta}+2\sum A\bar{\gamma}_{01}{}^2\\ &+2u_1{}^{(0)}\{(\Gamma_{300}p_0'p_1'^2)\}+2u_2{}^{(0)}\{(\Delta_{300}p_0'p_1'^2)\}\\ &+2u_3{}^{(0)}\{(\Theta_{300}p_0'p_1'^2)\}+2u_4{}^{(0)}\{(\Phi_{300}p_0'p_1'^2)\}\\ &+4u_1{}^{(0)}\{\alpha_1\bar{\gamma}_{01}+\beta_1\bar{\delta}_{01}+\gamma_1\bar{\theta}_{01}+\delta_1\bar{\phi}_{01}\}+4u_2{}^{(0)}\{\xi_1\bar{\gamma}_{01}+\eta_1\bar{\delta}_{01}+\zeta_1\bar{\theta}_{01}+\varpi_1\bar{\phi}_{01}\}\\ &+4u_3{}^{(0)}\{\phi_1\bar{\gamma}_{01}+\chi_1\bar{\delta}_{01}+\psi_1\bar{\theta}_{01}+\omega_1\bar{\phi}_{01}\}+4u_4{}^{(0)}\{\kappa_1\bar{\gamma}_{01}+\lambda_1\bar{\delta}_{01}+\mu_1\bar{\theta}_{01}+\nu_1\bar{\phi}_{01}\}.\end{aligned}$$

When all these values are inserted in the condition that emerges from the second-order approximation in the arc-relation, it can be expressed in the form

$$\begin{aligned}u_1{}^{(0)}\{2P_0+x^2(\Gamma_{300}p_0'p_1'^2)\}&+u_2{}^{(0)}\{2Q_0+x^2(\Delta_{300}p_0'p_1'^2)\}\\ &+u_3{}^{(0)}\{2R_0+x^2(\Theta_{300}p_0'p_1'^2)\}+u_4{}^{(0)}\{2S_0+x^2(\Phi_{400}p_0'p_1'^2)\}\\ &=\tfrac{1}{3}x^2\sum(\alpha\beta,\gamma\delta)s_{\alpha\beta}s_{\gamma\delta}.\end{aligned}$$

311. Thus there are five relations, composed of this condition and of the four equations arising out of the four parameters, for the determination of the five quantities W, P_0, Q_0, R_0, S_0. Let the four parametric equations be multiplied by $u_1{}^{(0)}$, $u_2{}^{(0)}$, $u_3{}^{(0)}$, $u_4{}^{(0)}$, respectively, and the sums be added; on using the condition from the arc-relation, and also the further relation

$$u_1{}^{(0)}p_0'+u_2{}^{(0)}q_0'+u_3{}^{(0)}r_0'+u_4{}^{(0)}t_0'=\sum Ap_0'^2=1,$$

we find

$$W+\tfrac{1}{6}wx^2\sum(\alpha\beta,\gamma\delta)s_{\alpha\beta}s_{\gamma\delta}$$

$$=\frac{w^2x}{3\Omega}\left[u_1^{(0)}\sum_\mu\{a_{1\mu}K_\mu(0,10)\}+u_2^{(0)}\sum_\mu\{a_{2\mu}K_\mu(0,10)\}\right.$$
$$\left.+u_3^{(0)}\sum_\mu\{a_{3\mu}K_\mu(0,10)\}+u_4^{(0)}\sum_\mu\{a_{4\mu}K_\mu(0,10)\}\right]$$
$$=\tfrac{1}{3}w^2x\{p_0'K_1(0,10)+q_0'K_2(0,10)+r_0'K_3(0,10)+t_0'K_4(0,10)\}.$$

But (§ 306)

$$K_\mu(0,10)=p_0'(\mu 1)_{10}+q_0'(\mu 2)_{10}+r_0'(\mu 3)_{10}+t_0'(\mu 4)_{10},$$

and, for all subscripts, we have

$$(\mu\lambda)_{kj}=-(\lambda\mu)_{kj};$$

consequently, the right-hand side of the equation in W is zero, and we have

$$W=-\tfrac{1}{6}wx^2\sum(\alpha\beta,\gamma\delta)s_{\alpha\beta}s_{\gamma\delta}.$$

Also

$$ws_{12}=\begin{vmatrix} p_1', & q_1' \\ wp_0', & wq_0' \end{vmatrix}=y\begin{vmatrix} p_1', & q_1' \\ p_2', & q_2' \end{vmatrix}=y\xi_{12},$$

and so for the other orientation-variables, these variables ξ_{12}, ξ_{23}, ξ_{31}, ξ_{14}, ξ_{24}, ξ_{34}, now being the orientation-variables for an orientation determined by the two directions OUA and OVB at O. Hence

$$W=-\tfrac{1}{6}\frac{x^2y^2}{w}\sum(\alpha\beta,\gamma\delta)\xi_{\alpha\beta}\xi_{\gamma\delta}.$$

The Riemann measure of curvature of the domain, denoted by K, has been expressed (§ 305) in the form

$$K=\frac{\sum(\alpha\beta,\gamma\delta)\xi_{\alpha\beta}\xi_{\gamma\delta}}{\sum(A_{\alpha\gamma}A_{\beta\delta}-A_{\alpha\delta}A_{\beta\gamma})\xi_{\alpha\beta}\xi_{\gamma\delta}};$$

and the denominator in K

$$=(\sum Ap_1'^2)(Ap_2'^2)-(\sum Ap_1'p_2')^2=\sin^2\widehat{12},$$

where $\widehat{12}$ is the angle AOB between the directions p_1', q_1', r_1', t_1', and p_2', q_2', r_2', t_2', in the domain. Accordingly, we have

$$W=-\frac{1}{6w}Kx^2y^2\sin^2\widehat{12},$$

accurately up to the third order of small quantities; and so, up to this order inclusive, the length of the third side of the geodesic triangle UOV, which has sides of lengths x and y at an inclination $\widehat{12}$, is

$$w\left\{1-\tfrac{1}{6}K\frac{x^2y^2\sin^2\widehat{12}}{x^2+y^2-2xy\cos\widehat{12}}\right\},$$

where

$$w^2 = x^2 + y^2 - 2xy \cos \widehat{12}.$$

If we denote by p the perpendicular drawn from the vertex to the base of a plane triangle, with linear sides x and y including an angle $\widehat{12}$, (alternatively denoted by ϵ), the foregoing expression for the length of the third side of the geodesic triangle becomes

$$w(1 - \tfrac{1}{6}K\mathrm{p}^2),$$

w being the length of the third side of the plane triangle.

Further, when the value of W is substituted in the equations derived through the parameters, we have

$$P_0 = -\tfrac{1}{2}x^2(\Gamma_{300}p_0'p_1'^2) + \tfrac{1}{6}p_0'K\frac{x^2y^2}{w^2}\sin^2\epsilon + \frac{xy}{3\Omega}\sum_\mu \{a_{1\mu}K_\mu(0, 12)\},$$

$$Q_0 = -\tfrac{1}{2}x^2(\Delta_{300}p_0'p_1'^2) + \tfrac{1}{6}q_0'K\frac{x^2y^2}{w^2}\sin^2\epsilon + \frac{xy}{3\Omega}\sum_\mu \{a_{2\mu}K_\mu(0, 12)\},$$

$$R_0 = -\tfrac{1}{2}x^2(\Theta_{300}p_0'p_1'^2) + \tfrac{1}{6}r_0'K\frac{x^2y^2}{w^2}\sin^2\epsilon + \frac{xy}{3\Omega}\sum_\mu \{a_{3\mu}K_\mu(0, 12)\},$$

$$S_0 = -\tfrac{1}{2}x^2(\Phi_{300}p_0'p_1'^2) + \tfrac{1}{6}t_0'K\frac{x^2y^2}{w^2}\sin^2\epsilon + \frac{xy}{3\Omega}\sum_\mu \{a_{4\mu}K_\mu(0, 12)\};$$

and the direction-variables of the domainal geodesic UV, at U in the direction UV, are

$$p' = p_0' - x\bar{\gamma}_{01} + P_0, \quad q' = q_0' - x\bar{\delta}_{01} + Q_0, \quad r' = r_0' - x\bar{\theta}_{01} + R_0, \quad t' = t_0' - x\bar{\phi}_{01} + S_0,$$

where

$$wp_0' = yp_2' - xp_1', \quad wq_0' = yq_2' - xq_1', \quad wr_0' = yr_2' - xr_1', \quad wt_0' = yt_2' - xt_1'.$$

The direction-variables of the same domainal geodesic UV, at V in the direction VU, are

$$-p_0' + y\bar{\gamma}_{02} + [P_0], \quad -q_0' + y\bar{\delta}_{02} + [Q_0], \quad -r_0' + y\bar{\theta}_{02} + [R_0], \quad -t_0' + y[\bar{\phi}_{02}] + [S_0],$$

where

$$[P_0] = \tfrac{1}{2}y^2(\Gamma_{300}p_0'p_2'^2) - \tfrac{1}{6}p_0'K\frac{x^2y^2}{w^2}\sin^2\epsilon + \frac{xy}{3\Omega}\sum_\mu \{a_{1\mu}K_\mu(0, 21)\},$$

$$[Q_0] = \tfrac{1}{2}y^2(\Delta_{300}p_0'p_2'^2) - \tfrac{1}{6}q_0'K\frac{x^2y^2}{w^2}\sin^2\epsilon + \frac{xy}{3\Omega}\sum_\mu \{a_{2\mu}K_\mu(0, 21)\},$$

$$[R_0] = \tfrac{1}{2}y^2(\Theta_{300}p_0'p_2'^2) - \tfrac{1}{6}r_0'K\frac{x^2y^2}{w^2}\sin^2\epsilon + \frac{xy}{3\Omega}\sum_\mu \{a_{3\mu}K_\mu(0, 21)\},$$

$$[S_0] = \tfrac{1}{2}y^2(\Phi_{300}p_0'p_2'^2) - \tfrac{1}{6}t_0'K\frac{x^2y^2}{w^2}\sin^2\epsilon + \frac{xy}{3\Omega}\sum_\mu \{a_{4\mu}K_\mu(0, 21)\},$$

with the same significance for the actual symbols w, p_0', q_0', r_0', t_0', as before.

Small geodesic triangle and the sphericity of the domain.

312. To complete the elements of the geodesic triangle UOV, we require a knowledge of the internal angles at U and V; we denote these angles by U and V respectively.

The direction-variables of the geodesic OUA at U in the direction UA are of the type

$$p_1' + xp_1'' + \tfrac{1}{2}x^2 p_1''',$$

up to the second order of small quantities inclusive, while those of the geodesic $SUVT$ at U in the direction UV are of the type

$$p_0' - x\bar{\gamma}_{01} + P_0,$$

with the foregoing significance of the quantities P_0, also up to the second order of small quantities. Moreover, the angle U is estimated at the place U, where the values of the primary magnitudes are of the form

$$A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2},$$

again up to the second order of small quantities. Hence

$$\begin{aligned}-\cos U &= \cos AUV\\ &= \sum A_U(p_1' + xp_1'' + \tfrac{1}{2}x^2p_1''')(p_0' - x\bar{\gamma}_{01} + P_0)\\ &= \sum\left[\left(A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2}\right)(p_1' + xp_1'' + \tfrac{1}{2}x^2p_1''')(p_0' - x\bar{\gamma}_{01} + P_0)\right];\end{aligned}$$

and having regard to the magnitudes involved, we can deduce the value of $\cos U$ to the second order of small quantities.

The finite terms on the right-hand side, together,

$$= \sum Ap_1'p_0',$$

which we shall write $-\cos U_0$, so that

$$U_0 = \pi - \widehat{10};$$

and U_0 is the finite part of U.

The terms, of the first order of small quantities, are the complete coefficient of x in the expression for $\cos U$ and therefore are xZ, where

$$Z = \sum\frac{dA}{ds_1}p_0'p_1' + \sum Ap_0'p_1'' - \sum Ap_1'\bar{\gamma}_{01}.$$

The value of the first term in Z is derived from the formula for $\sum\sum\frac{dA}{ds_j}p_e'p_i'$ in § 307, by making $e=0$, $i=1=j$, and therefore

$$\begin{aligned}&= u_1^{(0)}\bar{\gamma}_{11} + u_2^{(0)}\bar{\delta}_{11} + u_3^{(0)}\bar{\theta}_{11} + u_4^{(0)}\bar{\phi}_{11}\\ &\qquad + u_1^{(1)}\bar{\gamma}_{01} + u_2^{(1)}\bar{\delta}_{01} + u_3^{(1)}\bar{\theta}_{01} + u_4^{(1)}\bar{\phi}_{01}\\ &= -u_1^{(0)}p_1'' - u_2^{(0)}q_1'' - u_3^{(0)}r_1'' - u_4^{(0)}t_1''\\ &\qquad + u_1^{(1)}\bar{\gamma}_{01} + u_2^{(1)}\bar{\delta}_{01} + u_3^{(1)}\bar{\theta}_{01} + u_4^{(1)}\bar{\phi}_{01}\\ &= -\sum Ap_0'p_1'' + \sum Ap_1'\bar{\gamma}_{01}:\end{aligned}$$

that is, Z vanishes. Thus the terms of the first order of small quantities in the expression for $\cos U$ disappear.

The aggregate of the terms of the second order of small quantities in the expression for $-\cos U$

$$=x^2\sum\frac{dA}{ds_1}p_0'p_1''-\ x^2\sum Ap_1''\bar{\gamma}_{01}-x^2\sum\frac{dA}{ds_1}p_1'\bar{\gamma}_{01}$$

$$+\sum Ap_1'P_0+\tfrac{1}{2}x^2\sum Ap_0'p_1'''+\tfrac{1}{2}x^2\sum\frac{d^2A}{ds_1{}^2}p_0'p_1'.$$

For the first of the six sums, we have (by § 307)

$$\begin{aligned}\sum\frac{dA}{ds_1}p_0'p_1''&\\
=\ & p_1''[A\bar{\gamma}_{01}+H\bar{\delta}_{01}+G\bar{\theta}_{01}+L\bar{\phi}_{01}+\alpha_1u_1{}^{(0)}+\xi_1u_2{}^{(0)}+\phi_1u_3{}^{(0)}+\kappa_1u_4{}^{(0)}]\\
&+q_1''[H\bar{\gamma}_{01}+B\bar{\delta}_{01}+F\bar{\theta}_{01}+M\bar{\phi}_{01}+\beta_1u_1{}^{(0)}+\eta_1u_2{}^{(0)}+\chi_1u_3{}^{(0)}+\lambda_1u_4{}^{(0)}]\\
&+r_1''[G\bar{\gamma}_{01}+F\bar{\delta}_{01}+C\bar{\theta}_{01}+N\bar{\phi}_{01}+\gamma_1u_1{}^{(0)}+\zeta_1u_2{}^{(0)}+\psi_1u_3{}^{(0)}+\mu_1u_4{}^{(0)}]\\
&+t_1''[L\bar{\gamma}_{01}+M\bar{\delta}_{01}+N\bar{\theta}_{01}+D\bar{\phi}_{01}+\delta_1u_1{}^{(0)}+\varpi_1u_2{}^{(0)}+\omega_1u_3{}^{(0)}+\nu_1u_4{}^{(0)}]\\
=\ &\sum Ap_1''\bar{\gamma}_{01}+u_1{}^{(0)}(\alpha_1p_1''+\beta_1q_1''+\gamma_1r_1''+\delta_1t_1'')\\
&\qquad+u_2{}^{(0)}(\xi_1p_1''+\eta_1q_1''+\zeta_1r_1''+\varpi_1t_1'')\\
&\qquad+u_3{}^{(0)}(\phi_1p_1''+\chi_1q_1''+\psi_1r_1''+\omega_1t_1'')\\
&\qquad+u_4{}^{(0)}(\kappa_1p_1''+\lambda_1q_1''+\mu_1r_1''+\nu_1t_1'').\end{aligned}$$

The second term in the first line of the expression for $-\cos U$ is left unmodified. For the sum in the third term of that first line, we have (again by § 307)

$$\begin{aligned}\sum\frac{dA}{ds_1}p_1'\bar{\gamma}_{01}&\\
=\ &\bar{\gamma}_{01}[-Ap_1''-Hq_1''-Gr_1''-Lt_1''+\alpha_1u_1{}^{(1)}+\xi_1u_2{}^{(1)}+\phi_1u_3{}^{(1)}+\kappa_1u_4{}^{(1)}]\\
&+\bar{\delta}_{01}[-Hp_1''-Bq_1''-Fr_1''-Mt_1''+\beta_1u_1{}^{(1)}+\eta_1u_2{}^{(1)}+\chi_1u_3{}^{(1)}+\lambda_1u_4{}^{(1)}]\\
&+\bar{\theta}_{01}[-Gp_1''-Fq_1''-Cr_1''-Nt_1''+\gamma_1u_1{}^{(1)}+\zeta_1u_2{}^{(1)}+\psi_1u_3{}^{(1)}+\mu_1u_4{}^{(1)}]\\
&+\bar{\phi}_{01}[-Lp_1''-Mq_1''-Nr_1''-Dt_1''+\delta_1u_1{}^{(1)}+\varpi_1u_2{}^{(1)}+\omega_1u_3{}^{(1)}+\nu_1u_4{}^{(1)}]\\
=\ &-\sum Ap_1''\bar{\gamma}_{01}+u_1{}^{(1)}(\alpha_1\bar{\gamma}_{01}+\beta_1\bar{\delta}_{01}+\gamma_1\bar{\theta}_{01}+\delta_1\bar{\phi}_{01})\\
&\qquad+u_2{}^{(1)}(\xi_1\bar{\gamma}_{01}+\eta_1\bar{\delta}_{01}+\zeta_1\bar{\theta}_{01}+\varpi_1\bar{\phi}_{01})\\
&\qquad+u_3{}^{(1)}(\phi_1\bar{\gamma}_{01}+\chi_1\bar{\delta}_{01}+\psi_1\bar{\theta}_{01}+\omega_1\bar{\phi}_{01})\\
&\qquad+u_4{}^{(1)}(\kappa_1\bar{\gamma}_{01}+\lambda_1\bar{\delta}_{01}+\mu_1\bar{\theta}_{01}+\nu_1\bar{\phi}_{01}).\end{aligned}$$

For the present, the first term in the second line of the expression for $-\cos U$ is left unaltered. For the coefficient of $\tfrac{1}{2}x^2$ in the second term in that second line, we have

$$\begin{aligned}\sum Ap_0'p_1'''&=u_1{}^{(0)}p_1'''+u_2{}^{(0)}q_1'''+u_3{}^{(0)}r_1'''+u_4{}^{(0)}t_1'''\\
&=-u_1{}^{(0)}(\Gamma_{300}p_1'^3)-u_2{}^{(0)}(\Delta_{300}p_1'^3)-u_3{}^{(0)}(\Theta_{300}p_1'^3)-u_4{}^{(0)}(\Phi_{300}p_1'^3).\end{aligned}$$

Finally, for the coefficient of $\frac{1}{2}x^2$ in the third term in the second line of the expression for $-\cos U$, we use the result of § 308, making $i=0$, $j=1$, $k=1$, so that all the variables $t_{\gamma\delta}$ vanish; and we have

$$\begin{aligned}\sum \frac{d^2A}{ds_1^2} p_0' p_1' \\
= \;& u_1^{(0)}(\Gamma_{300}p_1'^3) + u_2^{(0)}(\varDelta_{300}p_1'^3) + u_3^{(0)}(\Theta_{300}p_1'^3) + u_4^{(0)}(\Phi_{300}p_1'^3) \\
& + u_1^{(1)}(\Gamma_{300}p_0'p_1'^2) + u_2^{(1)}(\varDelta_{300}p_0'p_1'^2) + u_3^{(1)}(\Theta_{300}p_0'p_1'^2) + u_4^{(1)}(\Phi_{300}p_0'p_1'^2) \\
& -2\left(\sum Ap_1''\bar{\gamma}_{01}\right) \\
& + 2u_1^{(1)}\{\alpha_1\bar{\gamma}_{01} + \beta_1\bar{\delta}_{01} + \gamma_1\bar{\theta}_{01} + \delta_1\bar{\phi}_{01}\} - 2u_1^{(0)}\{\alpha_1 p_1'' + \beta_1 q_1'' + \gamma_1 r_1'' + \delta_1 t_1''\} \\
& + 2u_2^{(1)}\{\xi_1\bar{\gamma}_{01} + \eta_1\bar{\delta}_{01} + \zeta_1\bar{\theta}_{01} + \varpi_1\bar{\phi}_{01}\} - 2u_2^{(0)}\{\xi_1 p_1'' + \eta_1 q_1'' + \zeta_1 r_1'' + \varpi_1 t_1''\} \\
& + 2u_3^{(1)}\{\phi_1\bar{\gamma}_{01} + \chi_1\bar{\delta}_{01} + \psi_1\bar{\theta}_{01} + \omega_1\bar{\phi}_{01}\} - 2u_3^{(0)}\{\phi_1 p_1'' + \chi_1 q_1'' + \psi_1 r_1'' + \omega_1 t_1''\} \\
& + 2u_4^{(1)}\{\kappa_1\bar{\gamma}_{01} + \lambda_1\bar{\delta}_{01} + \mu_1\bar{\theta}_{01} + \nu_1\bar{\phi}_{01}\} - 2u_4^{(0)}\{\kappa_1 p_1'' + \lambda_1 q_1'' + \mu_1 r_1'' + \nu_1 t_1''\}.\end{aligned}$$

Let these values of the various sets of terms be substituted in the foregoing aggregate of terms of the second order, which (to that order of small quantities) is equal to $-\cos U + \cos U_0$; then, after reduction, we find

$$\begin{aligned}-\cos U + \cos U_0 = & \sum Ap_1'P_0 \\
& + \tfrac{1}{2}x^2\{u_1^{(1)}(\Gamma_{300}p_0'p_1'^2) + u_2^{(1)}(\varDelta_{300}p_0'p_1'^2) + u_3^{(1)}(\Theta_{300}p_0'p_1'^2) + u_4^{(1)}(\Phi_{300}p_0'p_1'^2)\}.\end{aligned}$$

Let the values of the magnitudes P_0, Q_0, R_0, S_0, be inserted in the first summation on the right-hand side. The terms involving quantities of the type Γ_{300} disappear. The aggregate of terms involving K

$$\begin{aligned}&= \tfrac{1}{6}\left(K\frac{x^2y^2}{w^2}\sin^2\epsilon\right)\sum Ap_1'p_0' \\
&= -\tfrac{1}{6}K\frac{x^2y^2}{w^2}\sin^2\epsilon\cos U_0:\end{aligned}$$

or, as

$$x\sin\epsilon = w\sin V_0, \quad y\sin\epsilon = w\sin U_0,$$

this aggregate of terms

$$= -\tfrac{1}{6}Kxy\sin U_0\sin V_0\cos U_0.$$

Finally, the aggregate of terms involving the quantities $K_\mu(0,\,10)$

$$\begin{aligned}&= \frac{wx}{3\Omega}\sum\left[Ap_1'\left\{\sum_\mu a_{1\mu}K_\mu(0,\,10)\right\}\right] \\
&= \tfrac{1}{3}wx\{p_1'K_1(0,\,10) + q_1'K_2(0,\,10) + r_1'K_3(0,\,10) + t_1'K_4(0,\,10)\}:\end{aligned}$$

or, when the values of $K_\mu(0,\,10)$ are inserted and the terms are collected, this aggregate

$$\begin{aligned}&= \tfrac{1}{3}wx\sum(\alpha\beta,\,\gamma\delta)\,s_{\alpha\beta}s_{\gamma\delta} \\
&= \tfrac{1}{3}wx\frac{y^2}{w^2}\sum(\alpha\beta,\,\gamma\delta)\,\xi_{\alpha\beta}\xi_{\gamma\delta},\end{aligned}$$

with the former notation for the orientation-variables $\xi_{\alpha\beta}$ of the orientation of the domain at O. When the Riemann measure K is introduced, this becomes

$$=\tfrac{1}{3}\frac{xy^2}{w}K\sin^2\epsilon=\tfrac{1}{3}Kxy\sin\epsilon\sin U_0.$$

Thus the equation becomes

$$-\cos U+\cos U_0=\tfrac{1}{3}Kxy\sin\epsilon\sin U_0-\tfrac{1}{6}Kxy\sin U_0\sin V_0\cos U_0,$$

accurate up to the second order of small quantities inclusive.

It follows that $U-U_0$ is a small quantity, the most significant part of which is of the second order; and therefore, as

$$\cos U-\cos U_0=-(U-U_0)\sin U_0$$

up to this second order of small quantities, we obtain

$$U-U_0=\tfrac{1}{3}Kxy\sin\epsilon-\tfrac{1}{6}Kxy\cos U_0\sin V_0,$$

as giving the most important term in $U-U_0$, for a small geodesic triangle in the domain.

We immediately infer the corresponding result

$$V-V_0=\tfrac{1}{3}Kxy\sin\epsilon-\tfrac{1}{6}Kxy\cos V_0\sin U_0.$$

Now the angles of this small geodesic triangle are ϵ, U, V; and, for a domain as for a region, we use the phrase *angular excess* of a geodesic triangle to denote the excess of the sum of the angles of the geodesic triangle over the sum of the angles of a plane rectilinear triangle. Thus the angular excess of the geodesic triangle

$$\begin{aligned}&=\epsilon+U+V-\pi\\&=U-U_0+V-V_0\\&=\tfrac{1}{2}Kxy\sin\epsilon,\end{aligned}$$

because $\epsilon+U_0+V_0=\pi$; and therefore it follows that the area of the small geodesic triangle in the domain

$$=\frac{1}{K}\text{(angular excess of the triangle)}.$$

Having regard to the corresponding property of a sphere in triple homaloidal space, we call K the *sphericity*; or the Riemann measure of curvature of the domain in any orientation is the sphericity of the domain in that orientation.

In this result, the quantity K denotes the Riemann measure, at O, of the superficial curvature of the domain estimated in the orientation at O determined by the directions of the two domainal geodesics OUA and OVB. The sphericity is a variable magnitude depending upon position in the domain as well as upon orientation at any position. At U, for an orientation determined by the directions of the two geodesics UA and UV, there is a double variation from O. Were the orientation at U the same as at O, the magnitude of K at U would differ from its value at O by an amount, the most significant part of which is a small quantity of

the first order. But, as will appear later (§ 381), the orientation at U differs from the orientation at O; the domainal geodesic UV meets, in U and in V, the surface which is geodesic at O to the domain, but it does not lie in that surface; and thus the surface, which (through the directions UA and UV) is geodesic to the domain at U, is not the same as the surface VOU which is geodesic to the domain at O. The differences are small magnitudes, because the domainal geodesic triangle UOV is small; and they do not enter into the obtained expression for the angular excess of the domainal geodesic triangle, because ultimately only the most significant term in that excess has been retained.

Should the approximations be carried further, to the next order of small quantities, such differences would appear in the results. Even without further approximations, the matter has to be considered in connection with the parallelism of geodesics in the domain (Chapter XXXII).

Principal values and principal orientations of the sphericity at any place.

313. To find the maximum and the minimum values of the sphericity of a domain at a place O, among those which are provided by all the possible superficial orientations of the domain, we make K, where

$$K=\sum (ij,\, kl) s_{ij} s_{kl},$$

a maximum or a minimum for all the variables s_{ij}, which are subject to the organic condition

$$1=U=\sum (A_{ik}A_{jl}-A_{il}A_{jk})s_{ij}s_{kl}=\sum \{(ij,\, kl)_A s_{ij} s_{kl}\},$$

and to the identical relation

$$s_{23}s_{34}+s_{31}s_{24}+s_{12}s_{34}=0,$$

the existence of the latter affecting the possible forms of K which are equal to one another.

There are six critical equations *:

$$\frac{\partial K}{\partial s_{23}}=\theta\frac{\partial U}{\partial s_{23}}+\mu s_{14}, \qquad \frac{\partial K}{\partial s_{14}}=\theta\frac{\partial U}{\partial s_{14}}+\mu s_{23},$$

$$\frac{\partial K}{\partial s_{31}}=\theta\frac{\partial U}{\partial s_{31}}+\mu s_{24}, \qquad \frac{\partial K}{\partial s_{24}}=\theta\frac{\partial U}{\partial s_{24}}+\mu s_{31},$$

$$\frac{\partial K}{\partial s_{12}}=\theta\frac{\partial U}{\partial s_{12}}+\mu s_{34}, \qquad \frac{\partial K}{\partial s_{34}}=\theta\frac{\partial U}{\partial s_{34}}+\mu s_{12},$$

* If we had to deal with mixed concomitants of the system, involving line-variables, surface-variables, and region-variables, the critical equations would have a different form. Utilising Lie's method of continuous groups, we should obtain the small variations of the variables s_{ij} consequent upon the general continuous variations of the magnitudes of the type p', the general continuous variations of the magnitudes of the region-variables being contragredient to those of the line-variables. Where the only variables, which occur, are the surface-variables, it is sufficient to proceed as in the text.

where θ and μ are multipliers left undetermined in the construction of the critical equations.

The value of θ can be obtained at once. Multiply the six equations by $s_{23}, s_{31}, s_{12}, s_{14}, s_{24}, s_{34}$, respectively, and add the products: when account is taken of the conditions and of the value of K, the result is

$$K=\theta,$$

so that the quantity μ remains for consideration.

The persistence of the identical relation allows a magnitude, which involves the surface-variables, to assume a variety of expressions, apparently distinct in form, intrinsically equivalent in value. Thus a quantity such as K can have the form

$$K+\lambda(s_{23}s_{14}+s_{31}s_{24}+s_{12}s_{34}),$$

where λ is entirely arbitrary, without altering its value. To avoid the lack of definiteness in form, it is convenient to choose that form of a quantity Ξ which satisfies the universal covariantive differential equation

$$\frac{\partial^2\Xi}{\partial s_{23}\partial s_{14}}+\frac{\partial^2\Xi}{\partial s_{31}\partial s_{24}}+\frac{\partial^2\Xi}{\partial s_{12}\partial s_{34}}=0;$$

and the form, thus selected, is taken to be central form of reference for the value. It happens that K satisfies the equation, because

$$(23, 14)+(31, 24)+(12, 34)=0,$$

and that U also satisfies the equation, because the relation

$$(23, 14)_A+(31, 24)_A+(12, 34)_A=0$$

is satisfied identically.

Accordingly, we can take $K+\lambda(s_{23}s_{14}+s_{31}s_{24}+s_{12}s_{34})$ as the form instead of K without affecting the value of the sphericity, where λ is quite arbitrary, though free from the variables s_{ij}; and in this form, the critical equations for the maximum or the minimum of the sphericity become

$$\frac{\partial K}{\partial s_{23}}+\lambda s_{14}=K\frac{\partial U}{\partial s_{23}}+\mu s_{14},$$

with five others of the same type. As λ is an arbitrary quantity at our disposal, we choose $\lambda=\mu$; and now the critical equations are

$$\frac{\partial K}{\partial s_{ij}}-K\frac{\partial U}{\partial s_{ij}}=0,$$

for all the six significant combinations of i and j.

The six equations are homogeneous and linear in the six variables s_{ij}. It follows that K is determined by the sextic equation

$$\|\,(ij, kl)-K(ij, kl)_A\,\|=0,$$

the coefficient of K^6 in which is Ω^3; and therefore there are six principal values of the sphericity of the domain.

Moreover, each principal value of the sphericity belongs to a particular orientation; and therefore the domain has six principal orientations. Any two of these superficial orientations are perpendicular to one another; for if s_{ij} denote the variables for a principal value K_1, and if t_{ij} denote them for a principal value K_2, we have

$$\sum t_{ij} \frac{\partial K}{\partial s_{ij}} - K_1 \sum t_{ij} \frac{\partial U}{\partial s_{ij}} = 0,$$

$$\sum s_{ij} \frac{\partial K}{\partial t_{ij}} - K_2 \sum s_{ij} \frac{\partial U}{\partial t_{ij}} = 0.$$

But

$$\sum t_{ij} \frac{\partial K}{\partial s_{ij}} = 2 \sum (ij, kl) s_{ij} t_{kl} = \sum s_{ij} \frac{\partial K}{\partial t_{ij}},$$

$$\sum t_{ij} \frac{\partial U}{\partial s_{ij}} = 2 \sum \{(ij, kl)_A s_{ij} t_{kl}\} = \sum s_{ij} \frac{\partial U}{\partial t_{ij}};$$

and therefore

$$\sum (ij, kl) s_{ij} t_{kl} = 0, \quad \sum \{(ij, kl)_A s_{ij} t_{kl}\} = 0,$$

the second of which inferences shews that the superficial orientations s_{ij} and t_{ij} are perpendicular.

The six principal orientations determine four principal lines which are orthogonal to one another, the whole set constituting an orthogonal system of coordinate axes and coordinate planes within the tangent block of the domain.

Before proceeding further with the properties of the sphericity in a domain, we require the properties of regions (especially geodesic regions) and of surfaces (especially geodesic surfaces).

CHAPTER XXVII

Regions in a Domain : Geodesic Regions

Parametric region : its tangent flat.

314. When a region is wholly contained in a domain, one simple analytical representation of the region is provided by a single relation

$$\epsilon(p, q, r, t)=0$$

between the parameters of the domain, as in § 269. We shall now consider some of the properties of such a region, not now regarded as a configuration in the plenary homaloidal space of the domain, but regarded as a configuration within the domain.

Any direction in the region, being a direction

$$dy=y_1\,dp+y_2\,dq+y_3\,dr+y_4\,dt$$

in the domain with increments restricting the range to the region, must have these increments subject to the relation

$$0=\epsilon_1\,dp+\epsilon_2\,dq+\epsilon_3\,dr+\epsilon_4\,dt.$$

Consequently, any point on the line through O in the specified direction is given by the typical equation

$$\begin{aligned}\bar{y}-y&=\theta\,dy\\ &=\lambda y_1+\mu y_2+\nu y_3+\kappa y_4,\end{aligned}$$

where

$$\lambda=\theta\,dp, \quad \mu=\theta\,dq, \quad \nu=\theta\,dr, \quad \kappa=\theta\,dt,$$

that is, the parameters λ, μ, ν, κ, are subject to the condition

$$\epsilon_1\lambda+\epsilon_2\mu+\epsilon_3\nu+\epsilon_4\kappa=0.$$

Now the equations, typified by the foregoing equation in $\bar{y}-y$, are those of a triple homaloidal space, that is, of a flat; it is the locus of lines through O touching the part of the domain given by the region, and therefore it is the tangent flat of the region. The equations of the tangent flat are typified by

$$\bar{y}-y=\lambda y_1+\mu y_2+\nu y_3+\kappa y_4,$$

the four parameters in which are subject to the single condition

$$\epsilon_1\lambda+\epsilon_2\mu+\epsilon_3\nu+\epsilon_4\kappa=0.$$

Let a perpendicular, of length Π and of direction-cosines typified by Y_0, be drawn upon this tangent flat from a point Q of the region, with coordinates $\xi_1, \xi_2, \ldots$, typified by ξ, near O and at a small arc-distance δ along a regional curve OQ; and let the foot of this perpendicular be the point with the foregoing typical coordinate $\bar{y}$. Then

$$\begin{aligned} Y_0\Pi &= \xi - \bar{y} \\ &= \xi - (y + \lambda y_1 + \mu y_2 + \nu y_3 + \kappa y_4), \end{aligned}$$

and

$$\Pi^2 = \sum \{\xi - (y + \lambda y_1 + \mu y_2 + \nu y_3 + \kappa y_4)\}^2.$$

Because the line thus drawn is perpendicular to the tangent flat, the length of the line will be obtained by selecting the values of λ, μ, ν, κ, which will secure a minimum value of Π^2, these values of λ, μ, ν, κ, being subject to the foregoing condition. Hence the critical equations are

$$\sum y_\iota \{\xi - (y + \lambda y_1 + \mu y_2 + \nu y_3 + \kappa y_4)\} = Q\epsilon_\iota,$$

for $\iota = 1, 2, 3, 4$, where Q is a multiplier left undetermined in the construction of the critical equations.

In the first place, the equations can be written in the form

$$\Pi \sum y_\iota Y_0 = Q\epsilon_\iota;$$

and therefore

$$\Pi[\sum \{Y_0(y_1\lambda + y_2\mu + y_3\mu + y_4\nu)\}] = Q(\epsilon_1\lambda + \epsilon_2\mu + \epsilon_3\nu + \epsilon_4\kappa) = 0:$$

or, as $y_1\lambda + y_2\mu + y_3\mu + y_3\nu$ is the direction in the flat, the perpendicular is at right angles to that direction.

Again, the equations can be transformed so as to become, for $\iota = 1, 2, 3, 4$,

$$A_{1\iota}\lambda + A_{2\iota}\mu + A_{3\iota}\nu + A_{4\iota}\kappa + Q\epsilon_\iota = \sum \{y_\iota(\xi - y)\}.$$

Now, for the point Q in the region near O, we have

$$\xi - y = y'\delta + \tfrac{1}{2}y_0''\delta^2 + \ldots,$$

where the unexpressed terms contain powers of δ higher than the second, and where y_0'' implies continued arc-derivation along the curve drawn in the region in the direction y'. We have

$$y' = y_1p' + y_2q' + y_3r' + y_4t';$$

and therefore the coefficient of δ in the value of $\sum \{y_\iota(\xi - y)\}$

$$= A_{1\iota}p' + A_{2\iota}q' + A_{3\iota}r' + A_{4\iota}t'.$$

Again, denoting by p_0'', q_0'', r_0'', t_0'', the second variations of the domainal parameters taken along the regional curve, we have

$$y_0'' = y_1p_0'' + y_2q_0'' + y_3r_0'' + y_4t_0'' + \sum_j \sum_k y_{jk}x_j'x_k',$$

with the customary convention $x_k' = p', q', r', t'$, according as $k=1, 2, 3, 4$. Hence the coefficient of $\frac{1}{2}\delta^2$ in the value of $\sum \{y_i(\xi - y)\}$

$$\begin{aligned}
&= A_{1i}p_0'' + A_{2i}q_0'' + A_{3i}r_0'' + A_{4i}t_0'' + \sum_j \sum_k \{(\sum y_i y_{jk}) x_j' x_k'\} \\
&= A_{1i}\{p_0'' + \sum_j \sum_k (\Gamma_{jk} x_j' x_k')\} \\
&\qquad + A_{2i}\{q_0'' + \sum_j \sum_k (\varDelta_{jk} x_j' x_k')\} \\
&\qquad\quad + A_{3i}\{r_0'' + \sum_j \sum_k (\Theta_{jk} x_j' x_k')\} \\
&\qquad\qquad + A_{4i}\{t_0'' + \sum_j \sum_k \Phi_{jk} x_j' x_k'\}.
\end{aligned}$$

Consequently, when powers of δ higher than the second are neglected, the equations become, for $i=1, 2, 3, 4$,

$$\begin{aligned}
&A_{1i}[\lambda - p'\delta - \tfrac{1}{2}\delta^2\{p_0'' + \sum_j \sum_k (\Gamma_{jk} x_j' x_k')\}] \\
&\quad + A_{2i}[\mu - q'\delta - \tfrac{1}{2}\delta^2\{q_0'' + \sum_j \sum_k (\varDelta_{jk} x_j' x_k')\}] \\
&\qquad + A_{3i}[\nu - r'\delta - \tfrac{1}{2}\delta^2\{r_0'' + \sum_j \sum_k (\Theta_{jk} x_j' x_k')\}] \\
&\qquad\quad + A_{4i}[\kappa - t'\delta - \tfrac{1}{2}\delta^2\{t_0'' + \sum_j \sum_k (\Phi_{jk} x_j' x_k')\}] = -Q\epsilon_i.
\end{aligned}$$

When the four equations are resolved, they give

$$\begin{aligned}
\lambda - p'\delta - \tfrac{1}{2}\delta^2\{p_0'' + \sum_j \sum_k (\Gamma_{jk} x_j' x_k')\} &= -\frac{Q}{\Omega}(a\epsilon_1 + h\epsilon_2 + g\epsilon_3 + l\epsilon_4), \\
\mu - q'\delta - \tfrac{1}{2}\delta^2\{q_0'' + \sum_j \sum_k (\varDelta_{jk} x_j' x_k')\} &= -\frac{Q}{\Omega}(h\epsilon_1 + b\epsilon_2 + f\epsilon_3 + m\epsilon_4), \\
\nu - r'\delta - \tfrac{1}{2}\delta^2\{r_0'' + \sum_j \sum_k (\Theta_{jk} x_j' x_k')\} &= -\frac{Q}{\Omega}(g\epsilon_1 + f\epsilon_2 + c\epsilon_3 + n\epsilon_4), \\
\kappa - t'\delta - \tfrac{1}{2}\delta^2\{t_0'' + \sum_j \sum_k (\Phi_{jk} x_j' x_k')\} &= -\frac{Q}{\Omega}(l\epsilon_1 + m\epsilon_2 + n\epsilon_3 + d\epsilon_4).
\end{aligned}$$

Now the parameters $\lambda, \mu, \nu, \kappa$, are subject to the condition

$$\epsilon_1\lambda + \epsilon_2\mu + \epsilon_3\nu + \epsilon_4\kappa = 0;$$

and the direction p', q', r', t', touches the region $\epsilon(p, q, r, t)=0$, so that

$$\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0.$$

Hence, when the four resolved equations are multiplied by $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, and the products are added, we find

$$\tfrac{1}{2}\delta^2 E = \frac{Q}{\Omega}\sum a\epsilon_1^2,$$

where

$$\begin{aligned}
E = \epsilon_1\{p_0'' + \sum_j \sum_k (\Gamma_{jk} x_j' x_k')\} + \epsilon_2\{q_0'' + \sum_j \sum_k (\varDelta_{jk} x_j' x_k')\} \\
+ \epsilon_3\{r_0'' + \sum_j \sum_k (\Theta_{jk} x_j' x_k')\} + \epsilon_4\{t_0'' + \sum_j \sum_k (\Phi_{jk} x_j' x_k')\}.
\end{aligned}$$

As the second arc-derivations are effected along a curve lying in the region

$$\epsilon(p, q, r, t)=0,$$

we have

$$\epsilon_1 p_0''+\epsilon_2 q_0''+\epsilon_3 r_0''+\epsilon_4 t_0''+\sum_j\sum_k \epsilon_{jk} x_j' x_k'=0\,;$$

and therefore, with the symbols defined in § 269 by

$$\bar{\epsilon}_{jk}=\epsilon_{jk}-\epsilon_1 \Gamma_{jk}-\epsilon_2 \Delta_{jk}-\epsilon_3 \Theta_{jk}-\epsilon_4 \Phi_{jk},$$

we have

$$-E=\sum_j\sum_k \bar{\epsilon}_{jk} x_j' x_k'.$$

Also (§ 269),

$$\sum a\epsilon_1{}^2=\Omega\epsilon_n{}^2\,;$$

and therefore

$$Q=-\tfrac{1}{2}\frac{\delta^2}{\epsilon_n{}^2}\sum_j\sum_k \bar{\epsilon}_{jk} x_j' x_k'\,;$$

thus a value for the undetermined quantity Q is obtained.

Consequently, the values of λ, μ, ν, κ, Q, can be regarded as known, up to the second order of small quantities. Now

$$\begin{aligned} Y_0\Pi &= \xi-(y+y_1\lambda+y_2\mu+y_3\nu+y_4\kappa)\\ &= y'\delta+\tfrac{1}{2}y_0''\delta^2-(y_1\lambda+y_2\mu+y_3\nu+y_4\kappa), \end{aligned}$$

when powers of δ higher than the second are omitted. When the values of λ, μ, ν, κ, are substituted in $y_1\lambda+y_2\mu+y_3\nu+y_4\kappa$, the term involving the first power of δ

$$=(y_1p'+y_2q'+y_3r'+y_4t')\delta=y'\delta\,;$$

the term involving δ^2 has, for the coefficient of $\tfrac{1}{2}\delta^2$, the quantity

$$\begin{aligned} &y_1\{p_0''+\sum_j\sum_k(\Gamma_{jk}x_j'x_k')\}+y_2\{q_0''+\sum_j\sum_k(\Delta_{jk}x_j'x_k')\}\\ &+y_3\{\gamma_0''+\sum_j\sum_k(\Theta_{jk}x_j'x_k')\}+y_4\{t_0''+\sum_j\sum_k(\Phi_{jk}x_j'x_k')\}\,; \end{aligned}$$

and the term involving Q

$$\begin{aligned} &=-\frac{Q}{\Omega}\sum y_i(a_{1i}\epsilon_1+a_{2i}\epsilon_2+a_{3i}\epsilon_3+a_{4i}\epsilon_4)\\ &=-Q\epsilon_n\left(y_1\frac{dp}{dn}+y_2\frac{dq}{dn}+y_3\frac{dr}{dn}+y_4\frac{dt}{dn}\right)=-Q\epsilon_n\frac{dy}{dn}, \end{aligned}$$

by the formulæ in § 269, where $\frac{dy}{dn}$ is the typical direction-cosine of the domainal normal to the region. Hence

$$\begin{aligned} Y_0\Pi=\tfrac{1}{2}\delta^2[y_0''&-y_1\{p_0''+\sum_j\sum_k(\Gamma_{jk}x_j'x_k')\}-y_2\{q_0''+\sum_j\sum_k(\Delta_{jk}x_j'x_k')\}\\ &-y_3\{r_0''+\sum_j\sum_k(\Theta_{jk}x_j'x_k')\}-y_4\{t_0''+\sum\sum_k(\Phi_{jk}x_j'x_k')\}]+Q\epsilon_n\frac{dy}{dn}. \end{aligned}$$

Along the curve in the region, in the direction p', q', r', t', we have

$$y_0''=y_1p_0''+y_2q_0''+y_3r_0''+y_4t_0''+\sum_j\sum_k y_{jk}x_j'x_k',$$

so that the coefficient of $\frac{1}{2}\delta^2$

$$=\sum_j\sum_k\{(y_{jk}-y_1\Gamma_{jk}-y_2\Delta_{jk}-y_3\Theta_{jk}-y_4\Phi_{jk})x_j'x_k'\}$$

$$=\sum_j\sum_k\eta_{jk}x_j'x_k'=\frac{Y}{\rho},$$

where Y is the typical direction-cosine and ρ is the radius of curvature of the domainal geodesic in the direction p', q', r', t'. We write

$$-\frac{1}{\gamma}=\frac{1}{\epsilon_n}\sum_j\sum_k\bar{\epsilon}_{jk}x_j'x_k',$$

and shall obtain later (§ 316) the significance of this quantity γ; and thus the equation for Π becomes

$$Y_0\Pi=\tfrac{1}{2}\delta^2\left(\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn}\right).$$

Let the limit of the magnitude $2\Pi/\delta^2$, as δ diminishes to zero, be denoted by $1/\rho_0$; then

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{1}{\gamma}\frac{dy}{dn}.$$

Geodesics in a domainal region.

315. To obtain the intrinsic equations of geodesics of a region

$$\epsilon(p, q, r, t)=0$$

contained in the domain, we have to make the integral

$$\sum\left\{A\left(\frac{dp}{du}\right)^2\right\}^{\frac{1}{2}}du$$

a minimum, for all values of p, q, r, t, subject to the parametric equation $\epsilon=0$ defining the region. There are four critical equations which, after the customary march of the analysis, can be expressed in the form

$$p_\epsilon''+\sum\Gamma_{11}p'^2=W\sum a\epsilon_1=W\epsilon_n\Omega\frac{dp}{dn},$$

$$q_\epsilon''+\sum\Delta_{11}p'^2=W\sum h\epsilon_1=W\epsilon_n\Omega\frac{dq}{dn},$$

$$r_\epsilon''+\sum\Theta_{11}p'^2=W\sum g\epsilon_1=W\epsilon_n\Omega\frac{dr}{dn},$$

$$t_\epsilon''+\sum\Phi_{11}p'^2=W\sum l\epsilon_1=W\epsilon_n\Omega\frac{dt}{dn},$$

with the notation of § 269, the quantity W being the multiplier left undetermined in the construction of the critical equations; while the quantities p_ϵ'', q_ϵ'', r_ϵ'', t_ϵ'', are the second arc-derivatives of the parameters along the geodesic of the region $\epsilon=0$ drawn in the direction p', q', r', t'.

To obtain W, we take the second arc-derivative along a regional direction, so that

$$\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0,$$
$$\epsilon_1 p_\epsilon'' + \epsilon_2 q_\epsilon'' + \epsilon_3 r_\epsilon'' + \epsilon_4 t_\epsilon'' + \sum_j \sum_k \epsilon_{jk} x_j' x_k' = 0,$$

with the convention $x_i' = p'$, q', r', t', according as $i=1, 2, 3, 4$. Let the four intrinsic equations of regional geodesics in the domain be multiplied by ϵ_1, ϵ_2, ϵ_3, ϵ_4, respectively, and let the products be added; then with the definitions of the quantities $\bar{\epsilon}_{jk}$ on p. 254, the resulting relation, by the use of the second derivative of the ϵ-equation, becomes

$$-\sum_j \sum_k (\bar{\epsilon}_{jk} x_j' x_k') = W \epsilon_n \Omega \left(\epsilon_1 \frac{dp}{dn} + \epsilon_2 \frac{dq}{dn} + \epsilon_3 \frac{dr}{dn} + \epsilon_4 \frac{dt}{dn}\right)$$
$$= W \Omega \epsilon_n^2 :$$

or if we write, as in § 314,

$$-\frac{1}{\gamma} = \frac{1}{\epsilon_n} \sum_j \sum_k \bar{\epsilon}_{jk} x_j' x_k',$$

we have

$$W \epsilon_n \Omega = \frac{1}{\gamma}.$$

The significance of γ will be determined later (p. 388); meanwhile, the intrinsic equations of a regional geodesic in the domain become

$$\left.\begin{aligned}
p_\epsilon'' + \sum \Gamma_{11} p'^2 &= \frac{1}{\gamma}\frac{dp}{dn} \\
q_\epsilon'' + \sum \Delta_{11} p'^2 &= \frac{1}{\gamma}\frac{dq}{dn} \\
r_\epsilon'' + \sum \Theta_{11} p'^2 &= \frac{1}{\gamma}\frac{dr}{dn} \\
t_\epsilon'' + \sum \Phi_{11} p'^2 &= \frac{1}{\gamma}\frac{dt}{dn}
\end{aligned}\right\},$$

the quantities $\dfrac{dp}{dn}$, $\dfrac{dq}{dn}$, $\dfrac{dr}{dn}$, $\dfrac{dt}{dn}$, being the direction-variables of the domainal normal to the region.

These equations, apparently four in number, amount to only two independent equations, additional to the equation

$$\epsilon(p, q, r, t)=0$$

of the region and to the permanent domainal equation

$$\sum Ap'^2=1,$$

each of which can be regarded as a permanent integral of the intrinsic equations of the geodesics of the region $\epsilon=0$ in the domain.

Domainal flexure of a regional geodesic.

316. We define the *domainal flexure* of a regional geodesic in the domain as its arc-rate of deviation from the domainal geodesic drawn in the same direction, the two geodesics thus touching one another.

Along the domainal geodesic through p', q', r', t', at O, take a point P at a small arc-distance δ from O; and along the regional geodesic in the same direction, take a point Q at an equal small arc-distance δ from O. The desired result can be obtained in two ways: by orbicular representation: and by space-deviation.

The directions of the two geodesics at O are the same: their common typical direction-cosine is y'. At P, the typical direction-cosine of the domainal geodesic is $y'+y''\delta$, when squares and higher powers of δ are neglected; and at Q, the typical direction-cosine of the regional geodesic is $y'+y_\epsilon''\delta$, to the same approximation. Hence, in an orbicular representation (that is, in a configuration $\sum y'^2=1$), the components of the deviation of the regional geodesic from the domainal geodesic are typified by

$$\delta(y_\epsilon''-y'').$$

Let $1/\gamma_\epsilon$ denote the magnitude of the domainal flexure, and let $\overline{Y}$ temporarily denote the typical direction-cosine of the deviation, so that, as the deviation, by definition, is connected with the flexure by the relation

$$\frac{1}{\gamma_\epsilon}=\frac{1}{\delta}(\text{deviation}),$$

we have

$$\frac{\overline{Y}}{\gamma_\epsilon}=y_\epsilon''-y''.$$

But

$$y''=\frac{Y}{\rho}=\sum_j\sum_k y_{jk}x_j'x_k'+y_1p''+y_2q''+y_3r''+y_4t'',$$

$$y_\epsilon''=\sum_j\sum_k y_{jk}x_j'x_k'+y_1p_\epsilon''+y_2q_\epsilon''+y_3r_\epsilon''+y_4t_\epsilon'';$$

and therefore

$$\begin{aligned}\frac{\overline{Y}}{\gamma_\epsilon} &= y_1(p_\epsilon''-p'')+y_2(q_\epsilon''-q'')+y_3(r_\epsilon''-r'')+y_4(t_\epsilon-t'')\\ &= y_1(p_\epsilon''+\sum \Gamma_{11}p'^2)+y_2(q_\epsilon''+\sum \Delta_{11}q'^2)\\ &\quad +y_3(r_\epsilon''+\sum \Theta_{11}p'^2)+y_4(t_\epsilon''+\sum \Phi_{11}p'^2)\\ &= \frac{1}{\gamma}\left(y_1\frac{dp}{dn}+y_2\frac{dq}{dn}+y_3\frac{dr}{dn}+y_4\frac{dt}{dn}\right)=\frac{1}{\gamma}\frac{dy}{dn}.\end{aligned}$$

As this equation is typical, we have

$$\overline{Y}=\frac{dy}{dn}$$

as typical of all the direction-cosines: that is, the direction of the domainal flexure of a regional geodesic is along the line of the domainal normal to the region. We also have

$$\frac{1}{\gamma}=\frac{1}{\gamma_\epsilon}=-\frac{1}{\epsilon_n}\sum_j\sum_k \bar{\epsilon}_{jk}x_j'x_k';$$

and therefore the quantity $\frac{1}{\gamma}$, in the intrinsic equations of a regional geodesic, is the domainal flexure of that geodesic. We thus have

$$y_\epsilon''-y''=\frac{\overline{Y}}{\gamma_\epsilon}=\frac{1}{\gamma_\epsilon}\frac{dy}{dn}.$$

Let Y_ϵ denote the typical direction-cosine of the radius of circular curvature of the regional geodesic and $1/\rho_\epsilon$ denote the magnitude of that circular curvature, so that

$$y_\epsilon''\rho_\epsilon=Y_\epsilon.$$

Also we have $y''\rho=Y$, for the domainal geodesic in the same direction p', q', r', t'; and therefore

$$\frac{Y_\epsilon}{\rho_\epsilon}=\frac{Y}{\rho}+\frac{1}{\gamma_\epsilon}\frac{dy}{dn}.$$

There is another mode of obtaining the same result without reference to the orbicular representation, though ultimately using the same analysis. The typical space-coordinate of the point P on the domainal geodesic is

$$y+y'\delta+\tfrac{1}{2}y''\delta^2,$$

when powers of δ higher than the second are neglected; and, to the same order of approximation, the typical space-coordinate of the point Q on the regional geodesic is

$$y+y'\delta+\tfrac{1}{2}y_\epsilon''\delta^2.$$

Hence the typical component of the space-deviation of the regional geodesic from the domainal geodesic at points P and Q, at an equal arc-distance δ from O, is

$$=\tfrac{1}{2}(y_\epsilon'' - y'')\delta^2.$$

Let the space-deviation be denoted by ∇, and the typical direction-cosine of its direction be denoted by $\overline{Y}$; then

$$\overline{Y}\nabla = \tfrac{1}{2}(y_\epsilon'' - y'')\delta^2,$$

$$\delta^2 = 2\gamma_\epsilon \nabla,$$

and therefore

$$\frac{\overline{Y}}{\gamma_\epsilon} = y_\epsilon'' - y'',$$

the same central equation as before, leading to the general characteristic relation

$$\frac{Y_\epsilon}{\rho_\epsilon} = \frac{Y}{\rho} + \frac{1}{\gamma_\epsilon}\frac{dy}{dn}.$$

Various inferences can be derived from this general relation.

(i) In the first place, the domainal normal to the region lies within the tangent block of the domain; the prime normal of a domainal geodesic is orthogonal to the block, and is therefore at right angles to every direction in the block; hence

$$\sum Y\frac{dy}{dn} = 0.$$

As $\sum Y_\epsilon^2 = 1$, $\sum Y^2 = 1$, $\sum \left(\frac{dy}{dn}\right)^2 = 1$, it follows that the circular curvature of the domainal geodesic is connected with the circular curvature and the domainal flexure of the regional geodesic by the equation

$$\frac{1}{\rho_\epsilon^2} = \frac{1}{\rho^2} + \frac{1}{\gamma_\epsilon^2}.$$

(ii) In the next place, it now appears that the quantity γ in § 314 is the same as this radius of domainal flexure of the regional geodesic. Hence a comparison of the results yields the relation

$$\frac{Y_0}{\rho_0} = \frac{Y_\epsilon}{\rho_\epsilon},$$

typical of all the directions: that is,

$$Y_\epsilon = Y_0, \quad \rho_\epsilon = \rho_0.$$

It therefore follows that the limiting position of the perpendicular, drawn from Q upon the tangent flat at O to the region, coincides (as the small arc QO tends to zero when Q draws near to O) with the direction of the prime normal of the regional geodesic drawn in the direction OQ.

(iii) Further, the relations

$$\left\| \; Y_\epsilon, \; Y, \; \frac{dy}{dn} \; \right\| = 0$$

are satisfied by the direction-cosines of the prime normal of the regional geodesic, of the radius of domainal flexure of that geodesic, and of the prime normal of the domainal geodesic; and therefore these three directions lie in one plane. The last two are perpendicular to one another: hence, if ψ denote the angle between the prime normals of the geodesics, we have

$$\cos\psi = \sum Y Y_\epsilon, \quad \sin\psi = \sum \frac{dy}{dn} Y_\epsilon.$$

Consequently, we have the relations

$$\frac{\cos\psi}{\rho_\epsilon} = \frac{1}{\rho}, \quad \frac{\sin\psi}{\rho_\epsilon} = \frac{1}{\gamma_\epsilon},$$

$$Y_\epsilon = Y\cos\psi + \frac{dy}{dn}\sin\psi,$$

$$\frac{1}{\rho_\epsilon} = \frac{\cos\psi}{\rho} + \frac{\sin\psi}{\gamma_\epsilon},$$

all derivable from the equation

$$\frac{Y_\epsilon}{\rho_\epsilon} = \frac{Y}{\rho} + \frac{1}{\gamma_\epsilon}\frac{dy}{dn}.$$

A simple geometrical construction is immediate. In the plane containing the two prime normals of the geodesics and the radius of domainal flexure of the regional geodesic, let OC represent (in magnitude and direction) the radius of circular curvature of the domainal geodesic; and let OF_ϵ represent (also in magnitude and direction) the radius of domainal flexure of the regional geodesic: thus the angle COF_ϵ is a right angle. Let a perpendicular OC_ϵ be drawn on the line CF_ϵ; then OC_ϵ represents (in magnitude and direction) the radius of circular curvature of the regional geodesic.

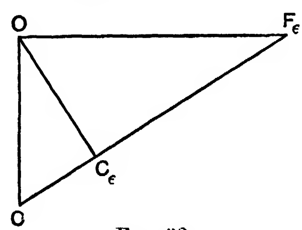

FIG. 32.

317. We shall require a knowledge of the angles between the gremial principal lines of a domainal geodesic touching a region, and the domainal normal of the region (that is, the direction of the radius of domainal flexure of all regional geodesics at a point).

(i) The typical direction-cosine of this normal is $\dfrac{dy}{dn}$. The tangent of the domainal geodesic is in the same direction as that of the regional geodesic, and

therefore it is at right angles to the domainal normal: we easily verify the analytical condition

$$\sum y' \frac{dy}{dn} = \sum Ap' \frac{dp}{dn} = 0.$$

The prime normal of the domainal geodesic is orthogonal to the tangent block of the domain, and therefore is at right angles to every direction within the block. One such direction is the domainal normal, and therefore the prime normal of the domainal geodesic is at right angles to the direction of the radius of domainal flexure of the regional geodesic in the same direction. The analytical relation is

$$\sum Y \frac{dy}{dn} = 0,$$

at once verified.

(ii) Let χ_3 denote the angle between the domainal normal and the binormal of the domainal geodesic touching the region, the direction-variables of the geodesic being p', q', r', t'; then

$$\cos \chi_3 = \sum l_3 \frac{dy}{dn}.$$

Now (§ 284)

$$l_3 = \lambda y_1 + \mu y_2 + \nu y_3 + \varpi y_4,$$

where

$$\frac{\lambda}{\sigma} = \frac{p'}{\rho} - \frac{\bar{v}_1}{\Omega}, \quad \frac{\mu}{\sigma} = \frac{q'}{\rho} - \frac{\bar{v}_2}{\Omega}, \quad \frac{\nu}{\sigma} = \frac{r'}{\rho} - \frac{\bar{v}_3}{\Omega}, \quad \frac{\varpi}{\sigma} = \frac{t'}{\rho} - \frac{\bar{v}_4}{\Omega},$$

while

$$\frac{dy}{dn} = y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn} + y_4 \frac{dt}{dn}.$$

Hence

$$\begin{aligned} \cos \chi_3 &= \sum \left\{ \lambda \left(A \frac{dp}{dn} + H \frac{dq}{dn} + G \frac{dr}{dn} + L \frac{dt}{dn} \right) \right\} \\ &= \frac{1}{\epsilon_n} (\epsilon_1 \lambda + \epsilon_2 \mu + \epsilon_3 \nu + \epsilon_4 \varpi), \end{aligned}$$

or, as

$$\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0,$$

we have

$$\begin{aligned} \cos \chi_3 &= -\frac{\sigma}{\Omega \epsilon_n} (\epsilon_1 \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{v}_3 + \epsilon_4 \bar{v}_4) \\ &= -\frac{\sigma}{\Omega \epsilon_n} \sum a v_1 \epsilon_1, \end{aligned}$$

which accordingly gives the inclination between the domainal normal and the binormal of a domainal geodesic touching the region $\epsilon = 0$. It also assigns a geometrical significance to the concomitant $\sum a v_1 \epsilon_1$.

If, in particular, the direction of the domainal geodesic be one of the principal directions which (p. 326) provide a maximum or a minimum among the values of the circular curvatures, then

$$\cos\chi_3 = -1:$$

that is, the binormal of the domainal geodesic in question lies along the domainal normal and therefore coincides (in direction) with the radius of domainal flexure of the regional geodesic drawn in the same direction.

(iii) Let χ_4 denote the angle between the trinormal of the domainal geodesic touching the region and the domainal normal to the region; then

$$\cos\chi_4 = \sum l_4 \frac{dy}{dn}.$$

Now (§ 286)

$$l_4 = y_1\alpha + y_2\beta + y_3\gamma + y_4\delta,$$

the values of α, β, γ, δ, being determinate; and therefore

$$\cos\chi_4 = \sum \alpha\left(A\frac{dp}{dn} + H\frac{dq}{dn} + G\frac{dr}{dn} + L\frac{dt}{dn}\right)$$

$$= \frac{1}{\epsilon_n}(\epsilon_1\alpha + \epsilon_2\beta + \epsilon_3\gamma + \epsilon_4\delta).$$

But (*l.c.*)

$$\frac{\alpha}{\tau} = p'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) - \frac{\bar{v}_1}{\Omega}\sigma' - \frac{\bar{w}_1}{\Omega}\sigma,$$

with corresponding values for β, γ, δ; also $\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0$; and therefore

$$\frac{1}{\tau}\cos\chi_4 = -\frac{\sigma'}{\Omega\epsilon_n}\sum\epsilon_1\bar{v}_1 - \frac{\sigma}{\Omega\epsilon_n}\sum\epsilon_1\bar{w}_1$$

$$= -\frac{\sigma'}{\Omega\epsilon_n}\sum a v_1\epsilon_1 - \frac{\sigma}{\Omega\epsilon_n}\sum a w_1\epsilon_1$$

$$= \frac{\sigma'}{\sigma}\cos\chi_3 - \frac{\sigma}{\Omega\epsilon_n}\sum a w_1\epsilon_1;$$

so that

$$\frac{1}{\sigma\tau}\cos\chi_4 = -\frac{d}{ds}\left(\frac{1}{\sigma}\right)\cos\chi_3 - \frac{1}{\Omega\epsilon_n}\sum a w_1\epsilon_1.$$

We thus have an expression for $\cos\chi_4$ and, concurrently, a geometrical significance for the covariant $\sum a w_1\epsilon_1$.

(iv) Let χ_5 denote the angle between the quartinormal of the domainal geodesic touching the region and the domainal normal of the region; then

$$\cos\chi_5 = \sum l_5 \frac{dy}{dn}.$$

Now (§ 288)

$$l_5 = \epsilon_0 y_1 + \eta y_2 + \iota y_3 + \omega y_4,$$

the values * of ϵ_0, η, ι, ω, being determinate; and therefore

$$\cos\chi_5 = \sum \epsilon_0 \left(A\frac{dp}{dn} + H\frac{dq}{dn} + G\frac{dr}{dn} + L\frac{dt}{dn}\right)$$

$$= \frac{1}{\epsilon_n}(\epsilon_1\epsilon_0 + \epsilon_2\eta + \epsilon_3\iota + \epsilon_4\omega).$$

When the values of the coefficients in l_5 are substituted, we have an expression for $\cos\chi_5$ in the form

$$\Omega^{\frac{1}{2}}\frac{\epsilon_n}{\sigma^2\tau}\cos\chi_5 = \begin{vmatrix} \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

Ex. By using the values of the coefficients in l_5 as given in § 289, shew that

$$\frac{1}{\sigma\tau\kappa}\cos\chi_5 + \sigma\frac{d}{ds}\left(\frac{1}{\sigma^2\tau}\right)\cos\chi_4 + \left\{\frac{d^2}{ds^2}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma\tau^2}\right\}\cos\chi_3 = -\frac{1}{\Omega\epsilon_n}\sum a z_1\epsilon_1.$$

This result, and the result in the text, provide the geometrical significance of two concomitants of the complete system.

318. Now consider the tangent flat of the domainal region and the domainal normal to the region. The flat naturally possesses three leading lines, which may be taken to be the directions of the three parametric curves, connected with the variations of the parameters p, q, r, as retained in § 332 to be the parameters of the domainal region; and the domainal normal is at right angles to each of these three directions. Hence the four directions, thus constituted, are four directionally independent lines: they lie, all of them, in the tangent block of the region; and therefore they are directionally equivalent to any set of four leading lines in that block. But four such lines are provided by the tangent, the binormal, the trinormal, and the quartinormal of any domainal geodesic: and therefore, in the present connection, by the specified four principal lines of a domainal geodesic touching the region. Accordingly, the typical quantities

$$y_1,\ y_2,\ y_3,\ \frac{dy}{dn},$$

must be linearly expressible in terms of the typical direction-cosines y', l_3, l_4, l_5, the coefficients in each individual set being the same throughout the members of the set.

* To avoid possible confusion with the functional symbol ϵ of the region, ϵ_0 is substituted here for the symbol ϵ of § 288.

Expressions for y_1, y_2, y_3, in terms of y', l_3, l_4, l_5, have already been given (§ 292). There is no change in the form of the expressions when the domainal geodesic touches the region $\epsilon=0$; there is, however, the limitation on the variations of p', q', r', t', due to the condition

$$\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0,$$

which does not affect the form of the expressions.

In order to represent $\frac{dy}{dn}$ linearly in a like form, we assume

$$\frac{dy}{dn} = Py' + Ql_3 + Rl_4 + Sl_5,$$

where P, Q, R, S, are the same for all the equations of which this relation is typical. Manifestly

$$\sum y' \frac{dy}{dn} = P, \quad \sum l_3 \frac{dy}{dn} = Q, \quad \sum l_4 \frac{dy}{dn} = R, \quad \sum l_5 \frac{dy}{dn} = S\,;$$

that is,

$$P=0, \quad Q=\cos\chi_3, \quad R=\cos\chi_4, \quad S=\cos\chi_5,$$

where the values of $\cos\chi_3$, $\cos\chi_4$, $\cos\chi_5$, are to be regarded as known.

Consequently, the values of y_1, y_2, y_3, remain as obtained in § 292; and we have

$$\frac{dy}{dn} = l_3\cos\chi_3 + l_4\cos\chi_4 + l_5\cos\chi_5,$$

with the foregoing values of the cosines.

Principal values of domainal linear flexure: principal directions of flexure in a region.

319. The domainal flexure of a regional geodesic, drawn in the direction p', q', r', t', touching the region in the domain, is given by

$$-\frac{\epsilon_n}{\gamma_\epsilon} = \sum_j \sum_k \bar{\epsilon}_{jk} x_j' x_k',$$

these direction-variables x' $(=p', q', r', t')$ being such as to satisfy

$$\sum Ap'^2 = 1, \quad \epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0.$$

This flexure has principal directions and principal values, associated with the maximum and minimum values of $1/\gamma_\epsilon$ and the directions determining those values. To determine these values and directions, we have to make the expression

$$\sum_j \sum_k \bar{\epsilon}_{jk} x_j' x_k'$$

a maximum or a minimum, among the values obtainable through quantities

p', q', r', t', satisfying the two conditions. When, in accordance with the notation of p. 255, we write

$$\bar{\epsilon}_i = \bar{\epsilon}_{1i} p' + \bar{\epsilon}_{2i} q' + \bar{\epsilon}_{3i} r' + \bar{\epsilon}_{4i} t',$$

for $i=1, 2, 3, 4$, the critical equations are

$$\bar{\epsilon}_i = \lambda u_i + \mu \epsilon_i,$$

for the same values of i, the quantities λ and μ being left undetermined in the formation of these critical equations. Let these be multiplied by p', q', r', t', respectively, and the products be added; then we have

$$\frac{\epsilon_n}{\gamma_\epsilon} = \lambda \sum u_i x_i' + \mu \sum \epsilon_i x_i' = \lambda.$$

Thus the four equations become

$$\bar{\epsilon}_i + \frac{\epsilon_n}{\gamma_\epsilon} u_i = \mu \epsilon_i,$$

for $i=1, 2, 3, 4$; and there is, in addition, the relation

$$\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0.$$

There are therefore five equations, linear and homogeneous in the five quantities p', q', r', t', μ; when these are eliminated, they lead to the cubic equation

$$\begin{vmatrix} \bar{\epsilon}_{11} + \frac{\epsilon_n}{\gamma_\epsilon} A, & \bar{\epsilon}_{12} + \frac{\epsilon_n}{\gamma_\epsilon} H, & \bar{\epsilon}_{13} + \frac{\epsilon_n}{\gamma_\epsilon} G, & \bar{\epsilon}_{14} + \frac{\epsilon_n}{\gamma_\epsilon} L, & \epsilon_1 \\ \bar{\epsilon}_{21} + \frac{\epsilon_n}{\gamma_\epsilon} H, & \bar{\epsilon}_{22} + \frac{\epsilon_n}{\gamma_\epsilon} B, & \bar{\epsilon}_{23} + \frac{\epsilon_n}{\gamma_\epsilon} F, & \bar{\epsilon}_{24} + \frac{\epsilon_n}{\gamma_\epsilon} M, & \epsilon_2 \\ \bar{\epsilon}_{31} + \frac{\epsilon_n}{\gamma_\epsilon} G, & \bar{\epsilon}_{32} + \frac{\epsilon_n}{\gamma_\epsilon} F, & \bar{\epsilon}_{33} + \frac{\epsilon_n}{\gamma_\epsilon} C, & \bar{\epsilon}_{34} + \frac{\epsilon_n}{\gamma_\epsilon} N, & \epsilon_3 \\ \bar{\epsilon}_{41} + \frac{\epsilon_n}{\gamma_\epsilon} L, & \bar{\epsilon}_{42} + \frac{\epsilon_n}{\gamma_\epsilon} M, & \bar{\epsilon}_{43} + \frac{\epsilon_n}{\gamma_\epsilon} N, & \bar{\epsilon}_{44} + \frac{\epsilon_n}{\gamma_\epsilon} D, & \epsilon_4 \\ \epsilon_1 \quad , & \epsilon_2 \quad , & \epsilon_3 \quad , & \epsilon_4 \quad , & 0 \end{vmatrix} = 0.$$

Let E denote the determinant

$$| \bar{\epsilon}_{11}, \bar{\epsilon}_{22}, \bar{\epsilon}_{33}, \bar{\epsilon}_{44} |,$$

similar to Ω when represented in the form

$$| A, B, C, D |;$$

and let c_{ij} denote the minor of $\bar{\epsilon}_{ij}$ in E, as a_{ij} denotes the minor of A_{ij} in Ω. The term in the foregoing equation independent of $1/\gamma_\epsilon$ is

$$-\sum_i \sum c_{ij} \epsilon_i \epsilon_j;$$

the term involving $1/\gamma_\epsilon^3$ is

$$-\frac{\epsilon_n^3}{\gamma_\epsilon^3} \sum_i \sum a_{ij} \epsilon_i \epsilon_j,$$

which is equal to

$$-\Omega\frac{\epsilon_n^5}{\gamma_\epsilon^3};$$

and the coefficients of $\dfrac{\epsilon_n}{\gamma_\epsilon}$ and of $\left(\dfrac{\epsilon_n}{\gamma_\epsilon}\right)^2$ respectively are

$$-\theta_1\left[\sum_i\sum_j c_{ij}\epsilon_i\epsilon_j\right], \quad -\theta_2\left[\sum_i\sum_j a_{ij}\epsilon_i\epsilon_j\right],$$

where θ_1 and θ_2 denote symbolical operators such that

$$\theta_1=\sum_l\sum_m A_{lm}\frac{\partial}{\partial\bar{\epsilon}_{lm}}, \quad \theta_2=\sum_l\sum_m \bar{\epsilon}_{lm}\frac{\partial}{\partial A_{lm}}.$$

Hence the equation has the form

$$\sum_i\sum_j c_{ij}\epsilon_i\epsilon_j+\frac{\epsilon_n}{\gamma_\epsilon}\theta_1\left[\sum_i\sum_j c_{ij}\epsilon_i\epsilon_j\right]+\frac{\epsilon_n^2}{\gamma_\epsilon^2}\theta_2\left[\sum_i\sum_j a_{ij}\epsilon_i\epsilon_j\right]+\Omega\frac{\epsilon_n^5}{\gamma_\epsilon^3}=0,$$

which accordingly gives the principal values of $1/\gamma_\epsilon$, the domainal flexure of regional geodesics.

The direction-variables of the principal directions are then given by the three equations

$$\frac{1}{\epsilon_1}\left(\bar{\epsilon}_1+\frac{\epsilon_n}{\gamma_\epsilon}u_1\right)=\frac{1}{\epsilon_2}\left(\bar{\epsilon}_2+\frac{\epsilon_n}{\gamma_\epsilon}u_2\right)=\frac{1}{\epsilon_3}\left(\bar{\epsilon}_3+\frac{\epsilon_n}{\gamma_\epsilon}u_3\right)=\frac{1}{\epsilon_4}\left(\bar{\epsilon}_4+\frac{\epsilon_n}{\gamma_\epsilon}u_4\right),$$

combined with the permanent relation $\sum Ap'^2=1$.

(i) The cubic equation can be expressed simply, when the quaternary quadratics $\sum_j\sum_k A_{jk}x_j'x_k'$ and $\sum_j\sum_k \bar{\epsilon}_{jk}x_j'x_k'$ are expressed in umbral notation. For this purpose, let

$$\bar{a}_{x'}=\bar{a}_1p'+\bar{a}_2q'+\bar{a}_3r'+\bar{a}_4t'=\sum\bar{a}_ix_i',$$

with umbral significance for the quantities $\bar{a}_i$, such that

$$\sum_j\sum_k A_{jk}x_j'x_k'=\bar{a}_{x'}^2=\bar{b}_{x'}^2=\bar{c}_{x'}^2,$$

where the symbols $\bar{b}$, $\bar{c}$, are equivalent to the symbol $\bar{a}$. Also let

$$\bar{e}_{x'}=\bar{e}_1p'+\bar{e}_2q'+\bar{e}_3r'+\bar{e}_4t'=\sum\bar{e}_ix_i',$$

with umbral significance for the quantities $\bar{e}_i$, such that

$$\sum_j\sum_k \bar{\epsilon}_{jk}x_j'x_k'=\bar{e}_{x'}^2=\bar{f}_{x'}^2=\bar{g}_{x'}^2,$$

where the symbols $\bar{f}$, $\bar{g}$, are equivalent to the symbols $\bar{e}$. Then the cubic equation is expressible in the umbral form

$$\tfrac{1}{6}(\bar{e}\bar{f}\bar{g}\epsilon)^2+\tfrac{1}{2}\frac{\epsilon_n}{\gamma_\epsilon}(\bar{e}\bar{f}\bar{a}\epsilon)^2+\tfrac{1}{2}\frac{\epsilon_n^2}{\gamma_\epsilon^2}(\bar{e}\bar{a}\bar{b}\epsilon)^2+\tfrac{1}{6}\frac{\epsilon_n^3}{\gamma_\epsilon^3}(\bar{a}\bar{b}\bar{c}\epsilon)^2=0,$$

the quantity $(\bar{e}\bar{f}\bar{g}\epsilon)$ denoting the umbral determinant

$$\begin{vmatrix} \bar{e}_1, & \bar{f}_1, & \bar{g}_1, & \epsilon_1 \\ \bar{e}_2, & \bar{f}_2, & \bar{g}_2, & \epsilon_2 \\ \bar{e}_3, & \bar{f}_3, & \bar{g}_3, & \epsilon_3 \\ \bar{e}_4, & \bar{f}_4, & \bar{g}_4, & \epsilon_4 \end{vmatrix},$$

and so for the other like quantities, where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, are not umbral.

It follows that, as there are three principal values of γ_ϵ, there are three regional measures of domainal flexure of geodesics. We take these to be E_1, E_2, E_3, defined by the equations

$$E_1 = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3}, \quad E_2 = \frac{1}{\gamma_2\gamma_3} + \frac{1}{\gamma_3\gamma_1} + \frac{1}{\gamma_1\gamma_2}, \quad E_3 = \frac{1}{\gamma_1\gamma_2\gamma_3};$$

and the values of these three regional measures are given by

$$E_1 = -\frac{1}{2\Omega\epsilon_n{}^3}(\bar{a}\bar{b}\bar{e}\epsilon)^2,$$

$$E_2 = \frac{1}{2\Omega\epsilon_n{}^4}(\bar{e}\bar{f}\bar{a}\epsilon)^2,$$

$$E_3 = -\frac{1}{6\Omega\epsilon_n{}^5}(\bar{e}\bar{f}\bar{g}\epsilon)^2.$$

(ii) Further, there are three principal directions of domainal flexure in the region, belonging respectively to the three principal values of that domainal flexure. Let any two of such principal directions have x_1', x_2', x_3', x_4'; and z_1', z_2', z_3', z_4'; for their direction-variables; with $1/\beta_\epsilon$ and μ_1, $1/\delta_\epsilon$ and μ_2, respectively for the corresponding values of the domainal flexures and of the quantity μ. Then we have

$$\left(\sum_j \bar{\epsilon}_{ij}x_j'\right) + \frac{\epsilon_n}{\beta_\epsilon}\left(\sum_j A_{ij}x_j'\right) = \mu_1\epsilon_i$$

(for $i=1, 2, 3, 4$) in connection with the first direction, and

$$\left(\sum_j \bar{\epsilon}_{ij}z_j'\right) + \frac{\epsilon_n}{\delta_\epsilon}\left(\sum_j A_{ij}z_j'\right) = \mu_2\epsilon_i$$

(for the same values of i) in connection with the second direction. When the equations for the first direction are multiplied by z_1', z_2', z_3', z_4', respectively, and the results are added, then

$$\left(\sum_j \sum_k \bar{\epsilon}_{jk}x_j'z_k'\right) + \frac{\epsilon_n}{\beta_\epsilon}\left(\sum_j \sum_k A_{jk}x_j'z_k'\right) = \mu_1(\epsilon_1 z_1' + \epsilon_2 z_2' + \epsilon_3 z_3' + \epsilon_4 z_4') = 0;$$

and when the equations for the second direction are multiplied by x_1', x_2', x_3', x_4', respectively, and the results are added, then

$$\left(\sum_j \sum_k \bar{\epsilon}_{jk}x_j'z_k'\right) + \frac{\epsilon_n}{\delta_\epsilon}\left(\sum_j \sum_k A_{jk}x_j'z_k'\right) = \mu_2(\epsilon_1 x_1' + \epsilon_2 x_2' + \epsilon_3 x_3' + \epsilon_4 x_4') = 0.$$

Now, in a general configuration subject to no special intrinsic conditions, the principal domainal flexures at any point of a region are unequal, so that β_ϵ and δ_ϵ are unequal; we therefore have

$$\sum_j \sum_k \bar{\epsilon}_{jk} x_j' z_k' = 0, \quad \sum_j \sum_k A_{jk} x_j' z_k' = 0.$$

The second of these inferences shews that the selected directions are at right angles—a conclusion which establishes the stated property. The first of them is a relation satisfied by any two of the principal directions of domainal flexure in the region.

The three principal directions in the region, which provide the principal domainal flexures of regional geodesics, make an orthogonal set of lines within the tangent flat of the region; when combined with the domainal normal to the region, they constitute an orthogonal set of lines within the tangent block of the domain. Also, we note that, along any one of the three principal directions of domainal flexure of regional geodesics, the equations

$$\left\| \begin{matrix} \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ u_1, & u_2, & u_3, & u_4 \end{matrix} \right\| = 0$$

are satisfied.

Ex. Prove that the value of μ, to be associated with a principal direction, satisfies the equations

$$\mu = \frac{1}{\Omega \epsilon_n^2} \sum a \epsilon_1 \bar{\epsilon}_1, \quad \mu^2 + \frac{1}{\gamma_\epsilon^2} = \frac{1}{\Omega \epsilon_n^2} \sum a \bar{\epsilon}_1^2.$$

320. The preceding results, as regards principal directions and magnitudes of domainal linear flexure of regional geodesics, can be expressed in a different (but equivalent) form. Instead of retaining the relation

$$\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0$$

concurrently with the other relations, and thus maintaining a symmetry among p', q', r', t', we use it to eliminate one of the direction-variables t'. Then if

$$\left.\begin{aligned} \epsilon_4^2 A_0 &= \epsilon_4^2 A - 2\epsilon_1\epsilon_4 L + \epsilon_1^2 D, & \epsilon_4^2 F_0 &= \epsilon_4^2 F - \epsilon_2\epsilon_4 N - \epsilon_3\epsilon_4 M + \epsilon_2\epsilon_3 D \\ \epsilon_4^2 B_0 &= \epsilon_4^2 B - 2\epsilon_2\epsilon_4 M + \epsilon_2^2 D, & \epsilon_4^2 G_0 &= \epsilon_4^2 G - \epsilon_3\epsilon_4 L - \epsilon_1\epsilon_4 N + \epsilon_3\epsilon_1 D \\ \epsilon_4^2 C_0 &= \epsilon_4^2 C - 2\epsilon_3\epsilon_4 N + \epsilon_3^2 D, & \epsilon_4^2 H_0 &= \epsilon_4^2 H - \epsilon_1\epsilon_4 M - \epsilon_2\epsilon_4 L + \epsilon_1\epsilon_2 D \end{aligned}\right\},$$

and, for all values of $i, j, = 1, 2, 3$,

$$\epsilon_4^2 \bar{\eta}_{ij} = \epsilon_4^2 \bar{\epsilon}_{ij} - \epsilon_i \epsilon_4 \bar{\epsilon}_{j4} - \epsilon_j \epsilon_4 \bar{\epsilon}_{i4} + \epsilon_i \epsilon_j \bar{\epsilon}_{44},$$

the permanent relation for arcs in the direction p', q', r', (now the direction-variables in the modified expressions) in the region is

$$\sum A_0 p'^2 = 1,$$

while the domainal flexure of the regional geodesic in that direction is given by

$$-\frac{\epsilon_n}{\gamma_\epsilon}=\bar{\eta}_{11}p'^2+2\bar{\eta}_{12}p'q'+\bar{\eta}_{22}q'^2+2\bar{\eta}_{13}p'r'+2\bar{\eta}_{23}q'r'+\bar{\eta}_{33}r'^2.$$

The principal values of the domainal flexure of geodesics are the roots of the cubic equation

$$\begin{vmatrix} \bar{\eta}_{11}+\frac{\epsilon_n}{\gamma_\epsilon}A_0, & \bar{\eta}_{12}+\frac{\epsilon_n}{\gamma_\epsilon}H_0, & \bar{\eta}_{13}+\frac{\epsilon_n}{\gamma_\epsilon}G_0 \\ \bar{\eta}_{12}+\frac{\epsilon_n}{\gamma_\epsilon}H_0, & \bar{\eta}_{22}+\frac{\epsilon_n}{\gamma_\epsilon}B_0, & \bar{\eta}_{23}+\frac{\epsilon_n}{\gamma_\epsilon}F_0 \\ \bar{\eta}_{13}+\frac{\epsilon_n}{\gamma_\epsilon}G_0, & \bar{\eta}_{23}+\frac{\epsilon_n}{\gamma_\epsilon}F_0, & \bar{\eta}_{33}+\frac{\epsilon_n}{\gamma_\epsilon}C_0 \end{vmatrix}=0.$$

The inferences, connected with this result, follow as before.

Superficial measures of dominal flexure.

321. The preceding investigation provides the measures of linear domainal flexure of a region. There are also superficial measures of the domainal flexure of regional geodesics; and they arise in the same way as the superficial measures of curvature of a region in a plenary quadruple homaloidal space *.

Consider a superficial orientation in the region. It may be constructed by means of two domainal directions touching the region; when the variables of these directions are x_1', x_2', x_3', x_4', and z_1', z_2', z_3', z_4', there are orientation-variables s_{ij} as in § 270. Or it may be provided as the intersection of the parametric region $\epsilon=0$ with another parametric region $v(p, q, r, t)=0$; and then there are orientation-variables l_{ij}

$$\begin{Vmatrix} \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ v_1, & v_2, & v_3, & v_4 \end{Vmatrix},$$

the relations between the two sets of variables (when they represent the same orientation) being given on p. 257.

To obtain the maximum and the minimum values of the domainal flexure of regional geodesics which, in direction, originate in the specified superficial orientation, we have to make the quaternary quadratic

$$\sum \bar{\epsilon}_{11}p'^2,$$

which is the value of $-\epsilon_n/\gamma_\epsilon$, a maximum or a minimum for values of p', q', r', t', satisfying the three conditions

$$\sum Ap'^2=1,$$

$$\epsilon_1p'+\epsilon_2q'+\epsilon_3r'+\epsilon_4t'=0,\quad v_1p'+v_2q'+v_3r'+v_4t'=0.$$

* *G.F.D.*, vol. ii, §§ 318-323.

The critical equations for the maximum or minimum are

$$\bar{\epsilon}_i+\theta u_i+\lambda\epsilon_i+\mu v_i=0,$$

for $i=1, 2, 3, 4$, the values of the multipliers θ, λ, μ, being undetermined in the formation of the critical equations.

The value of θ is at once obtainable. We multiply the four equations by p', q', r', t', and add the products : the result is

$$-\frac{\epsilon_n}{\gamma_\epsilon}+\theta=0,$$

on using the conditions. Thus there are four equations

$$\bar{\epsilon}_i+\frac{\epsilon_n}{\gamma_\epsilon}u_i+\lambda\epsilon_i+\mu v_i=0,$$

and these are linear and homogeneous in the six quantities p', q', r', t', λ, μ ; also there are the two conditions

$$\epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t'=0, \quad v_1 p'+v_2 q'+v_3 r'+v_4 t'=0,$$

likewise linear. When the six magnitudes are eliminated determinantally among the six equations, we have

$$\begin{vmatrix}
\bar{\epsilon}_{11}+\frac{\epsilon_n}{\gamma_\epsilon}A, & \bar{\epsilon}_{12}+\frac{\epsilon_n}{\gamma_\epsilon}H, & \bar{\epsilon}_{13}+\frac{\epsilon_n}{\gamma_\epsilon}G, & \bar{\epsilon}_{14}+\frac{\epsilon_n}{\gamma_\epsilon}L, & \epsilon_1, & v_1 \\
\bar{\epsilon}_{12}+\frac{\epsilon_n}{\gamma_\epsilon}H, & \bar{\epsilon}_{22}+\frac{\epsilon_n}{\gamma_\epsilon}B, & \bar{\epsilon}_{23}+\frac{\epsilon_n}{\gamma_\epsilon}F, & \bar{\epsilon}_{24}+\frac{\epsilon_n}{\gamma_\epsilon}M, & \epsilon_2, & v_2 \\
\bar{\epsilon}_{13}+\frac{\epsilon_n}{\gamma_\epsilon}G, & \bar{\epsilon}_{23}+\frac{\epsilon_n}{\gamma_\epsilon}F, & \bar{\epsilon}_{33}+\frac{\epsilon_n}{\gamma_\epsilon}C, & \bar{\epsilon}_{34}+\frac{\epsilon_n}{\gamma_\epsilon}N, & \epsilon_3, & v_3 \\
\bar{\epsilon}_{14}+\frac{\epsilon_n}{\gamma_\epsilon}L, & \bar{\epsilon}_{24}+\frac{\epsilon_n}{\gamma_\epsilon}M, & \bar{\epsilon}_{34}+\frac{\epsilon_n}{\gamma_\epsilon}N, & \bar{\epsilon}_{44}+\frac{\epsilon_n}{\gamma_\epsilon}D, & \epsilon_4, & v_4 \\
\epsilon_1 \quad , & \epsilon_2 \quad , & \epsilon_3 \quad , & \epsilon_4 \quad , & 0, & 0 \\
v_1 \quad , & v_2 \quad , & v_3 \quad , & v_4 \quad , & 0, & 0
\end{vmatrix}=0,$$

manifestly a quadratic equation in γ_ϵ. When expressed in expanded form, the equation has the form

$$\begin{aligned}
&\sum(\bar{\epsilon}_{ik}\bar{\epsilon}_{jl}-\bar{\epsilon}_{il}\bar{\epsilon}_{jk})s_{ij}s_{kl} \\
&\quad+\frac{\epsilon_n}{\gamma_\epsilon}\sum(A_{ik}\bar{\epsilon}_{jl}-A_{il}\bar{\epsilon}_{jk}-A_{jk}\bar{\epsilon}_{il}+A_{jl}\bar{\epsilon}_{ik})s_{ij}s_{kl} \\
&\quad+\frac{\epsilon_n^2}{\gamma_\epsilon^2}\sum(A_{ik}A_{jl}-A_{il}A_{jk})s_{ij}s_{kl}=0,
\end{aligned}$$

where only the ratios of the superficial variables occur. The coefficient of $\epsilon_n^2/\gamma_\epsilon^2$ is $\sin^2\omega$, where ω is the angle between the two guiding directions in the orientation ;

and it can be made unity if the orientation-variables are taken in their canonical form (§ 270).

Thus there are two principal values of the domainal flexure of those regional geodesics which originate in any superficial orientation in the region; and there are corresponding principal directions of such flexure in the orientation. These two directions are at right angles. Denoting their direction-variables by p_1', q_1', r_1', t_1', and p_2', q_2', r_2', t_2', respectively, we have, on multiplying the equations for the first direction by p_2', q_2', r_2', t_2', and adding,

$$\sum \bar{\epsilon}_1{}^{(1)} p_2' + \frac{\epsilon_n}{(\gamma_\epsilon)_1} \sum u_\iota{}^{(1)} p_2' + \lambda \sum \epsilon_1 p_2' + \mu \sum v_1 p_2' = 0,$$

that is,

$$\sum \bar{\epsilon}_{11} p_1' p_2' + \frac{\epsilon_n}{(\gamma_\epsilon)_1} \sum A p_1' p_2' = 0.$$

Similarly, multiplying the equations for the second direction by p_1', q_1', r_1', t_1', and adding, we have

$$\sum \bar{\epsilon}_{11} p_1' p_2' + \frac{\epsilon_n}{(\gamma_\epsilon)_2} \sum A p_1' p_2' = 0.$$

Hence $\sum A p_1' p_2' = 0$, as the principal domainal flexures are unequal in general: consequently the two principal directions are at right angles.

Manifestly there are two measures of domainal flexure of regional geodesics originating in the orientation, being the product and the sum of the principal domain flexures for the orientation. They correspond to Gauss's measures of curvature of a surface in triple homaloidal space: and they can be denoted by $\bar{K}_s$ and $\bar{H}_s$, where

$$\bar{K}_s = \frac{1}{\epsilon_n{}^2} \frac{\sum (\bar{\epsilon}_{ik}\bar{\epsilon}_{jl} - \bar{\epsilon}_{il}\bar{\epsilon}_{jk}) s_{ij} s_{kl}}{\sum (A_{ik}A_{jl} - A_{il}A_{jk}) s_{ij} s_{kl}},$$

$$\bar{H}_s = -\frac{1}{\epsilon_n} \frac{\sum (A_{ik}\bar{\epsilon}_{jl} - A_{il}\bar{\epsilon}_{jk} - A_{jk}\bar{\epsilon}_{il} + A_{jl}\bar{\epsilon}_{ik}) s_{ij} s_{kl}}{\sum (A_{ik}A_{jl} - A_{il}A_{jk}) s_{ij} s_{kl}}.$$

Moreover, there are principal values of these measures, defined as the maximum and the minimum of $\bar{K}_s$ and of $\bar{H}_s$ respectively, among all the values which can arise through the various admissible orientations. By analysis, similar to that used for the corresponding investigations * in circular curvature of geodesics in a primary region (so that the domain is a block), it can be proved that

(i) for $\bar{K}_s$ and for $\bar{H}_s$, the principal orientations in the region have, for their leading lines, each pair out of the three principal directions of domainal flexure of regional geodesics: and

* *G.F.D.*, vol. ii, 320-322.

(ii) the principal values of $\overline{K}_s$ are

$$\frac{1}{\gamma_2\gamma_3}, \quad \frac{1}{\gamma_3\gamma_1}, \quad \frac{1}{\gamma_1\gamma_2},$$

while the principal values of $\overline{H}_s$ are

$$-\left(\frac{1}{\gamma_2}+\frac{1}{\gamma_3}\right), \quad -\left(\frac{1}{\gamma_3}+\frac{1}{\gamma_1}\right), \quad -\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right),$$

where $\gamma_1, \gamma_2, \gamma_3$, are the principal radii of domainal flexure of geodesics in the domainal region.

It is almost superfluous to note that the principal directions of domainal flexure of regional geodesics are quite distinct from the principal directions of circular curvature of those geodesics *.

Sphericity of a domainal region in any superficial orientation.

322. The investigation of the sphericity of a domainal region in any orientation within the region can be made by the method of § 320.

We write

$$\epsilon_1 + c_1\epsilon_4 = 0, \quad \epsilon_2 + c_2\epsilon_4 = 0, \quad \epsilon_3 + c_3\epsilon_4 = 0,$$

so that the direction-variable t', when connected with a region $\epsilon = 0$ in the domain, is given by

$$t' = -c_1 p' - c_2 q' - c_3 r',$$

the variables p', q', r', being retained for direction in the region. Similarly, for orientation-variables of an orientation in the region (and therefore for orientation in the domain), we retain s_{23}, s_{31}, s_{12}; and the other orientation-variables in the domain are given by

$$s_{14} = -c_2 s_{12} + c_3 s_{31}, \quad s_{24} = c_1 s_{12} - c_3 s_{23}, \quad s_{34} = c_2 s_{23} - c_1 s_{31}.$$

Now the circular curvature of a regional geodesic and its domainal flexure are connected with the circular curvature of the domainal geodesic in the same direction by the typical relation

$$\frac{Y_\epsilon}{\rho_\epsilon} = \frac{Y}{\rho} + \frac{1}{\gamma}\frac{dy}{dn}.$$

When we take account of the direction-variables of the regional geodesic with special reference to the region alone, we have a typical equation of the form

$$\frac{Y_\epsilon}{\rho_\epsilon} = \zeta_{11}p'^2 + 2\zeta_{12}p'q' + \zeta_{22}q'^2 + 2\zeta_{13}p'q' + 2\zeta_{23}q'r' + \zeta_{33}r'^2.$$

* Even for the simpler instance of geodesics on surfaces within a primary region, the inclination of a principal direction of circular curvature to a principal direction of flexure is a covariant of the whole configuration: see *G.D.F.*, vol. ii, § 362.

Also we have

$$\frac{Y}{\rho}=\sum \eta_{11}p'^2,$$

or, again including only the regional direction-variables,

$$\frac{Y}{\rho}=\bar{\eta}_{11}p'^2+2\bar{\eta}_{12}p'q'+\bar{\eta}_{22}q'^2+2\bar{\eta}_{13}p'r'+2\bar{\eta}_{23}q'r'+\bar{\eta}_{33}r'^2,$$

where

$$\bar{\eta}_{ij}=\eta_{ij}-c_i\eta_{j4}-c_j\eta_{i4}+c_ic_j\eta_{44},$$

for all the combinations $i, j, =1, 2, 3$. Similarly, we have

$$\begin{aligned}-\frac{\epsilon_n}{\gamma}&=\sum \bar{\epsilon}_{11}p'^2\\&=E_{11}p'^2+2E_{12}p'q'+E_{22}q'^2+2E_{13}p'r'+2E_{23}q'r'+E_{33}r'^2,\end{aligned}$$

where

$$E_{ij}=\bar{\epsilon}_{ij}-c_i\bar{\epsilon}_{j4}-c_j\bar{\epsilon}_{i4}+c_ic_j\bar{\epsilon}_{44},$$

for all the i-j combinations. Finally, the permanent arc-relation becomes

$$\sum \mathbf{A}_{11}p'^2=1$$

where

$$\mathbf{A}_{ij}=A_{ij}-c_iA_{j4}-c_jA_{i4}+c_ic_jA_{44}.$$

Thus the equation connecting the curvatures becomes

$$\sum \zeta_{11}p'^2=\sum \bar{\eta}_{11}p'^2-\frac{1}{\epsilon_n}\frac{dy}{dn}\sum E_{11}p'^2;$$

and there is no homogeneous relation among the three direction-variables p', q', r', so that we have

$$\zeta_{ij}=\bar{\eta}_{ij}-\frac{1}{\epsilon_n}\frac{dy}{dn}E_{ij},$$

for all the combinations $i, j, =1, 2, 3$.

Let K_r denote the sphericity (the Riemann measure of curvature) of the region in the orientation with variables s_{23}, s_{31}, s_{12}, so that

$$K_r=\frac{\sum\{\sum(\zeta_{11}\zeta_{22}-\zeta_{12}^2)\}s_{23}^2}{\sum(\mathbf{A}_{11}\mathbf{A}_{22}-\mathbf{A}_{12}^2)s_{23}^2}.$$

If x_1', x_2', x_3', x_4', and z_1', z_2', z_3', z_4', are two directions defining the orientation s_{23}, s_{31}, s_{12}, where

$$x_4'=-c_1x_1'-c_2x_2'-c_3x_3', \quad z_4'=-c_1z_1'-c_2z_2'-c_3z_3',$$

we have

$$\sum(\mathbf{A}_{11}\mathbf{A}_{22}-\mathbf{A}_{12}^2)s_{23}^2=\left(\sum_1^3 \mathbf{A}_{11}x_1'^2\right)\left(\sum_1^3 \mathbf{A}_{11}z_1'^2\right)-\left(\sum_1^3 \mathbf{A}_{11}x_1'z_1'\right)^2.$$

But with the values of **A**, we have

$$\sum_1^3 \mathbf{A}_{11}x_1'^2 = \sum A_{11}x_1'^2,$$

$$\sum_1^3 \mathbf{A}_{11}z_1'^2 = \sum A_{11}z_1'^2,$$

$$\sum_1^3 \mathbf{A}_{11}x_1'z_1' = \sum A_{11}x_1'z_1',$$

where, on the right-hand sides, the summation extends over the four direction-variables in the domain; and therefore

$$\sum (\mathbf{A}_{11}\mathbf{A}_{22}-\mathbf{A}_{12}^2)s_{23}^2 = (\sum A_{11}x_1'^2)(\sum A_{11}z_1'^2) - (\sum A_{11}x_1'z_1')^2$$
$$= \sum_i \sum_j \sum_k \sum_l \{(A_{ik}A_{jl} - A_{il}A_{jk})s_{ij}s_{kl}\},$$

the summation now extending over all the orientation-variables in the domainal expression of the orientation.

Again, with the definitions of the quantities ζ_{ij}, we have

$$\sum \zeta_{\alpha\beta}\zeta_{\gamma\delta} = \sum \left(\bar{\eta}_{\alpha\beta} - \frac{1}{\epsilon_n}\frac{dy}{dn}E_{\alpha\beta}\right)\left(\bar{\eta}_{\gamma\delta} - \frac{1}{\epsilon_n}\frac{dy}{dn}E_{\gamma\delta}\right),$$

the summation extending over all the dimensions of the plenary homaloidal space. But, because

$$\frac{dy}{dn} = y_1\frac{dp}{dn} + y_2\frac{dq}{dn} + y_3\frac{dr}{dn} + y_4\frac{dt}{dn},$$

and because

$$\sum y_\kappa \eta_{\lambda\mu} = 0,$$

for all values of κ, λ, μ, we have

$$\sum \bar{\eta}_{\theta\nu}\frac{dy}{dn} = \sum \frac{dy}{dn}(\eta_{\theta\nu} - c_\theta\eta_{4\nu} - c_\nu\eta_{4\theta} + c_\theta c_\nu\eta_{44}) = 0,$$

for all values of θ and ν. Hence, as $\sum \left(\frac{dy}{dn}\right)^2 = 1$, we have

$$\sum \zeta_{\alpha\beta}\zeta_{\gamma\delta} = \sum \bar{\eta}_{\alpha\beta}\bar{\eta}_{\gamma\delta} + \frac{1}{\epsilon_n^2}E_{\alpha\beta}E_{\gamma\delta},$$

for all values of α, β, γ, δ. Consequently the numerator of the expression for K_r becomes transformed so as to give

$$\sum \{\sum (\zeta_{11}\zeta_{22} - \zeta_{12}^2)\} s_{23}^2$$
$$= \sum \{\sum (\bar{\eta}_{11}\bar{\eta}_{22} - \bar{\eta}_{12}^2)\} s_{23}^2 + \frac{1}{\epsilon_n^2}\sum (E_{11}E_{22} - E_{12}^2)s_{23}^2.$$

The second term on the right-hand side

$$=\frac{1}{\epsilon_n^2}\left[\left(\sum_1^3 E_{11}x_1'^2\right)\left(\sum_1^3 E_{11}z_1'^2\right)-\left(\sum_1^3 E_{11}x_1'z_1'\right)^2\right];$$

also

$$\sum_1^3 E_{11}x_1'^2=\sum\bar{\epsilon}_{11}x_1'^2,\quad \sum_1^3 E_{11}z_1'^2=\sum\bar{\epsilon}_{11}z_1'^2,\quad \sum_1^3 E_{11}x_1'z_1'=\sum\bar{\epsilon}_{11}x_1'z_1',$$

where, as for the permanent arc-relation, the summations on the right-hand sides extend over the four variables in the domainal expression of each direction; and therefore the second term on the right-hand side

$$=\frac{1}{\epsilon_n^2}[(\sum\bar{\epsilon}_{11}x_1'^2)(\sum\bar{\epsilon}_{11}z_1'^2)-(\sum\epsilon_{11}x_1'z_1')^2]$$

$$=\frac{1}{\epsilon_n^2}\sum\{(\bar{\epsilon}_{ik}\bar{\epsilon}_{jl}-\bar{\epsilon}_{il}\bar{\epsilon}_{jk})s_{ij}s_{kl}\}$$

$$=\bar{K}_s\sum\{(A_{ik}A_{jl}-A_{il}A_{jk})s_{ij}s_{kl}\},$$

where (§ 321) the quantity $\bar{K}_s$ is the Gauss (specific) measure of domainal flexure of regional geodesics in the specified orientation. Moreover, this Gauss measure of circular curvatures of geodesics on a surface in triple homaloidal space (as also for geodesics in any primary amplitude) is the simplified form of the Riemann measure for the surface (or the primary amplitude); and we therefore regard this quantity $\bar{K}_s$ as a Riemann measure of domainal flexure.

For the first term in the numerator of the expression for K_r, which

$$=\sum\{\sum(\bar{\eta}_{11}\bar{\eta}_{22}-\bar{\eta}_{12}^2)\}s_{12}^2,$$

we have to proceed in a different manner, because the internal summations, such as $\sum(\eta_{11}\eta_{22}-\bar{\eta}_{12}^2)$, extend over the dimensions of the plenary space. When account is taken of the values of the quantities $\bar{\eta}_{ij}$ in terms of the quantities $\eta_{\alpha\beta}$, we find

$$\begin{aligned}\sum(\bar{\eta}_{ik}\bar{\eta}_{jl}-\bar{\eta}_{il}\bar{\eta}_{jk})&\\ =(ij, kl)-c_i(4j, kl)-c_j(i4, kl)&-c_k(ij, 4l)-c_l(ij, k4)\\ &+c_ic_k(4j, 4l)+(c_jc_k)(i4, 4l)+c_ic_l(4j, k4)+c_jc_l(i4, k4),\end{aligned}$$

a general relation which expresses the values of the regional quantities k_{11}, k_{22}, k_{33}, k_{23}, k_{31}, k_{12}, of § 162, in terms of the Riemann four-index symbols of the domain. Let these values be substituted in the coefficients of s_{23}^2, s_{31}^2, s_{12}^2, $s_{31}s_{12}$, $s_{12}s_{23}$, $s_{23}s_{31}$, and let the coefficients of the different four-index symbols be collected. Obviously we at once have

$$\begin{aligned}(12, 12)s_{12}^2+(23, 23)s_{23}^2+(31, 31)s_{12}^2&\\ +2(31, 12)s_{31}s_{12}&+2(12, 23)s_{12}s_{23}+2(23, 31)s_{23}s_{31},\end{aligned}$$

as the aggregate of terms, free from the quantities c and free also from any of the four-index symbols where the number 4 occurs.

The coefficient of (12, 14)

$$\begin{aligned} &= -2c_2 s_{12}^2, \text{ out of the term in } s_{12}^2, \\ &\quad + 2c_3 s_{31} s_{12}, \ \ldots\ldots\ldots\ldots\ldots\ldots \ s_{31}s_{12}, \\ &= 2s_{12}s_{14}, \end{aligned}$$

and so on for all the four-index symbols. The complete value of the term

$$= \sum (ij, kl) s_{ij} s_{kl},$$

with complete summation for the numbers i, j, k, l: that is, it is

$$= K_D \sum \{(A_{ik}A_{jl} - A_{il}A_{jk}) s_{ij} s_{kl}\},$$

where (§ 275) the quantity K_D denotes the sphericity (the Riemann measure of curvature) of the domain in the specified orientation.

When these values are inserted in the expression for K_r, we find

$$K_r = K_D + \overline{K}_s:$$

or the sphericity of a domainal region in any orientation in the region is the sum of (i), the sphericity of the domain in that orientation and (ii), the Riemann measure of domainal flexure of the region in the orientation.

The corresponding theorem, concerning the sphericity of a parametric surface in a region, has already (§ 219) been established.

Regions geodesic to a domain at a place.

323. Among the regions contained by a domain, one special class demands detailed consideration: it is the class which, on the analogy of geodesic surfaces, are called geodesic regions and which are defined in a manner analogous to that used for the surfaces. Let OA, OB, OC, be any three domainal geodesics such that their three directions at O do not lie in a plane; and take the aggregate of all the domainal geodesics through O, the directions of which originate in the flat determined by the tangents to OA, OB, OC, as leading lines. This aggregate of domainal geodesics through O determines a region which is said to be geodesic to the domain at O. Manifestly every regional geodesic of this region through O is the same as the domainal geodesic through O in the same direction and its domainal flexure is zero; but we cannot assert, and it is not a general property, that regional geodesics of the region, which do not pass through O, are domainal geodesics. In order that a parametric region $\epsilon = 0$ may be geodesic to the domain at O, it is clear that conditions will have to be satisfied.

Certain preliminary results are necessary, supplementary to those (§ 269) already obtained. The direction-variables of the domainal normal to the region are

$$\Omega \epsilon_n \frac{dp}{dn} = a\epsilon_1 + h\epsilon_2 + g\epsilon_3 + l\epsilon_4,$$

with corresponding values for $\frac{dq}{dn}$, $\frac{dr}{dn}$, $\frac{dt}{dn}$, where the dilatation ϵ_n is given by

$$\Omega\epsilon_n^2 = \sum a\epsilon_1^2.$$

We have had quantities $\bar{\epsilon}_{ij}$, such that

$$\bar{\epsilon}_{ij} = \epsilon_{ij} - \epsilon_1\Gamma_{ij} - \epsilon_2\Delta_{ij} - \epsilon_3\Theta_{ij} - \epsilon_4\Phi_{ij}$$

for all values of i and j; and have proved that

$$\frac{\partial\epsilon_n}{\partial p} = \bar{\epsilon}_{11}\frac{dp}{dn} + \bar{\epsilon}_{12}\frac{dq}{dn} + \bar{\epsilon}_{13}\frac{dr}{dn} + \bar{\epsilon}_{14}\frac{dt}{dn},$$

with like values for $\frac{\partial\epsilon_n}{\partial q}$, $\frac{\partial\epsilon_n}{\partial r}$, $\frac{\partial\epsilon_n}{\partial t}$, so that

$$\epsilon_{nn} = \frac{d^2\epsilon}{dn^2} = \sum \bar{\epsilon}_{11}\left(\frac{dp}{dn}\right)^2,$$

$$\frac{d\epsilon_n}{ds} = \sum \bar{\epsilon}_{11}\frac{dp}{dn}\frac{dp}{ds},$$

whether the domainal direction ds lies within the region or not. We have had (§ 269) quantities

$$\bar{\epsilon}_i = \bar{\epsilon}_{1i}p' + \bar{\epsilon}_{2i}q' + \bar{\epsilon}_{3i}r' + \bar{\epsilon}_{4i}t',$$

for $i = 1, 2, 3, 4$; and we shall find it convenient to have quantities $\bar{\nu}_1$, $\bar{\nu}_2$, $\bar{\nu}_3$, $\bar{\nu}_4$, defined by

$$\bar{\nu}_i = \bar{\epsilon}_{1i}\frac{dp}{dn} + \bar{\epsilon}_{2i}\frac{dq}{dn} + \bar{\epsilon}_{3i}\frac{dr}{dn} + \bar{\epsilon}_{4i}\frac{dt}{dn}.$$

Differentiating the relation

$$\epsilon_n\frac{dp}{dn} = \frac{a}{\Omega}\epsilon_1 + \frac{h}{\Omega}\epsilon_2 + \frac{g}{\Omega}\epsilon_3 + \frac{l}{\Omega}\epsilon_4$$

with respect to p, and using the results of § 268, we find

$$\frac{\partial}{\partial p}\left(\epsilon_n\frac{dp}{dn}\right) = \frac{1}{\Omega}(a\bar{\epsilon}_{11} + h\bar{\epsilon}_{12} + g\bar{\epsilon}_{13} + l\bar{\epsilon}_{14}) - \epsilon_n\left(\Gamma_{11}\frac{dp}{dn} + \Gamma_{12}\frac{dq}{dn} + \Gamma_{13}\frac{dr}{dn} + \Gamma_{14}\frac{dt}{dn}\right);$$

and similarly

$$\frac{\partial}{\partial q}\left(\epsilon_n\frac{dp}{dn}\right) = \frac{1}{\Omega}(a\bar{\epsilon}_{12} + h\bar{\epsilon}_{22} + g\bar{\epsilon}_{23} + l\bar{\epsilon}_{24}) - \epsilon_n\left(\Gamma_{12}\frac{dp}{dn} + \Gamma_{22}\frac{dq}{dn} + \Gamma_{23}\frac{dr}{dn} + \Gamma_{24}\frac{dt}{dn}\right),$$

$$\frac{\partial}{\partial r}\left(\epsilon_n\frac{dp}{dn}\right) = \frac{1}{\Omega}(a\bar{\epsilon}_{13} + h\bar{\epsilon}_{23} + g\bar{\epsilon}_{33} + l\bar{\epsilon}_{34}) - \epsilon_n\left(\Gamma_{13}\frac{dp}{dn} + \Gamma_{23}\frac{dq}{dn} + \Gamma_{33}\frac{dr}{dn} + \Gamma_{34}\frac{dt}{dn}\right),$$

$$\frac{\partial}{\partial t}\left(\epsilon_n\frac{dp}{dn}\right) = \frac{1}{\Omega}(a\bar{\epsilon}_{14} + h\bar{\epsilon}_{24} + g\bar{\epsilon}_{34} + l\bar{\epsilon}_{44}) - \epsilon_n\left(\Gamma_{14}\frac{dp}{dn} + \Gamma_{24}\frac{dq}{dn} + \Gamma_{34}\frac{dr}{dn} + \Gamma_{44}\frac{dt}{dn}\right).$$

Consequently

$$\frac{d}{dn}\left(\epsilon_n \frac{dp}{dn}\right)=\frac{1}{\Omega}(a\bar{\nu}_1+h\bar{\nu}_2+g\bar{\nu}_3+l\bar{\nu}_4)-\epsilon_n\sum\Gamma_{11}\left(\frac{dp}{dn}\right)^2,$$

and therefore

$$\frac{d^2p}{dn^2}=\frac{1}{\Omega\epsilon_n}(a\bar{\nu}_1+h\bar{\nu}_2+g\bar{\nu}_3+l\bar{\nu}_4)-\left\{\sum\Gamma_{11}\left(\frac{dp}{dn}\right)^2\right\}-\frac{\epsilon_{nn}}{\epsilon_n}\frac{dp}{dn}.$$

Similarly

$$\frac{d^2q}{dn^2}=\frac{1}{\Omega\epsilon_n}(h\bar{\nu}_1+b\bar{\nu}_2+f\bar{\nu}_3+m\bar{\nu}_4)-\left\{\sum\Delta_{11}\left(\frac{dp}{dn}\right)^2\right\}-\frac{\epsilon_{nn}}{\epsilon_n}\frac{dq}{dn},$$

$$\frac{d^2r}{dn^2}=\frac{1}{\Omega\epsilon_n}(g\bar{\nu}_1+f\bar{\nu}_2+c\bar{\nu}_3+n\bar{\nu}_4)-\left\{\sum\Theta_{11}\left(\frac{dp}{dn}\right)^2\right\}-\frac{\epsilon_{nn}}{\epsilon_n}\frac{dr}{dn},$$

$$\frac{d^2t}{dn^2}=\frac{1}{\Omega\epsilon_n}(l\bar{\nu}_1+m\bar{\nu}_2+n\bar{\nu}_3+d\bar{\nu}_4)-\left\{\sum\Phi_{11}\left(\frac{dp}{dn}\right)^2\right\}-\frac{\epsilon_{nn}}{\epsilon_n}\frac{dt}{dn}.$$

As a verification of these values, we have

$$\begin{aligned}\frac{d^2\epsilon}{dn^2}&=\sum\left\{\epsilon_{11}\left(\frac{dp}{dn}\right)^2\right\}+\epsilon_1\frac{d^2p}{dn^2}+\epsilon_2\frac{d^2q}{dn^2}+\epsilon_3\frac{d^2r}{dn^2}+\epsilon_4\frac{d^2t}{dn^2}\\&=\sum\left\{\bar{\epsilon}_{11}\left(\frac{dp}{dn}\right)^2\right\}+\epsilon_1\left[\frac{d^2p}{dn^2}+\left\{\sum\Gamma_{11}\left(\frac{dp}{dn}\right)^2\right\}\right]+\epsilon_2\left[\frac{d^2q}{dn^2}+\left\{\sum\Delta_{11}\left(\frac{dp}{dn}\right)^2\right\}\right]\\&\qquad+\epsilon_3\left[\frac{d^2r}{dn^2}+\left\{\sum\Theta_{11}\left(\frac{dp}{dn}\right)^2\right\}\right]+\epsilon_4\left[\frac{d^2t}{dn^2}+\left\{\sum\Phi_{11}\left(\frac{dp}{dn}\right)^2\right\}\right],\end{aligned}$$

which, on substitution for quantities in the last four terms, leads to the former result

$$\frac{d^2\epsilon}{dn^2}=\sum\left\{\bar{\epsilon}_{11}\left(\frac{dp}{dn}\right)^2\right\}.$$

Also, with the convention $x_i=p, q, r, t$, for $i=1, 2, 3, 4$,

$$\frac{d}{ds}\left(\epsilon_n\frac{dp}{dn}\right)=\frac{1}{\Omega}(a\bar{\epsilon}_1+h\bar{\epsilon}_2+g\bar{\epsilon}_3+l\bar{\epsilon}_4)-\epsilon_n\left\{\sum_i\sum_j\Gamma_{ij}\frac{dx_i}{dn}\frac{dx_j}{ds}\right\},$$

$$\frac{d}{ds}\left(\epsilon_n\frac{dq}{dn}\right)=\frac{1}{\Omega}(h\bar{\epsilon}_1+b\bar{\epsilon}_2+f\bar{\epsilon}_3+m\bar{\epsilon}_4)-\epsilon_n\left\{\sum_i\sum_j\Delta_{ij}\frac{dx_i}{dn}\frac{dx_j}{ds}\right\},$$

$$\frac{d}{ds}\left(\epsilon_n\frac{dr}{dn}\right)=\frac{1}{\Omega}(g\bar{\epsilon}_1+f\bar{\epsilon}_2+c\bar{\epsilon}_3+n\bar{\epsilon}_4)-\epsilon_n\left\{\sum_i\sum_j\Theta_{ij}\frac{dx_i}{dn}\frac{dx_j}{ds}\right\},$$

$$\frac{d}{ds}\left(\epsilon_n\frac{dt}{dn}\right)=\frac{1}{\Omega}(l\bar{\epsilon}_1+m\bar{\epsilon}_2+n\bar{\epsilon}_3+d\bar{\epsilon}_4)-\epsilon_n\left\{\sum_i\sum_j\Phi_{ij}\frac{dx_i}{dn}\frac{dx_j}{ds}\right\};$$

and therefore as the value of $\dfrac{d\epsilon_n}{ds}$ is known, being $\sum\limits_i\sum\limits_j\bar{\epsilon}_{ij}\dfrac{dx_i}{dn}\dfrac{dx_j}{ds}$, we have

$$\frac{d}{ds}\left(\frac{dp}{dn}\right)=\frac{1}{\Omega\epsilon_n}(a\bar{\epsilon}_1+h\bar{\epsilon}_2+g\bar{\epsilon}_3+l\bar{\epsilon}_4)-\sum_i\sum_j\left\{\left(\Gamma_{ij}+\frac{\bar{\epsilon}_{ij}}{\epsilon_n}\frac{dp}{dn}\right)\frac{dx_i}{dn}\frac{dx_j}{ds}\right\},$$

$$\frac{d}{ds}\left(\frac{dq}{dn}\right)=\frac{1}{\Omega\epsilon_n}(h\bar{\epsilon}_1+b\bar{\epsilon}_2+f\bar{\epsilon}_3+m\bar{\epsilon}_4)-\sum_i\sum_j\left\{\left(\Delta_{ij}+\frac{\bar{\epsilon}_{ij}}{\epsilon_n}\frac{dq}{dn}\right)\frac{dx_i}{dn}\frac{dx_j}{ds}\right\},$$

$$\frac{d}{ds}\left(\frac{dr}{dn}\right)=\frac{1}{\Omega\epsilon_n}(g\bar{\epsilon}_1+f\bar{\epsilon}_2+c\bar{\epsilon}_3+n\bar{\epsilon}_4)-\sum_i\sum_j\left\{\left(\Theta_{ij}+\frac{\bar{\epsilon}_{ij}}{\epsilon_n}\frac{dr}{dn}\right)\frac{dx_i}{dn}\frac{dx_j}{ds}\right\},$$

$$\frac{d}{ds}\left(\frac{dt}{dn}\right)=\frac{1}{\Omega\epsilon_n}(l\bar{\epsilon}_1+m\bar{\epsilon}_2+n\bar{\epsilon}_3+d\bar{\epsilon}_4)-\sum_i\sum_j\left\{\left(\Phi_{ij}+\frac{\bar{\epsilon}_{ij}}{\epsilon_n}\frac{dt}{dn}\right)\frac{dx_i}{dn}\frac{dx_j}{ds}\right\},$$

where, again, ds denotes any domainal direction whether lying in the region or not.

Next,

$$\bar{\epsilon}_{ij}=\epsilon_{ij}-\epsilon_1\Gamma_{ij}-\epsilon_2\Delta_{ij}-\epsilon_3\Theta_{ij}-\epsilon_4\Phi_{ij};$$

and therefore differentiating with respect to x_k, and using the values of the derivatives of Γ, Δ, Θ, Φ, obtained in § 279, we have

$$\begin{aligned}\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}=\epsilon_{ijk}&-\epsilon_{1k}\Gamma_{ij}-\epsilon_{2k}\Delta_{ij}-\epsilon_{3k}\Theta_{ij}-\epsilon_{4k}\Phi_{ij}\\&-\epsilon_1\left\{\Gamma_{ijk}+\Gamma_i(kj)+\Gamma_j(ki)-\frac{1}{3\Omega}\sum_\mu a_{1\mu}[(\mu i, jk)+(\mu j, ik)]\right\}\\&-\epsilon_2\left\{\Delta_{ijk}+\Delta_i(kj)+\Delta_j(ki)-\frac{1}{3\Omega}\sum_\mu a_{2\mu}[(\mu i, jk)+(\mu j, ik)]\right\}\\&-\epsilon_3\left\{\Theta_{ijk}+\Theta_i(kj)+\Theta_j(ki)-\frac{1}{3\Omega}\sum_\mu a_{3\mu}[(\mu i, jk)+(\mu j, ik)]\right\}\\&-\epsilon_4\left\{\Phi_{ijk}+\Phi_i(kj)+\Phi_j(ki)-\frac{1}{3\Omega}\sum_\mu a_{4\mu}[(\mu i, jk)+(\mu j, ik)]\right\},\end{aligned}$$

with the significance of the symbols $\Gamma_i(kj)$ as on p. 277. Let quantities E_{ijk} be introduced under the definition

$$\begin{aligned}E_{ijk}=\epsilon_{ijk}&-\epsilon_1\{\Gamma_{ijk}+\Gamma_i(jk)+\Gamma_j(ki)+\Gamma_k(ij)\}\\&-\epsilon_2\{\Delta_{ijk}+\Delta_i(jk)+\Delta_j(ki)+\Delta_k(ij)\}\\&-\epsilon_3\{\Theta_{ijk}+\Theta_i(jk)+\Theta_j(ki)+\Theta_k(ij)\}\\&-\epsilon_4\{\Phi_{ijk}+\Phi_i(jk)+\Phi_j(ki)+\Phi_k(ij)\},\end{aligned}$$

where manifestly E_{ijk} is unaltered by interchanges of i, j, k, among one another; then we have

$$\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}=E_{ijk}-\bar{\epsilon}_{1k}\Gamma_{ij}-\bar{\epsilon}_{2k}\Delta_{ij}-\bar{\epsilon}_{3k}\Theta_{ij}-\bar{\epsilon}_{4k}\Phi_{ij}+\tfrac{1}{3}\epsilon_n\sum_\mu\frac{dx_\mu}{dn}[(\mu i, jk)+(\mu j, ik)],$$

the summation being taken for $\mu=1, 2, 3, 4$. On the right-hand side, the integers i and j (if different) can be interchanged without affecting its value.

We shall require the sum

$$\frac{\partial \bar{\epsilon}_{jk}}{\partial x_i}+\frac{\partial \bar{\epsilon}_{ki}}{\partial x_j}+\frac{\partial \bar{\epsilon}_{ij}}{\partial x_k}.$$

The coefficient of $\frac{1}{3}\epsilon_n \frac{dx_\mu}{dn}$ on the right-hand side

$$=[(\mu j, ki)+(\mu k, ij)]+[(\mu k, ij)+(\mu i, kj)]+(\mu i, jk)+(\mu j, ik)],$$

which vanishes identically. If then, by analogy with the symbols $\Gamma_i(jk)$, we write

$$\bar{\epsilon}_\alpha(\beta\gamma)=\bar{\epsilon}_{1\alpha}\Gamma_{\beta\gamma}+\bar{\epsilon}_{2\alpha}\Delta_{\beta\gamma}+\bar{\epsilon}_{3\alpha}\Theta_{\gamma\beta}+\bar{\epsilon}_{4\alpha}\Phi_{\beta\gamma},$$

we have

$$\frac{\partial \bar{\epsilon}_{jk}}{\partial x_i}+\frac{\partial \bar{\epsilon}_{ki}}{\partial x_j}+\frac{\partial \bar{\epsilon}_{ij}}{\partial x_k}=3E_{ijk}-\bar{\epsilon}_i(jk)-\bar{\epsilon}_j(ki)-\bar{\epsilon}_k(ij).$$

324. The preceding analysis can be used for the construction of a result which, holding for any region and not relevant to the consideration of geodesic regions, will be required later in connection with other issues. We have

$$\frac{dy}{dn}=y_1\frac{dp}{dn}+y_2\frac{dq}{dn}+y_3\frac{dr}{dn}+y_4\frac{dt}{dn};$$

and therefore

$$\frac{d}{ds}\left(\frac{dy}{dn}\right)=y_1'\frac{dp}{dn}+y_2'\frac{dq}{dn}+y_3'\frac{dr}{dn}+y_4'\frac{dt}{dn}$$
$$+y_1\frac{d}{ds}\left(\frac{dp}{dn}\right)+y_2\frac{d}{ds}\left(\frac{dq}{dn}\right)+y_3\frac{d}{ds}\left(\frac{dr}{dn}\right)+y_4\frac{d}{ds}\left(\frac{dt}{dn}\right).$$

Now

$$\begin{aligned} y_1'&=y_{11}p'+y_{12}q'+y_{13}r'+y_{14}t' \\ &=\eta_{11}p'+\eta_{12}q'+\eta_{13}r'+\eta_{14}t' \\ &\qquad +y_1\alpha+y_2\xi+y_3\phi+y_4\kappa \\ &=\eta_1+y_1\alpha+y_2\xi+y_3\phi+y_4\kappa, \end{aligned}$$

using the symbols of § 306, and so for the others; hence the first line in the foregoing relation

$$=\eta_1\frac{dp}{dn}+\eta_2\frac{dq}{dn}+\eta_3\frac{dr}{dn}+\eta_4\frac{dt}{dn}$$
$$+y_1\left(\sum\Gamma_{11}\frac{dp}{ds}\frac{dp}{dn}\right)+y_2\left(\sum\Delta_{11}\frac{dp}{ds}\frac{dp}{dn}\right)+y_3\left(\sum\Theta_{11}\frac{dp}{ds}\frac{dp}{dn}\right)+y_4\left(\sum\Phi_{11}\frac{dp}{ds}\frac{dp}{dn}\right).$$

Also we have, by the result on p. 408,

$$\frac{d}{ds}\left(\frac{dp}{dn}\right)+\frac{\epsilon_n'}{\epsilon_n}\frac{dp}{dn}=\frac{1}{\Omega\epsilon_n}(a\bar{\epsilon}_1+h\bar{\epsilon}_2+g\bar{\epsilon}_3+l\bar{\epsilon}_4)-\left(\sum\Gamma_{11}\frac{dp}{dn}\frac{dp}{ds}\right),$$

and similarly for the like derivatives of q, r, t. When these values are inserted, and the results are re-arranged, we have an equation

$$\frac{d}{ds}\left(\frac{dy}{dn}\right)+\frac{\epsilon_n'}{\epsilon_n}\frac{dy}{dn}=\eta_1\frac{dp}{dn}+\eta_2\frac{dq}{dn}+\eta_3\frac{dr}{dn}+\eta_4\frac{dt}{dn}$$
$$+\frac{1}{\Omega\epsilon_n}(a\between\bar{\epsilon}_1,\bar{\epsilon}_2,\bar{\epsilon}_3,\bar{\epsilon}_4\between y_1,y_2,y_3,y_4).$$

The analytical similarity, to the corresponding result (§ 208) in connection with a regional normal to a surface contained in the region, is complete.

Arc-variation of the domainal flexure along a regional geodesic.

325. The domainal flexure of the regional geodesic in the direction p', q', r', t', $(=x_1', x_2', x_3', x_4')$, has been obtained in the form

$$-\frac{\epsilon_n}{\gamma_\epsilon}=\sum_i\sum_j\bar{\epsilon}_{ij}x_i'x_j';$$

and therefore

$$-\frac{d}{ds}\left(\frac{\epsilon_n}{\gamma_\epsilon}\right)=\sum_i\sum_j\sum_k\left(\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}x_i'x_j'x_k'\right)+\sum_i\sum_j\{\bar{\epsilon}_{ij}(\bar{x}_i''x_j'+x_i'\bar{x}_j'')\},$$

where the second arc-derivatives $\bar{x}_i''$ and $\bar{x}_j''$ have to be taken in the region, because we are dealing with variations along the regional geodesic. We therefore have

$$\bar{x}_1''-x_1''=p_\epsilon''-p''$$
$$=\frac{1}{\gamma_\epsilon}\frac{dx_1}{dn},$$

by § 315, with three similar results. Thus

$$\sum_i\sum_j\bar{\epsilon}_{ij}(\bar{x}_i''x_j'+x_i'\bar{x}_j'')$$
$$=\sum_i\sum_j\{\bar{\epsilon}_{ij}(x_i''x_j'+x_i'x_j'')\}+\frac{2}{\gamma_\epsilon}\sum_i\sum_j\left(\bar{\epsilon}_{ij}\frac{dx_i}{dn}\frac{dx_j}{ds}\right)$$
$$=\sum_i\sum_j\{\bar{\epsilon}_{ij}(x_i''x_j'+x_i'x_j'')\}+\frac{2}{\gamma_\epsilon}\frac{d\epsilon_n}{ds};$$

so that

$$-\frac{d}{ds}\left(\frac{\epsilon_n}{\gamma_\epsilon}\right)-\frac{2}{\gamma_\epsilon}\frac{d\epsilon_n}{ds}=\sum_i\sum_j\sum_k\left(\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}x_i'x_j'x_k'\right)+\sum_i\sum_j\{\bar{\epsilon}_{ij}(x_i''x_j'+x_i'x_j'')\}.$$

The total coefficient of $x_i'x_j'x_k'$ on the right-hand side

$$\begin{aligned} = \;& 2\left(\frac{\partial\bar{\epsilon}_{jk}}{\partial x_i}+\frac{\partial\bar{\epsilon}_{ki}}{\partial x_j}+\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}\right) \\ & -4\left(\bar{\epsilon}_{1i}\Gamma_{jk}+\bar{\epsilon}_{1j}\Gamma_{ki}+\bar{\epsilon}_{1k}\Gamma_{ij}\right) \\ & -4\left(\bar{\epsilon}_{2i}\Delta_{jk}+\bar{\epsilon}_{2j}\Delta_{ki}+\bar{\epsilon}_{2k}\Delta_{ij}\right) \\ & -4\left(\bar{\epsilon}_{3i}\Theta_{jk}+\bar{\epsilon}_{3j}\Theta_{ki}+\bar{\epsilon}_{3k}\Theta_{ij}\right) \\ & -4\left(\bar{\epsilon}_{4i}\Phi_{jk}+\bar{\epsilon}_{4j}\Phi_{ki}+\bar{\epsilon}_{4k}\Phi_{ij}\right): \end{aligned}$$

the sum of the last four lines is

$$-4\{\bar{\epsilon}_i(jk)+\bar{\epsilon}_j(ki)+\bar{\epsilon}_k(ij)\};$$

and therefore, when we substitute the value obtained in § 323 for the expression in the first line, the total coefficient of $x_i'x_j'x_k'$ becomes

$$6E_{ijk}-6\{\bar{\epsilon}_i(jk)+\bar{\epsilon}_j(ki)+\bar{\epsilon}_k(ij)\}.$$

We denote this total coefficient by $6\bar{\epsilon}_{ijk}$, so that

$$-\epsilon_n\frac{d}{ds}\left(\frac{1}{\gamma_\epsilon}\right)-\frac{3}{\gamma_\epsilon}\frac{d\epsilon_n}{ds}=\sum_i\sum_j\sum_k\bar{\epsilon}_{ijk}x_i'x_j'x_k'.$$

We now have

$$\bar{\epsilon}_{ijk}=E_{ijk}-\bar{\epsilon}_i(jk)-\bar{\epsilon}_j(ki)-\bar{\epsilon}_k(ij);$$

when the value of E_{ijk} is substituted on the right-hand side, the value of $\bar{\epsilon}_{ijk}$ acquires the form

$$\begin{aligned} \bar{\epsilon}_{ijk}=\epsilon_{ijk} & -\bar{\epsilon}_i(jk)-\bar{\epsilon}_j(ki)-\bar{\epsilon}_k(ij) \\ & -\epsilon_1\{\Gamma_{ijk}+\Gamma_i(jk)+\Gamma_j(ki)+\Gamma_k(ij)\} \\ & -\epsilon_2\{\Delta_{ijk}+\Delta_i(jk)+\Delta_j(ki)+\Delta_k(ij)\} \\ & -\epsilon_3\{\Theta_{ijk}+\Theta_i(jk)+\Theta_j(ki)+\Theta_k(ij)\} \\ & -\epsilon_4\{\Phi_{ijk}+\Phi_i(jk)+\Phi_j(ki)+\Phi_k(ij)\}. \end{aligned}$$

Also, with this relation between E_{ijk} and $\bar{\epsilon}_{ijk}$, we have

$$\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}=\bar{\epsilon}_{ijk}+\bar{\epsilon}_i(jk)+\bar{\epsilon}_j(ki)+\tfrac{1}{3}\epsilon_n\sum_\mu\frac{dx_\mu}{dn}[(\mu i, jk)+(\mu j, ik)],$$

where, on the right-hand side, $x_1, x_2, x_3, x_4, =p, q, r, t$: and, for convenience of reference, we repeat the definition

$$\psi_i(jk)=\psi_{1i}\Gamma_{jk}+\psi_{2i}\Delta_{jk}+\psi_{3i}\Theta_{jk}+\psi_{4i}\Phi_{jk},$$

for $\psi=\Gamma, \Delta, \Theta, \Phi$, and $\bar{\epsilon}$.

Modified expressions for $\bar{\epsilon}_{ijk}$ can be obtained by introducing the symbols ϵ_{ij} instead of $\bar{\epsilon}_{ij}$ in the portion $\bar{\epsilon}_i(jk)+\bar{\epsilon}_j(ki)+\bar{\epsilon}_k(ij)$. But all the modified expressions are equivalent to one another; and the magnitudes $\bar{\epsilon}_{ij}$ recur more frequently than

ϵ_{ij}, just as the symbols η_{ij} recur more frequently than y_{ij} in connection with the circular curvature of domainal geodesics. For all forms, the arc-variation of the domainal flexure is given by

$$-\frac{1}{\epsilon_n^2}\frac{d}{ds}\left(\frac{\epsilon_n^3}{\gamma_\epsilon}\right)=\sum_i\sum_j\sum_k \bar{\epsilon}_{ijk}x_i'x_j'x_k'.$$

Again, we have

$$\epsilon_{nn}=\sum\bar{\epsilon}_{11}\left(\frac{dp}{dn}\right)^2,$$

and therefore

$$\frac{d^3\epsilon}{dn^3}=\frac{d\epsilon_{nn}}{dn}=2\left(\bar{\nu}_1\frac{d^2p}{dn^2}+\bar{\nu}_2\frac{d^2q}{dn^2}+\bar{\nu}_3\frac{d^2r}{dn^2}+\bar{\nu}_4\frac{d^2t}{dn^2}\right)+\sum\frac{d\bar{\epsilon}_{11}}{dn}\left(\frac{dp}{dn}\right)^2.$$

(i) To evaluate the first set of terms, we write

$$N=\frac{1}{\Omega\epsilon_n}(\sum a\bar{\nu}_1^2)-\frac{\epsilon_{nn}^2}{\epsilon_n};$$

and therefore

$$\nu_1\frac{d^2p}{dn^2}+\nu_2\frac{d^2q}{dn^2}+\nu_3\frac{d^2r}{dn^2}+\nu_4\frac{d^2t}{dn^2}-N$$
$$=-\nu_1\left\{\sum\Gamma_{11}\left(\frac{dp}{dn}\right)^2\right\}-\nu_2\left\{\sum\Delta_{11}\left(\frac{dq}{dn}\right)^2\right\}$$
$$-\nu_3\left\{\sum\Theta_{11}\left(\frac{dr}{dn}\right)^2\right\}-\nu_4\left\{\sum\Phi_{11}\left(\frac{dt}{dn}\right)^2\right\}$$

which may be written

$$-\sum_i\sum_j\sum_k\sum_l\left[\bar{\epsilon}_{kl}\{ij,l\}\frac{dx_i}{dn}\frac{dx_j}{dn}\frac{dx_k}{dn}\right].$$

(ii) For the summation, we use the relation

$$\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}=\bar{\epsilon}_{ijk}+\bar{\epsilon}_i(jk)+\bar{\epsilon}_j(ki)+\tfrac{1}{3}\epsilon_n\sum_\mu\frac{dx_\mu}{dn}[(\mu i,jk)+(\mu j,ki)];$$

and therefore, in that term which can be taken in the form

$$\sum_i\sum_j\sum_k\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}\frac{dx_i}{dn}\frac{dx_j}{dn}\frac{dx_k}{dn},$$

the total coefficient of the combination $\dfrac{dx_i}{dn}\dfrac{dx_j}{dn}\dfrac{dx_k}{dn}$ consists of three parts, arising from

$$2\frac{\partial\bar{\epsilon}_{ij}}{\partial x_k}+2\frac{\partial\bar{\epsilon}_{ki}}{\partial x_j}+2\frac{\partial\bar{\epsilon}_{jk}}{\partial x_i}.$$

The part arising from the Riemann four-index symbols has, for the coefficient of

$$\tfrac{2}{3}\epsilon_n \frac{dx_\mu}{dn},$$

the sum of the quantities

$$(\mu i, jk) + (\mu j, ki), \quad (\mu k, ij) + (\mu i, kj), \quad (\mu j, ki) + (\mu k, ij),$$

a sum which is zero : hence this part vanishes.

The part arising from the quantities $\bar{\epsilon}_a(\beta\gamma)$ is

$$\begin{aligned} &2\{\bar{\epsilon}_i(jk) + \bar{\epsilon}_j(ki)\} + 2\{\bar{\epsilon}_k(ij) + \bar{\epsilon}_i(kj)\} + 2\{\bar{\epsilon}_j(ki) + \bar{\epsilon}_k(ij)\} \\ &= 4\{\bar{\epsilon}_i(jk) + \bar{\epsilon}_j(ki) + \bar{\epsilon}_k(ij)\} \\ &= 4\sum_l [\bar{\epsilon}_{il}\{jk, l\} + \bar{\epsilon}_{jl}\{ki, l\} + \bar{\epsilon}_{kl}\{ij, l\}], \end{aligned}$$

the summation being for $l=1, 2, 3, 4$.

The part arising from the quantities $\bar{\epsilon}_{ijk}$ is $6\bar{\epsilon}_{ijk}$.

Consequently, the summation in the expression for $\dfrac{d^3\epsilon}{dn^3}$ is

$$\begin{aligned} &= \sum_i \sum_j \sum_k \bar{\epsilon}_{ijk} \frac{dx_i}{dn}\frac{dx_j}{dn}\frac{dx_k}{dn} \\ &\qquad + 2\sum_i \sum_j \sum_k \bar{\epsilon}_{kl}\{ij, l\} \frac{dx_i}{dn}\frac{dx_j}{dn}\frac{dx_k}{dn}, \end{aligned}$$

where the summations on the right-hand side are over the values of i, j, k, independently of one another and not in combination.

When the values thus obtained are substituted, we find

$$\frac{d^3\epsilon}{dn^3} + 2\frac{\epsilon_{nn}^2}{\epsilon_n} - \frac{2}{\Omega\epsilon_n}\sum a\bar{\nu}_1^2 = \sum_i \sum_j \sum_k \bar{\epsilon}_{ijk} \frac{dx_i}{dn}\frac{dx_j}{dn}\frac{dx_k}{dn},$$

with the convention $x_1, x_2, x_3, x_4 = p, q, r, t$, and the definition, for $i=1, 2, 3, 4$,

$$\bar{\nu}_i = \bar{\epsilon}_{1i}\frac{dp}{dn} + \bar{\epsilon}_{2i}\frac{dq}{dn} + \bar{\epsilon}_{3i}\frac{dr}{dn} + \bar{\epsilon}_{4i}\frac{dt}{dn}.$$

Ex. The values of the quantities

$$\frac{d}{ds}\left(\frac{d\epsilon}{dn}\right), \quad \frac{d}{dn}\left(\frac{d\epsilon}{dn}\right), \quad \frac{d^2}{dn^2}\left(\frac{d\epsilon}{dn}\right)$$

have been given in the text, because they are required in subsequent investigations that occur. The determination of the values of

$$\frac{d}{dn}\left\{\frac{d}{ds}\left(\frac{d\epsilon}{dn}\right)\right\}, \quad \frac{d}{ds}\left\{\frac{d}{dn}\left(\frac{d\epsilon}{dn}\right)\right\}, \quad \frac{d^2}{ds^2}\left(\frac{d\epsilon}{dn}\right),$$

is left as an exercise.

Parametric region touching a geodesic region.

326. Consider any domainal region represented by a parametric equation

$$\epsilon(p, q, r, t)=0;$$

directions in this region are subject to the sole condition

$$\epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t'=0.$$

Let a domainal geodesic be drawn through O in this direction; at any point O', at an arc-distance δ from O along the geodesic, the parameters P, Q, R, T, are given by expressions of the type

$$P=p+p'\delta+\tfrac{1}{2}p''\delta^2+\tfrac{1}{6}p'''\delta^3+\dots,$$

all the magnitudes on the right-hand side belonging to the domain, so that

$$-p''=\sum\Gamma_{11}p'^2, \quad -p'''=\sum\Gamma_{111}p'^3,$$

and so on, with like values for the arc-derivatives of q, r, t. If the domainal geodesic lies wholly in the region, the coordinates of O' must satisfy the parametric equation of the region; and then the relation

$$\epsilon(P, Q, R, T)=0$$

must be satisfied for all values of δ. If the domainal geodesic has only contact with the region, the order of that contact is measured by the number of successive powers of δ, the coefficients of which vanish.

Let the parametric equation at O' be expressed in powers of δ, in the form

$$E_0+E_1\delta+\tfrac{1}{2}E_2\delta^2+\tfrac{1}{6}E_3\delta^3+\dots=0;$$

we have to determine how far, in successive powers of δ, this equation is satisfied.

(i) The first term E_0 always vanishes, for it is $\epsilon(p, q, r, t)$.

(ii) The second term also vanishes; for

$$E_1=\epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t',$$

and we are discussing domainal geodesics in any direction p', q', r', t', touching the region. If, in particular, the region is to be geodesic to the domain at O, so that p', q', r', t', denote any direction contained in the flat, which is defined by three non-complanar directions of domainal geodesics and which has P_0, Q_0, R_0, T_0, for its variables of regional orientation, the equation

$$Z=P_0p'+Q_0q'+R_0r'+T_0t'=0$$

must be satisfied; and the two equations $E_1=0$, $Z=0$, then are the same. In any event, the equation $E_1=0$ is the condition of tangency to the region.

(iii) As regards the coefficient of $\frac{1}{2}\delta^2$, we have

$$E_2=\epsilon_1 p''+\epsilon_2 q''+\epsilon_3 r''+\epsilon_4 t''+\sum \epsilon_{11}p'^2$$
$$=\sum_i \sum_j \bar{\epsilon}_{ij} x_i' x_j',$$

with the notation (p. 254) for the symbols $\bar{\epsilon}_{ij}$ and the convention as to variables x_1', x_2', x_3', $x_4'=p'$, q', r', t'. Accordingly, the domainal geodesic has contact of the second order with the region, along the direction p', q', r', t', under the condition

$$\sum_i \sum_j \bar{\epsilon}_{ij} x_i' x_j'=0.$$

The condition may be satisfied in two different ways, both subject to the linear condition $E_1=0$. It may be satisfied only for those values of the direction-variables which simultaneously satisfy the two equations

$$E_1=0, \quad E_2=0;$$

that is, there * are four single infinitudes of directions, constituting a four-sheeted surface, along any one of which a domainal geodesic has inflexional contact with the region. Or the condition $E_2=0$ may be satisfied for all values of p', q', r', t', satisfying $E_1=0$; the residuary relations, expressing this property, will be developed later (§ 327). In either event, the equation $E_2=0$ is the condition that the domainal flexure of the regional geodesic, drawn in the direction p', q', r', t', shall vanish.

As our main concern is with geodesic regions, and with possible orders of contact between a parametric region and a geodesic region in the same orientation P_0, Q_0, R_0, T_0, we shall not discuss in any detail the significance of the first mode of satisfying the condition $E_2=0$. Obviously, any region within the domain possesses surfaces of the type indicated: they may be called the *inflexional surfaces* of the region.

(iv) The coefficient of $\frac{1}{6}\delta^3$ is given by

$$E_3=\sum \epsilon_1 p'''+3\sum \epsilon_{11}p'p''+\sum \epsilon_{300}p'^3,$$

with the customary significance for the signs of summation. When the values of quantities such as p''' and p'' are inserted, the total coefficient of

$$\frac{6}{i!\,j!\,k!} x_i' x_j' x_k'$$

on the right-hand side

$$\begin{aligned}=\epsilon_{ijk} &- (\epsilon_{1i}\Gamma_{jk}+\epsilon_{1j}\Gamma_{ki}+\epsilon_{1k}\Gamma_{ij})\\
&- (\epsilon_{2i}\Delta_{jk}+\epsilon_{2j}\Delta_{ki}+\epsilon_{2k}\Delta_{ij})\\
&- (\epsilon_{3i}\Theta_{jk}+\epsilon_{3j}\Theta_{ki}+\epsilon_{3k}\Theta_{ij})\\
&- (\epsilon_{4i}\Phi_{jk}+\epsilon_{4j}\Phi_{ki}+\epsilon_{4k}\Phi_{ij})\\
&- (\epsilon_1\Gamma_{ijk}+\epsilon_2\Delta_{ijk}+\epsilon_3\Theta_{ijk}+\epsilon_4\Phi_{ijk}),\end{aligned}$$

* See footnote, p. 417.

which is one of the transformed values of the quantity $\bar{\epsilon}_{ijk}$ in § 325. Hence

$$E_3 = \sum_i \sum_j \sum_k \bar{\epsilon}_{ijk} x_i' x_j' x_k'.$$

Accordingly, the domainal geodesic has contact of the third order with the region (the conditions for contact of the second order being supposed satisfied) for domainal directions which touch the region and satisfy the equation

$$\sum_i \sum_j \sum_k \bar{\epsilon}_{ijk} x_i' x_j' x_k' = 0.$$

Taken in association with the preceding necessary conditions, the condition may be satisfied in various ways, partly dependent upon the alternatives arising in connection with $E_2 = 0$.

(*a*) It may be satisfied when the quantities E_1, E_2, E_3, are not subject to relations among their coefficients. Then there are six sets of values of p', q', r', t', satisfying all these conditions: or, in any parametric region in the domain, there are six directions along which a domainal geodesic has contact of the third order with the region, and they lie on the inflexional surface of the region.

(*b*) If the equation $E_2 = 0$ is satisfied for all directions touching the region, so that $E_2 = 0$ is now not independent of $E_1 = 0$, then the requirement is satisfied for all values of p', q', r', t', which simultaneously satisfy the two equations

$$E_1 = 0, \quad E_3 = 0:$$

that is, there are * six single infinitudes of directions, constituting a six-sheeted surface, along any one of which a domainal geodesic has triple contact with the region.

(*c*) If the equation $E_2 = 0$ is satisfied for all directions, touching the region and therefore satisfying $E_1 = 0$, the equation $E_3 = 0$ may be satisfied for all these directions; residuary relations, expressing the property, will be framed later.

The foregoing inferences relate primarily to a region which is postulated by a parametric equation. But the analytical expression of a geodesic region can be framed otherwise. The parameters of a current point O' on a domainal geodesic are given by the quantities P, Q, R, T, as on p. 415. On the assumption that

* The result is obtained by using the expressions for p', q', r', t', of the type $p' = \lambda p_1' + \mu p_2' + \nu p_3'$, in terms of any three sets of directions of the region. These parametric forms, involving λ, μ, ν, satisfy the equation $E_1 = 0$ unconditionally. An equation $E_2 = 0$ then becomes a homogeneous quadratic relation in λ, μ, ν; and an equation $E_3 = 0$ becomes a homogeneous cubic relation in λ, μ, ν. There is also the (non-homogeneous) quadratic relation in λ, μ, ν, arising from the arc-relation; so that $E_2 = 0$ then leads to an equation of degree four in the two parameters λ and μ, after elimination of ν; and $E_3 = 0$ then leads to an equation of degree six, in like circumstances.

the geodesic region is to be defined in connection with the regional orientation P_0, Q_0, R_0, T_0, the equations

$$\sum Ap'^2=1, \quad Z=P_0p'+Q_0q'+R_0r'+T_0t'=0,$$

render the direction-variables p', q', r', t', equivalent to two independent magnitudes. Hence the parameters P, Q, R, T, of the current point O' involve these two magnitudes, as well as the variable δ of length along the geodesic: that is, three independent quantities. The locus of O' is therefore a region: it is the geodesic region of the domain in the orientation P_0, Q_0, R_0, T_0; and its equation

$$G(P, Q, R, T)=0$$

can be regarded as the result of eliminating the five quantities p', q', r', t', δ, from the six equations. Moreover, when the stated values of P, Q, R, T, are substituted in $G=0$ and the equation is then arranged in powers of δ, the coefficients of all powers of δ vanish.

The eliminant equation $G=0$ and the earlier parametric equation $\epsilon=0$ may be functionally equivalent to one another: in that event, the ϵ-region is geodesic to the domain. When the two equations are not equivalent, there still may be a partial agreement between them, arising from vanishing coefficients of successive powers of δ in the expanded form of the equation $\epsilon=0$; and the vanishing of such terms in the parametric equation can be taken as indications of orders of contact between the two regions.

The vanishing of the term E_0 merely expresses the fact that the parametric region contains the point O.

The vanishing of the term in the first power of δ gives

$$E_1=\epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t'=0.$$

When this relation holds for all values of p', q', r', t', that satisfy $Z=0$, the two regions have the same tangent flat; and this geometrical property can be called *contact of the first order* between the two regions. But if the relation does not hold for all such values, the three equations

$$\sum Ap'^2=1, \quad E_1=0, \quad Z=0,$$

are the parametric equations of the tangent plane at O to the surface of intersection of the parametric region and the geodesic region *. We shall assume that the two regions certainly have contact of the first order, so that they have the same regional orientation.

The vanishing of the term in the second power of δ gives

$$E_2=\sum \epsilon_{11} p'^2=0.$$

When this relation holds for all values of p', q', r', t', in the assumed regional

* This analytical possibility, with a different significance for Z, occurs in the discussion (§ 344) of geodesic surfaces in a domain.

orientation at O, it expresses the property that every domainal geodesic through O in the geodesic region osculates the parametric region. This geometrical property can be called *contact of the second order* between the two regions. But if the relation does not hold for all such values, the three equations

$$\sum Ap'^2=1, \quad E_1=0, \quad E_2=0,$$

can be regarded as equations of the cone of contact to the surface of intersection of the two regions at O which is a conical point on that surface. (All special or degenerate forms of the cone are included in the phrase.)

Similarly, the vanishing of the term in the third power of δ gives

$$E_3=\sum \bar{\epsilon}_{111}p'^3=0.$$

When this relation, as well as $E_2=0$, is satisfied for all values of p', q', r', t', in the regional orientation at O, it expresses the property that every domainal geodesic through O in the geodesic region has third-order contact with the parametric region. This geometrical property can be called *contact of the third order* between the two regions. But if the relation does not hold for all such values of p', q', r', t', there are various possibilities: thus, in the extreme instance, the equations

$$\sum Ap'^2=1, \quad E_1=0, \quad E_2=0, \quad E_3=0,$$

would determine those six generators of the tangent cone of the intersection of the two regions, which are the directions of domainal geodesics having inflexional contact with the parametric region.

In effect, this mode of estimating the relations between the two regions establishes, for a regional orientation in the domain at O, the geodesic region as the uniquely determinate region of reference for all regions with the orientation. It is the further extension of the mode which establishes (§ 210) a geodesic surface in a region as the surface of reference for all regional surfaces in the same orientation.

We proceed to the analytical significance of these conditions of contact of successive orders between the geodesic region and a parametric region.

Conditions for orders of contact between a parametric region and a geodesic region.

327. I. The first and governing equation is

$$E_1=\epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t'=0.$$

Let p_i', q_i', r_i', t_i', for $i=1, 2, 3$, be three different non-complanar directions in the domain; when the domainal geodesics are drawn through O in these directions, and (as in § 323) lead to the construction of a region geodesic at O to the domain, the foregoing equation $E_1=0$ must be satisfied by each of the three directions when $\epsilon=0$ represents this geodesic region. Accordingly, if P_0, Q_0, R_0, T_0, denote

the regional variables of orientation defined (§ 269) by three such directions, we have

$$\frac{\epsilon_1}{P_0}=\frac{\epsilon_2}{Q_0}=\frac{\epsilon_3}{R_0}=\frac{\epsilon_4}{T_0}=\Omega^{\frac{1}{2}}\frac{\epsilon_n}{\Xi^{\frac{1}{2}}},$$

where

$$\Xi=1-\cos^2\widehat{23}-\cos^2\widehat{31}-\cos^2\widehat{12}+2\cos\widehat{23}\cos\widehat{31}\cos\widehat{12},$$

the symbol $\widehat{ij}$ denoting the angle between the directions p_i', q_i', r_i', t_i', and p_j', q_j', r_j', t_j'.

II. The second-order condition is

$$E_2=\sum\bar{\epsilon}_{11}p'^2=0\,;$$

and it is to be satisfied for all directions p', q', r', t', in the regional orientation. Let the value of t', given by

$$t'=-\frac{\epsilon_1}{\epsilon_4}p'-\frac{\epsilon_2}{\epsilon_4}q'-\frac{\epsilon_3}{\epsilon_4}r',$$

be substituted in the condition, which then becomes a homogeneous quadratic relation in the linearly independent quantities p', q', r'. The condition, thus modified, is to be satisfied; hence the coefficients of the various combinations of p', q', r', must vanish. Consequently there are six relations among the coefficients $\bar{\epsilon}_{ij}$; and they can be arranged in the symmetric form

$$2\frac{\bar{\epsilon}_{ij}}{\epsilon_i\epsilon_j}=\frac{\bar{\epsilon}_{ii}}{\epsilon_i^2}+\frac{\bar{\epsilon}_{jj}}{\epsilon_j^2},$$

for the combinations i, j, $=1, 2, 3, 4$.

In the first place, these six relations at O are necessary to secure that the second-order condition is satisfied for all directions in the orientation. Conversely, when, by means of them, substitution is made for the six coefficients $\bar{\epsilon}_{ij}$ in E_2, we have

$$\sum\bar{\epsilon}_{11}p'^2=(\epsilon_1p'+\epsilon_2q'+\epsilon_3r'+\epsilon_4t')\left(\frac{\bar{\epsilon}_{11}}{\epsilon_1}p'+\frac{\bar{\epsilon}_{22}}{\epsilon_2}q'+\frac{\bar{\epsilon}_{33}}{\epsilon_3}r'+\frac{\bar{\epsilon}_{44}}{\epsilon_4}t'\right)$$

identically; thus the relations are also sufficient to secure that the second-order condition is satisfied for all admissible directions.

Ex. 1. Prove that, when the six relations for contact of the second order between the parametric ϵ-region and the geodesic region are satisfied, the equations

$$\bar{\epsilon}_{i1}p'+\bar{\epsilon}_{i2}q'+\bar{\epsilon}_{i3}r'+\bar{\epsilon}_{i4}t'=\frac{\epsilon_i}{\epsilon_n}\frac{d\epsilon_n}{ds},$$

(for $i=1, 2, 3, 4$), hold for all directions in the orientation.

Ex. 2. Verify, for the six relations of second-order contact between the parametric region and the geodesic region, their invariantive character under all functional transformations of the parametric equation in the form

$$E=f(\epsilon)=\text{constant}.$$

In the second place, owing to the six relations, not merely does the determinant ∇ vanish, where ∇ denotes the discriminant of the quaternary quadratic $\sum \bar{\epsilon}_{11} p'^2$ and is given by

$$\nabla = \begin{vmatrix} \bar{\epsilon}_{11}, & \bar{\epsilon}_{12}, & \bar{\epsilon}_{13}, & \bar{\epsilon}_{14} \\ \bar{\epsilon}_{21}, & \bar{\epsilon}_{22}, & \bar{\epsilon}_{23}, & \bar{\epsilon}_{24} \\ \bar{\epsilon}_{31}, & \bar{\epsilon}_{32}, & \bar{\epsilon}_{33}, & \bar{\epsilon}_{34} \\ \bar{\epsilon}_{41}, & \bar{\epsilon}_{42}, & \bar{\epsilon}_{43}, & \bar{\epsilon}_{44} \end{vmatrix};$$

but also every first minor of ∇ vanishes. Now ∇ has ten such minors; their simultaneous vanishing requires only three independent relations, which can be taken to be any trio of the set $\bar{E}_{ii}=0$, where $\bar{E}_{ii}$ denotes the minor of $\bar{\epsilon}_{ii}$.

These three relations are the conditions that the quaternary quadratic form $\sum \bar{\epsilon}_{11} p'^2$ should be the product of two factors, each linear in the variables p', q', r', t'. The whole set of six relations becomes necessary, in order that one of these linear factors should be $\epsilon_1{}' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t'$.

The foregoing relations are essentially the same as the analytical conditions under which a quadric surface in homaloidal triple space becomes special or degenerates. We may regard p', q', r', t', as homogeneous coordinates of a point in a three-dimensional space: then $E_1=0$ represents a plane, and $E_2=0$ represents a quadric, in that space. When $\nabla=0$ is the sole relation satisfied, the quadric is a cone. When the three independent relations hold, by means of the vanishing minors of ∇, the quadric degenerates into two planes. When all the six relations hold, $E_1=0$ is one of the planes into which the quadric has degenerated.

Ex. 3. Verify that the relations among the quantities $\bar{\epsilon}_{ij}$ can be expressed in the form

$$\bar{\epsilon}_{13} = \alpha\bar{\epsilon}_{33} + \beta\bar{\epsilon}_{34}, \quad \epsilon_{23} = \gamma\bar{\epsilon}_{33} + \delta\bar{\epsilon}_{34},$$
$$\bar{\epsilon}_{14} = \alpha\bar{\epsilon}_{34} + \beta\bar{\epsilon}_{44}, \quad \epsilon_{24} = \gamma\bar{\epsilon}_{34} + \delta\bar{\epsilon}_{44},$$
$$\bar{\epsilon}_{11} = \alpha^2\bar{\epsilon}_{33} + 2\alpha\beta\bar{\epsilon}_{34} + \beta^2\bar{\epsilon}_{44},$$
$$\bar{\epsilon}_{12} = \alpha\gamma\bar{\epsilon}_{33} + (\alpha\delta + \beta\gamma)\bar{\epsilon}_{34} + \beta\delta\bar{\epsilon}_{44},$$
$$\bar{\epsilon}_{22} = \gamma^2\bar{\epsilon}_{33} + 2\gamma\delta\bar{\epsilon}_{34} + \delta^2\bar{\epsilon}_{44}.$$

Ex. 4. For the special class of parametric regions, which have contact of the second order with the geodesic in the same regional orientation, and which are defined by the equations

$$\frac{\bar{\epsilon}_{11}}{\epsilon_1{}^2} = \frac{\bar{\epsilon}_{ij}}{\epsilon_i \epsilon_j} = \frac{\bar{\epsilon}_{44}}{\epsilon_4{}^2},$$

for all combinations i, j, $=1, 2, 3, 4$, prove that the common value of these equal fractions is

$$\frac{\epsilon_{nn}}{\epsilon_n{}^2},$$

and that the normal dilatation of the region, so defined, has a vanishing first arc-derivative for all directions in the regional orientation at O.

III. The third-order condition is

$$E_3=\sum \bar{\epsilon}_{111}p'^3=0\,;$$

and it is to be satisfied for all values of p', q', r', t', in the regional orientation. Let the value of t', given by

$$t'=-\frac{\epsilon_1}{\epsilon_4}p'-\frac{\epsilon_2}{\epsilon_4}q'-\frac{\epsilon_3}{\epsilon_4}r',$$

be substituted in the condition, which then becomes a homogeneous cubic relation in the linearly independent quantities p', q', r'. Proceeding as before with the second-order condition, we now find that ten relations must hold among the quantities $\bar{\epsilon}_{ijk}$; and these can be expressed in symmetric form, by the six relations

$$3\frac{\bar{\epsilon}_{iim}}{\epsilon_i^2\epsilon_m}-\frac{\bar{\epsilon}_{iii}}{\epsilon_i^3}=3\frac{\bar{\epsilon}_{imm}}{\epsilon_i\epsilon_m^2}-\frac{\bar{\epsilon}_{mmm}}{\epsilon_m^3}=2\frac{\zeta_{im}}{\epsilon_i\epsilon_m},$$

for the combinations i, m, $=1, 2, 3, 4$, and the four relations

$$3\bar{\epsilon}_{ijk}=\epsilon_i\zeta_{jk}+\epsilon_j\zeta_{ki}+\epsilon_k\zeta_{ij},$$

for the combinations i, j, k, $=1, 2, 3, 4$, where the symbols ζ are defined by the values of the respective pairs of equal quantities in the first six relations.

These ten relations are necessary, in order to secure the third-order condition; it is easy to verify that they are also sufficient to that end, by shewing that the quaternary cubic E_3 contains $\epsilon_1p'+\epsilon_2q'+\epsilon_3r'+\epsilon_4t'$ as a factor when the ten relations are satisfied.

The analogy (p. 421), between the second-order condition and a quadric in triple space, can be extended to the third-order condition. With the same interpretation assigned to p', q', r', t', we can regard the third-order condition as the equation of a cubic surface in triple space which, on account of the factorial quality of E_3, degenerates * into the plane $E_1=0$ and a quadric in that space.

Ex. Prove that if a quaternary cubic form C is the product of a quaternary linear form L and a quaternary quadratic form Q, the Hessian of C contains L as a factor.

Hence or otherwise shew that, when a parametric region and a parametric region in the same orientation have contact of the third order, the equation

$$\begin{vmatrix} c_{11}, & c_{12}, & c_{13}, & c_{14} \\ c_{21}, & c_{22}, & c_{23}, & c_{24} \\ c_{31}, & c_{32}, & c_{33}, & c_{34} \\ c_{41}, & c_{42}, & c_{43}, & c_{44} \end{vmatrix}=0$$

* For properties of cubic surfaces, see Baker's *Principles of Geometry* (1923), vol. iii, chap. iv.

is satisfied for all directions in the orientation, where

$$c_{ij} = \tfrac{1}{6} \frac{\partial^2 E_3}{\partial x_i' \partial x_j'},$$

with the customary convention as to the variables x.

NOTE. The umbral notation can be used for a simple construction of these conditions in the same manner as for the construction of the corresponding conditions of contact between parametric surfaces and geodesic surfaces in a region (p. 64).

We take sets of umbral symbols $a, b, c, \ldots$ associated with E_2, and sets of umbral symbols $\alpha, \beta, \gamma, \ldots$ associated with E_3, so that

$$E_2 = \sum \bar{\epsilon}_{11}\, p'^2 = (a_1 p' + a_2 q' + a_3 r' + a_4 t')^2 = a_{p'}^2,$$

$$E_3 = \sum \bar{\epsilon}_{111} p'^3 = (\alpha_1 p' + \alpha_2 q' + \alpha_3 r' + \alpha_4 t')^2 = \alpha_{p'}^3 ;$$

and we write

$$\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = Z\epsilon_4,$$

the symbols ϵ being non-umbral. Then

$$a_{p'} = G_1 p' + G_2 q' + G_3 r' + a_4 Z, \quad \alpha_{p'} = \Gamma_1 p' + \Gamma_2 q' + \Gamma_3 r' + \alpha_4 Z,$$

where, for $i = 1, 2, 3$,

$$G_i = a_i - \frac{\epsilon_i}{\epsilon_4} a_4, \quad \Gamma_i = \alpha_i - \frac{\epsilon_i}{\epsilon_4} \alpha_4 ;$$

and the quantities E_2, E_3, acquire the forms

$$E_2 = (G_1 p' + G_2 q' + G_3 r' + a_4 Z)^2, \quad E_3 = (\Gamma_1 p' + \Gamma_2 q' + \Gamma_3 r' + \alpha_4 Z)^3.$$

The directions, in the regional orientation at O which is common to the parametric region and the geodesic region, are given by $Z = 0$.

(i) The condition for second-order contact between the two regions is

$$(G_1 p' + G_2 q' + G_3 r')^2 = 0,$$

to be satisfied for all values of the three independent quantities p', q', r'. There are six necessary relations. In umbral form, they are

$$G_m{}^2 = 0, \quad G_m G_n = 0,$$

for all values $m, n, = 1, 2, 3$. When expressed in terms of the coefficients $\bar{\epsilon}_{ij}$, they are

$$\bar{\epsilon}_{mm} - 2\frac{\epsilon_m}{\epsilon_4}\bar{\epsilon}_{m4} + \frac{\epsilon_m{}^2}{\epsilon_4{}^2}\bar{\epsilon}_{44} = 0, \quad \bar{\epsilon}_{mn} - \frac{\epsilon_m}{\epsilon_4}\bar{\epsilon}_{n4} - \frac{\epsilon_n}{\epsilon_4}\bar{\epsilon}_{m4} + \frac{\epsilon_m \epsilon_n}{\epsilon_4{}^2}\bar{\epsilon}_{44} = 0 ;$$

and these can be modified to the forms in the text.

Moreover, when the six relations are satisfied, we have

$$E_2 = \frac{\partial E_2}{\partial t'} Z + \bar{\epsilon}_{44} Z^2,$$

identically; and therefore the six relations are sufficient to ensure that the condition $E_2 = 0$ for second-order contact is satisfied for all directions in the regional orientation, as given by $Z = 0$.

(ii) Similarly, the condition for third-order contact between the regions is

$$(\Gamma_1 p' + \Gamma_2 q' + \Gamma_3 r')^3 = 0,$$

to be satisfied for all values of the three independent quantities p', q', r'. There are ten necessary relations. In umbral form, they are

$$\Gamma_m{}^3=0, \quad \Gamma_m{}^2\Gamma_n=0, \quad \Gamma_1\Gamma_2\Gamma_3=0,$$

for all values m, n, $=1, 2, 3$. When expressed in terms of the coefficients of E_3, they are

$$\bar{\epsilon}_{mmm}-3\frac{\epsilon_m}{\epsilon_4}\bar{\epsilon}_{mm4}+3\frac{\epsilon_m{}^2}{\epsilon_4{}^2}\bar{\epsilon}_{m44}-\frac{\epsilon_m{}^3}{\epsilon_4{}^3}\bar{\epsilon}_{444}=0,$$

$$\bar{\epsilon}_{mmn}-\frac{1}{\epsilon_4}(2\epsilon_m\bar{\epsilon}_{mn4}+\epsilon_n\bar{\epsilon}_{mm4}) \\ +\frac{1}{\epsilon_4{}^2}(2\epsilon_m\epsilon_n\bar{\epsilon}_{m44}+\epsilon_m{}^2\bar{\epsilon}_{n44})-\frac{\epsilon_m{}^2\epsilon_n}{\epsilon_4{}^3}\bar{\epsilon}_{444}=0,$$

$$\bar{\epsilon}_{123}-\frac{1}{\epsilon_4}(\epsilon_1\bar{\epsilon}_{234}+\epsilon_2\bar{\epsilon}_{314}+\epsilon_3\bar{\epsilon}_{124}) \\ +\frac{1}{\epsilon_4{}^2}(\epsilon_2\epsilon_3\bar{\epsilon}_{144}+\epsilon_3\epsilon_1\bar{\epsilon}_{244}+\epsilon_1\epsilon_2\bar{\epsilon}_{344})-\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_4{}^3}\bar{\epsilon}_{444}=0;$$

and these can be modified to the forms in the text.

Also, when the ten relations are satisfied, we have

$$E_3=\frac{\partial E_3}{\partial t'}Z+\tfrac{1}{2}\frac{\partial^2 E_3}{\partial t'^2}Z^2+\bar{\epsilon}_{444}Z^3,$$

identically; and therefore the ten relations are sufficient to ensure that the condition for third-order contact is satisfied.

Quadruply orthogonal families of regions in a domain.

328. As a triple system of orthogonal surfaces can exist in a region (or a flat), so a quadruple system of orthogonal regions can exist in a domain (or a block); and just as no one of the families of surfaces in the triple system can be assumed arbitrarily (§ 262), so it will appear that no one of the families of regions in the quadruple system can be assumed arbitrarily.

Let a possible quadruple system of regions in a domain be represented by the equations

$$\alpha(p, q, r, t)=\alpha, \quad \epsilon(p, q, r, t)=\epsilon, \quad \iota(p, q, r, t)=\iota, \quad \kappa(p, q, r, t)=\kappa,$$

where the right-hand magnitudes α, ϵ, ι, κ, denote the parameters of the four families: and, as before, let

$$\begin{aligned} a\omega_1+h\omega_2+g\omega_3+l\omega_4&=\bar{\Omega}_1,\\ h\omega_1+b\omega_2+f\omega_3+m\omega_4&=\bar{\Omega}_2,\\ g\omega_1+f\omega_2+c\omega_3+n\omega_4&=\bar{\Omega}_3,\\ l\omega_1+m\omega_2+n\omega_3+d\omega_4&=\bar{\Omega}_4,\end{aligned}$$

for ω, Ω, $=\alpha$, A; ϵ, E; ι, I; κ, K; in turn. The six conditions of orthogonality of the system can be written in the forms

$$\sum a\alpha_1\epsilon_1=0, \quad \sum a\alpha_1\iota_1=0, \quad \sum a\alpha_1\kappa_1=0,$$
$$\sum a\iota_1\kappa_1=0, \quad \sum a\kappa_1\epsilon_1=0, \quad \sum a\epsilon_1\iota_1=0,$$

and also in the equivalent forms

$$\sum A\bar{A}_1\bar{E}_1=0,\quad \sum A\bar{A}_1\bar{I}_1=0,\quad \sum A\bar{A}_1\bar{K}_1=0,$$
$$\sum A\bar{I}_1\bar{K}_1=0,\quad \sum A\bar{K}_1\bar{E}_1=0,\quad \sum A\bar{E}_1\bar{I}_1=0.$$

All these primary conditions are of the first order; and they relate solely to the directions of domainal normals to the regions at their point of intersection.

For further conditions, which are to express continued orthogonality along intersections, we differentiate the primary conditions with respect to the domainal parameters. Proceeding from the relation

$$\sum \frac{a}{\Omega} a_1\epsilon_1=0,$$

and differentiating with respect to p, we have

$$\sum \frac{a}{\Omega}(a_1\epsilon_{11}+\epsilon_1 a_{11})-\sum \frac{a_1\epsilon_1}{\Omega}(2a\Gamma_{11}+2h\Gamma_{12}+2g\Gamma_{13}+2l\Gamma_{14})=0,$$

each summation extending over the ten terms corresponding to the combinations of the type $a_1\epsilon_1$, $a_1\epsilon_2+a_2\epsilon_1$, ..., $a_4\epsilon_4$. When this relation is re-arranged, it can be expressed in the form

$$\sum_m \bar{A}_m\bar{\epsilon}_{1m}+\sum_m \bar{E}_m\bar{a}_{1m}=0,$$

with the customary significance

$$\bar{\omega}_{ij}=\omega_{ij}-\omega_1\Gamma_{ij}-\omega_2\Delta_{ij}-\omega_3\Theta_{ij}-\omega_4\Phi_{ij},$$

for $\omega=a, \epsilon, \iota, \kappa$. When the same relation is differentiated with respect to q, to r, and to t, in turn, we similarly obtain the successive relations

$$\sum_m \bar{A}_m\bar{\epsilon}_{2m}+\sum_m \bar{E}_m\bar{a}_{2m}=0,$$
$$\sum_m \bar{A}_m\bar{\epsilon}_{3m}+\sum_m \bar{E}_m\bar{a}_{3m}=0,$$
$$\sum_m \bar{A}_m\bar{\epsilon}_{4m}+\sum_m \bar{E}_m\bar{a}_{4m}=0.$$

Let these four relations be multiplied by $\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4$, respectively, and the products be added; then

$$\sum_\lambda\sum_\mu(\bar{I}_\lambda\bar{A}_\mu\bar{\epsilon}_{\lambda\mu})+\sum_\lambda\sum_\mu(\bar{I}_\lambda\bar{E}_\mu\bar{a}_{\lambda\mu})=0.$$

Similarly, when the same four relations are multiplied by $\bar{K}_1, \bar{K}_2, \bar{K}_3, \bar{K}_4$, respectively, and when the results are added, there follows the relation

$$\sum_\lambda\sum_\mu(\bar{K}_\lambda\bar{A}_\mu\bar{\epsilon}_{\lambda\mu})+\sum_\lambda\sum_\mu(\bar{K}_\lambda\bar{E}_\mu\bar{a}_{\lambda\mu})=0.$$

Thus there are two relations of this type, linear in second derivatives of a and ϵ, and otherwise bilinear in the first derivatives of $a, \epsilon, \iota, \kappa$; and they are obtained by combining the derivatives of the single relation $\sum a a_1\epsilon_1=0$ of orthogonality.

Similarly, from the other five primary relations of orthogonality, we obtain the corresponding pairs of secondary relations

$$\left.\begin{aligned}\sum\sum(\bar{E}_\lambda\bar{A}_\mu\iota_{\lambda\mu})+\sum\sum(\bar{E}_\lambda\bar{I}_\mu\bar{a}_{\lambda\mu})=0\\ \sum\sum(\bar{K}_\lambda\bar{A}_\mu\bar{\iota}_{\lambda\mu})+\sum\sum(\bar{K}_\lambda\bar{I}_\mu\bar{a}_{\lambda\mu})=0\end{aligned}\right\},$$

$$\left.\begin{aligned}\sum\sum(\bar{E}_\lambda\bar{A}_\mu\bar{\kappa}_{\lambda\mu})+\sum\sum(\bar{E}_\lambda\bar{K}_\mu\bar{a}_{\lambda\mu})=0\\ \sum\sum(\bar{I}_\lambda\bar{A}_\mu\kappa_{\lambda\mu})+\sum\sum(\bar{I}_\lambda\bar{K}_\mu\bar{a}_{\lambda\mu})=0\end{aligned}\right\},$$

$$\left.\begin{aligned}\sum\sum(\bar{A}_\lambda\bar{I}_\mu\kappa_{\lambda\mu})+\sum\sum(\bar{A}_\lambda\bar{K}_\mu\iota_{\lambda\mu})=0\\ \sum\sum(\bar{E}_\lambda\bar{I}_\mu\bar{\kappa}_{\lambda\mu})+\sum\sum(\bar{E}_\lambda\bar{K}_\mu\bar{\iota}_{\lambda\mu})=0\end{aligned}\right\},$$

$$\left.\begin{aligned}\sum\sum(\bar{A}_\lambda\bar{K}_\mu\epsilon_{\lambda\mu})+\sum\sum(\bar{A}_\lambda\bar{E}_\mu\kappa_{\lambda\mu})=0\\ \sum\sum(\bar{I}_\lambda\bar{K}_\mu\bar{\epsilon}_{\lambda\mu})+\sum\sum(\bar{I}_\lambda\bar{E}_\mu\bar{\kappa}_{\lambda\mu})=0\end{aligned}\right\},$$

$$\left.\begin{aligned}\sum\sum(\bar{A}_\lambda\bar{E}_\mu\bar{\iota}_{\lambda\mu})+\sum\sum(\bar{A}_\lambda\bar{I}_\mu\epsilon_{\lambda\mu})=0\\ \sum\sum(\bar{K}_\lambda\bar{E}_\mu\bar{\iota}_{\lambda\mu})+\sum\sum(\bar{K}_\lambda\bar{I}_\mu\bar{\epsilon}_{\lambda\mu})=0\end{aligned}\right\}.$$

These lead at once to the twelve simpler forms:

$$\sum\sum(\bar{I}_\lambda\bar{K}_\mu\bar{a}_{\lambda\mu})=0,\quad \sum\sum(\bar{K}_\lambda\bar{E}_\mu\bar{a}_{\lambda\mu})=0,\quad \sum\sum(\bar{E}_\lambda\bar{I}_\mu\bar{a}_{\lambda\mu})=0;$$
$$\sum\sum(\bar{K}_\lambda\bar{A}_\mu\bar{\epsilon}_{\lambda\mu})=0,\quad \sum\sum(\bar{A}_\lambda\bar{I}_\mu\bar{\epsilon}_{\lambda\mu})=0,\quad \sum\sum(\bar{I}_\lambda\bar{K}_\mu\bar{\epsilon}_{\lambda\mu})=0;$$
$$\sum\sum(\bar{A}_\lambda\bar{E}_\mu\bar{\iota}_{\lambda\mu})=0,\quad \sum\sum(\bar{E}_\lambda\bar{K}_\mu\bar{\iota}_{\lambda\mu})=0,\quad \sum\sum(\bar{K}_\lambda\bar{A}_\mu\bar{\iota}_{\lambda\mu})=0;$$
$$\sum\sum(\bar{E}_\lambda\bar{I}_\mu\bar{\kappa}_{\lambda\mu})=0,\quad \sum\sum(\bar{I}_\lambda\bar{A}_\mu\bar{\kappa}_{\lambda\mu})=0,\quad \sum\sum(\bar{A}_\lambda\bar{E}_\mu\bar{\kappa}_{\lambda\mu})=0.$$

Now consider the nine equations, in the two groups,

$$\sum a_1\bar{E}_1=0,\qquad \sum a_1\bar{I}_1=0,\qquad \sum a_1\bar{K}_1=0,$$
$$\sum A\bar{I}_1\bar{K}_1=0,\quad \sum A\bar{K}_1\bar{E}_1=0,\quad \sum A\bar{E}_1\bar{I}_1=0,$$

these six being the primary conditions, and the three secondary conditions

$$\sum\sum(\bar{I}_1\bar{K}_1\bar{a}_{11})=0,\quad \sum\sum(\bar{K}_1\bar{E}_1\bar{a}_{11})=0,\quad \sum\sum(\bar{E}_1\bar{I}_1\bar{a}_{11})=0.$$

They involve nine magnitudes in all, made up of the three sets of ratios of the quantities $\bar{E}_\lambda$ to one another, of the quantities $\bar{I}_\lambda$ to one another, and of the quantities $\bar{K}_\lambda$ to one another; the coefficients of the nine magnitudes involve the quantities a_i, $\bar{a}_{ij}$, and the primary magnitudes of the domain; and therefore the equations are potentially sufficient to determine each of these nine ratios. Thus we shall have relations

$$\frac{\bar{E}_1}{P}=\frac{\bar{E}_2}{Q}=\frac{\bar{E}_3}{R}=\frac{\bar{E}_4}{T},$$

(with a like set of relations for $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$, $\bar{I}_4$, and a like set of relations for $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_3$, $\bar{K}_4$: their exact form is not immediately material for the present purpose); and so there are relations

$$\frac{\epsilon_1}{P}=\frac{\epsilon_2}{Q}=\frac{\epsilon_3}{R}=\frac{\epsilon_4}{T},$$

where P, Q, R, T, are functions of the quantities a_i and $\bar{a}_{ij}$, for $i, j, =1, 2, 3, 4$.

Consequently the Pfaffian equation

$$P\,dp+Q\,dq+R\,dr+T\,dt=0,$$

which is equivalent to $d\epsilon=0$, must satisfy the conditions of being an exact equation, which are

$$Q\left(\frac{\partial R}{\partial t}-\frac{\partial T}{\partial r}\right)+R\left(\frac{\partial T}{\partial q}-\frac{\partial Q}{\partial t}\right)+T\left(\frac{\partial Q}{\partial r}-\frac{\partial R}{\partial q}\right)=0,$$

$$R\left(\frac{\partial T}{\partial p}-\frac{\partial P}{\partial t}\right)+T\left(\frac{\partial P}{\partial r}-\frac{\partial R}{\partial p}\right)+P\left(\frac{\partial R}{\partial t}-\frac{\partial T}{\partial r}\right)=0,$$

$$T\left(\frac{\partial P}{\partial q}-\frac{\partial Q}{\partial p}\right)+P\left(\frac{\partial Q}{\partial t}-\frac{\partial T}{\partial q}\right)+Q\left(\frac{\partial T}{\partial p}-\frac{\partial P}{\partial t}\right)=0,$$

$$P\left(\frac{\partial Q}{\partial r}-\frac{\partial R}{\partial q}\right)+Q\left(\frac{\partial R}{\partial p}-\frac{\partial P}{\partial r}\right)+R\left(\frac{\partial P}{\partial q}-\frac{\partial Q}{\partial p}\right)=0,$$

being three linearly independent conditions in all.

Each of these conditions, when substitution of the values of P, Q, R, T, is made in terms of the quantities a_ι and $\bar{a}_{ij}$, becomes a partial differential equation of the third order satisfied by the magnitude a; and therefore the parameter of the family of regions satisfies three partial differential equations of the third order.

A like inference would follow from the consideration of the ratios $\bar{I}_1 : \bar{I}_2 : \bar{I}_3 : \bar{I}_4$, and also from the consideration of the ratios of $\bar{K}_1 : \bar{K}_2 : \bar{K}_3 : \bar{K}_4$. The symmetry of the whole system suggests that the three sets of equations, in each of the combined inferences, are equivalent to a single set of equations. Such a trio of equations can be obtained by an entirely different process, as follows: they arise in connection with the third-order conditions associated with the continued orthogonal intersections.

329. We proceed from a relation

$$\sum_\lambda \sum_\mu \bar{E}_\lambda \bar{I}_\mu \bar{a}_{\lambda\mu}=0,$$

and operate with ϑ_a where

$$\vartheta_a=\bar{A}_1\frac{\partial}{\partial p}+\bar{A}_2\frac{\partial}{\partial q}+\bar{A}_3\frac{\partial}{\partial r}+\bar{A}_4\frac{\partial}{\partial t}=\sum \bar{A}_k\frac{\partial}{\partial x_k},$$

with the usual convention, $x_1, x_2, x_3, x_4, =p, q, r, t$.

From the definition of $\bar{E}_\lambda$, we have

$$\begin{aligned}\frac{\partial \bar{E}_\lambda}{\partial p}&=\sum_m (a_{m\lambda}\epsilon_{m1})+\bar{E}_\lambda\frac{\Omega_1}{\Omega}-\sum_u\sum_m \epsilon_m[a_{um}\{1u,\lambda\}+a_{u\lambda}\{1u,m\}]\\&=\sum_m (a_{m\lambda}\bar{\epsilon}_{m1})+\bar{E}_\lambda\frac{\Omega_1}{\Omega}-\sum_u\sum_m[\epsilon_m a_{um}\{1u,\lambda\}]\\&=\sum_m (a_{m\lambda}\bar{\epsilon}_{m1})+\bar{E}_\lambda\frac{\Omega_1}{\Omega}-\sum_u \bar{E}_u\{1u,\lambda\}.\end{aligned}$$

Similarly for derivatives of $\bar{E}_\lambda$ with respect to q, r, t; and therefore

$$\begin{aligned}\vartheta_a(\bar{E}_\lambda) &= \sum_m a_{m\lambda}(\bar{A}_1\bar{\epsilon}_{m1} + \bar{A}_2\bar{\epsilon}_{m2} + \bar{A}_3\bar{\epsilon}_{m3} + \bar{A}_4\bar{\epsilon}_{m4}) \\ &\qquad + \bar{E}_\lambda\vartheta_a(\log\Omega) - \sum_i\sum_j \bar{A}_i\bar{E}_j\{ij, \lambda\} \\ &= -\sum_m a_{m\lambda}(\bar{E}_1\bar{a}_{m1} + \bar{E}_2\bar{a}_{m2} + \bar{E}_3 a_{m3} + \bar{E}_4 a_{m4}) \\ &\qquad + \bar{E}_\lambda\vartheta_a(\log\Omega) - \sum_i\sum_j \bar{A}_i\bar{E}_j\{ij, \lambda\} \\ &= -\sum_i\sum_j (a_{i\lambda}\bar{E}_j\bar{a}_{ij}) + \bar{E}_\lambda\vartheta_a(\log\Omega) - \sum_i\sum_j \bar{A}_i\bar{E}_j\{ij, \lambda\}.\end{aligned}$$

In the same way, we find

$$\vartheta_a(\bar{I}_\mu) = -\sum_l\sum_m (a_{l\mu}\bar{I}_m\bar{a}_{lm}) + \bar{I}_\mu\vartheta_a(\log\Omega) - \sum_l\sum_m \bar{A}_l\bar{I}_m\{lm, \mu\}.$$

Accordingly, when we operate with ϑ_a on the selected relation so as to obtain a third-order relation in the form

$$\sum_\lambda\sum_\mu [\bar{E}_\lambda\bar{I}_\mu\vartheta_a(\bar{a}_{\lambda\mu}) + \bar{I}_\mu\bar{a}_{\lambda\mu}\vartheta_a(\bar{E}_\lambda) + \bar{E}_\lambda\bar{a}_{\lambda\mu}\vartheta_a(\bar{I}_\mu)] = 0,$$

and when substitution is made for the quantities $\vartheta_a(\bar{E}_\lambda)$ and $\vartheta_a(\bar{I}_\mu)$ with the values just obtained, we have a relation which is homogeneous and bilinear in the quantities $\bar{E}$ and $\bar{I}$, the coefficients of the various combinations being functions of the derivatives of a alone.

Consider the coefficient of $-\bar{I}_h\bar{E}_k$, in so far as it arises from the summation of the second and the third terms. The coefficient of the term involving $\vartheta_a(\log\Omega)$ is

$$-2\bar{a}_{hk}\bar{I}_h\bar{E}_k;$$

in the total double summation, the aggregate of such terms vanishes because of the secondary conditions. The rest of the coefficient of $-\bar{I}_h\bar{E}_k$

$$\begin{aligned}&= \sum_m\sum_n (a_{mn}\bar{a}_{mh}\bar{a}_{nk}) + \sum_e\sum_f [\bar{A}_e\bar{a}_{hf}\{ek, f\}] \\ &\qquad + \sum_m\sum_n (a_{mn}\bar{a}_{mk}\bar{a}_{nh}) + \sum_e\sum_f [\bar{A}_e\bar{a}_{kf}\{eh, f\}] \\ &= 2\sum_m\sum_n (a_{mn}\bar{a}_{mh}\bar{a}_{nk}) + \sum_e\sum_f \bar{A}_e[\bar{a}_{hf}\{ek, f\} + \bar{a}_{kf}\{eh, f\}] \\ &= T_{hk},\end{aligned}$$

using T_{hk} to denote the immediately preceding magnitude. Thus the third-order relation is

$$\sum_\lambda\sum_\mu \bar{E}_\lambda\bar{I}_\mu[\vartheta_a(\bar{a}_{\lambda\mu}) - T_{\lambda\mu}] = 0.$$

It is necessary to develop the magnitude $\vartheta_a(\bar{a}_{\lambda\mu})$. Denoting the element of the

domainal arc normal to a by dN, we have, from the result on p. 409, the set of relations

$$\frac{\partial \bar{a}_{\lambda\mu}}{\partial x_k} = \bar{a}_{\lambda\mu k} + a_\lambda(\mu k) + a_\mu(\lambda k) + \tfrac{1}{3}\frac{da}{dN}\sum_s \frac{dx_s}{dN}[(s\lambda, \mu k) + (s\mu, \lambda k)],$$

where $\bar{a}_{\lambda\mu k}$ has the significance analogous to that of $\bar{\epsilon}_{ijk}$ on p. 412 ; where

$$\begin{aligned} a_i(jk) &= \bar{a}_{1i}\Gamma_{jk} + \bar{a}_{2i}\Delta_{jk} + \bar{a}_{3i}\Theta_{jk} + \bar{a}_{4i}\Phi_{jk} \\ &= \sum_\theta \bar{a}_{i\theta}\{jk, \theta\} ; \end{aligned}$$

and where (§ 269)

$$\Omega \frac{da}{dN}\frac{dx_s}{dN} = a_{1s}a_1 + a_{2s}a_2 + a_{3s}a_3 + a_{4s}a_4 = \bar{A}_s,$$

for all the values of s.

Thus the magnitude $\vartheta_a(\bar{a}_{\lambda\mu})$ consists of three aggregates of terms : (i), those which involve the magnitudes of the type $\bar{a}_{\lambda\mu k}$; (ii), those which are free from the magnitudes $\bar{a}_{\lambda\mu k}$ and from the four-index symbols ; and (iii), those which involve the four-index symbols.

The first aggregate, (i), which consists of the terms, involving the four quantities $\bar{a}_{\lambda\mu k}$ that conserve the symbols λ and μ,

$$\begin{aligned} &= \sum_\gamma \bar{A}_\gamma \bar{a}_{\lambda\mu\gamma} \\ &= \sum_\gamma \sum_\delta a_{\gamma\delta} a_\delta \bar{a}_{\lambda\mu\gamma}. \end{aligned}$$

The second aggregate, (ii), which consists of the terms free from the quantities $a_{\lambda\mu\nu}$ and from the four-index symbols,

$$\begin{aligned} &= \sum_e \bar{A}_e[a_\lambda(\mu e) + a_\mu(\lambda e)] \\ &= \sum_e \sum_f \bar{A}_e[a_{\lambda f}\{e\mu, f\} + a_{\mu f}\{e\lambda, f\}], \end{aligned}$$

which is the same as the second double-summation in the quantity $T_{\lambda\mu}$; and therefore, in the final coefficient of $E_\lambda I_\mu$ in the third-order condition, these two double-summations cancel.

The third aggregate, (iii), which consists of the terms involving the four-index symbols, is found after substitution for $\frac{dx_s}{dN}$ and collection of terms, to be

$$\begin{aligned} &= \frac{1}{3\Omega}\sum_\gamma \sum_\delta \bar{A}_\gamma \bar{A}_\delta[(\gamma\lambda, \mu\delta) + (\gamma\mu, \lambda\delta)]. \\ &= \frac{2}{3\Omega}\sum_\gamma \sum_\delta \bar{A}_\gamma \bar{A}_\delta(\gamma\lambda, \mu\delta) = \frac{2}{3\Omega} J_a(\lambda\mu), \end{aligned}$$

the summation in the quantity denoted by J_a extending over γ and δ independently.

When these results are collected, and we write

$$\bar{\bar{A}}_{\lambda\mu}=\sum_{\gamma}\sum_{\delta}\{a_{\gamma\delta}(a_{\gamma}\bar{a}_{\lambda\mu\delta}-2\bar{a}_{\gamma\lambda}\bar{a}_{\mu\delta})\}+\frac{2}{3\Omega}J_{a}(\lambda\mu),$$

the third-order condition becomes

$$\sum_{\lambda}\sum_{\mu}\bar{E}_{\lambda}\bar{I}_{\mu}\bar{\bar{A}}_{\lambda\mu}=0.$$

This result has been derived from the second-order condition $\sum_{\lambda}\sum_{\mu}\bar{E}_{\lambda}\bar{I}_{\mu}\bar{a}_{\lambda\mu}=0.$ When the second-order conditions

$$\sum_{\lambda}\sum_{\mu}\bar{I}_{\lambda}\bar{K}_{\mu}\,\bar{a}_{\lambda\mu}=0,\quad \sum_{\lambda}\sum_{\mu}\bar{K}_{\lambda}\bar{E}_{\mu}\,\bar{a}_{\lambda\mu}=0$$

are used in the same way, they lead to the third-order conditions

$$\sum_{\lambda}\sum_{\mu}\bar{I}_{\lambda}\bar{K}_{\mu}\bar{\bar{A}}_{\lambda\mu}=0,\quad \sum_{\lambda}\sum_{\mu}\bar{K}_{\lambda}\bar{E}_{\mu}\bar{\bar{A}}_{\lambda\mu}=0.$$

These are three conditions, each involving the third-order derivatives of a; and they are bilinear in the same sets of ratios of the quantities $\bar{E}$, the quantities $\bar{I}$, and the quantities $\bar{K}$, as the earlier conditions of the first order and the second order respectively. But those earlier conditions potentially sufficed for the determination of the three sets, each of three ratios; hence, when the hypothetical values are inserted in the three conditions of the third order affecting a alone, there result three partial differential equations of the third order which must be satisfied by the parameter a of a family of regions in a domain constituting a set in a quadruply orthogonal system of regions in the domain.

The same partial differential equations of the third order must be satisfied by the parameter of each of the four families of regions in the quadruply orthogonal system.

The explicit form of these differential equations remains for determination; and the complete symmetry of the system of equations, as between the three sets of quantities $\bar{E}$, $\bar{I}$, $\bar{K}$, to be eliminated from the a-equations, suggests that all the three equations, so far linear in the magnitudes $\bar{\bar{A}}_{\lambda\mu}$, may be combinable into a single equation, free from the radicals affecting and discriminating the quantities to be eliminated. The following stage in the mode of elimination may be noted.

330. Of the whole set of twelve equations, which are

$$\left.\begin{array}{l}\sum\bar{\bar{A}}_{11}\,\bar{I}_1\bar{K}_1=0\\ \sum\bar{\bar{A}}_{11}\bar{K}_1\bar{E}_1=0\\ \sum\bar{\bar{A}}_{11}\bar{E}_1\,\bar{I}_1=0\end{array}\right\},\quad \left.\begin{array}{l}\sum\bar{a}_{11}\,\bar{I}_1\bar{K}_1=0\\ \sum\bar{a}_{11}\bar{K}_1\bar{E}_1=0\\ \sum\bar{a}_{11}\bar{E}_1\,\bar{I}_1=0\end{array}\right\},\quad \left.\begin{array}{l}\sum A_{11}\,\bar{I}_1\bar{K}_1=0\\ \sum A_{11}\bar{K}_1\bar{E}_1=0\\ \sum A_{11}\bar{E}_1\,\bar{I}_1=0\end{array}\right\},\quad \left.\begin{array}{l}\sum a_1\,\bar{E}_1=0\\ \sum a_1\,\bar{I}_1=0\\ \sum a_1\bar{K}_1=0\end{array}\right\},$$

we first take the three equations:

$$\begin{aligned}(\textstyle\sum\bar{a}_{11}\bar{E}_1)\bar{I}_1+(\sum\bar{a}_{12}\bar{E}_1)\bar{I}_2+(\sum\bar{a}_{13}\bar{E}_1)\bar{I}_3+(\sum\bar{a}_{14}\bar{E}_1)\bar{I}_4&=0,\\ (\textstyle\sum A_{11}\bar{E}_1)\bar{I}_1+(\sum A_{12}\bar{E}_1)\bar{I}_2+(\sum A_{13}\bar{E}_1)\bar{I}_3+(\sum A_{14}\bar{E}_1)\bar{I}_4&=0,\\ a_1\bar{I}_1+\qquad a_2\bar{I}_2+\qquad a_3\bar{I}_3+\qquad a_4\bar{I}_4&=0.\end{aligned}$$

Apparently they should suffice to determine the ratios $\bar{I}_1 : \bar{I}_2 : \bar{I}_3 : \bar{I}_4$, in a form

$$\frac{\bar{I}_1}{P_a}=\frac{\bar{I}_2}{Q_a}=\frac{\bar{I}_3}{R_a}=\frac{\bar{I}_4}{T_a}.$$

But three equations, of precisely the same forms with quantities $\bar{K}$ substituted for quantities $\bar{I}$, occur in the set of twelve; and they would lead to ratios

$$\frac{\bar{K}_1}{P_a}=\frac{\bar{K}_2}{Q_a}=\frac{\bar{K}_3}{R_a}=\frac{\bar{K}_4}{T_a},$$

with the same magnitudes P_a, Q_a, R_a, T_a, as before. If therefore these forms of the ratios are determinate, we should have

$$\frac{\bar{I}_1}{\bar{K}_1}=\frac{\bar{I}_2}{\bar{K}_2}=\frac{\bar{I}_3}{\bar{K}_3}=\frac{\bar{I}_4}{\bar{K}_4},$$

that is,

$$\frac{\iota_1}{\kappa_1}=\frac{\iota_2}{\kappa_2}=\frac{\iota_3}{\kappa_3}=\frac{\iota_4}{\kappa_4},$$

relations that manifestly are impossible in the orthogonal system. Consequently the forms of the ratios cannot be determinate: and therefore

$$P_a=0, \quad Q_a=0, \quad R_a=0, \quad T_a=0.$$

The three selected equations, linear and homogeneous in $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$, $\bar{I}_4$, must therefore be equivalent to two only; and consequently there must exist quantities X and Z such that

$$\begin{aligned}
\sum (\bar{a}_{11}\bar{E}_1) &= Xa_1+Z\sum (A_{11}\bar{E}_1),\\
\sum (\bar{a}_{12}\bar{E}_1) &= Xa_2+Z\sum (A_{12}\bar{E}_1),\\
\sum (\bar{a}_{13}\bar{E}_1) &= Xa_3+Z\sum (A_{13}\bar{E}_1),\\
\sum (\bar{a}_{14}\bar{E}_1) &= Xa_4+Z\sum (A_{14}\bar{E}_1).
\end{aligned}$$

Also, we have the relation

$$\sum (a_1\bar{E}_1)=0.$$

Eliminating the five quantities $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$, $\bar{E}_4$, X, among these equations, we have Z as a root of the equation

$$\begin{vmatrix}
\bar{a}_{11}-ZA_{11}, & \bar{a}_{12}-ZA_{12}, & \bar{a}_{13}-ZA_{13}, & \bar{a}_{14}-ZA_{14}, & a_1\\
\bar{a}_{12}-ZA_{12}, & \bar{a}_{22}-ZA_{22}, & \bar{a}_{23}-ZA_{23}, & \bar{a}_{24}-ZA_{24}, & a_2\\
\bar{a}_{13}-ZA_{13}, & \bar{a}_{23}-ZA_{23}, & \bar{a}_{33}-ZA_{33}, & \bar{a}_{34}-ZA_{34}, & a_3\\
\bar{a}_{14}-ZA_{14}, & \bar{a}_{24}-ZA_{24}, & \bar{a}_{34}-ZA_{34}, & \bar{a}_{44}-ZA_{44}, & a_4\\
a_1 \quad , & a_2 \quad , & a_3 \quad , & a_4 \quad , & 0
\end{vmatrix}=0.$$

This equation is of the third degree in Z, and may be written in the form

$$\theta_0 Z^3-\theta_1 Z^2+\theta_2 Z-\theta_3=0,$$

where each of the four quantities θ is of the second degree in the quantities a_1, a_2, a_3, a_4, and θ_n is of degree n in the quantities $\bar{a}_{rs}$, for $n=0, 1, 2, 3$.

These equations, in form, are the same as the equations for the principal domainal flexures, in magnitude and in direction, belonging to the a-region (§ 319). In particular,

$$Z = -\frac{a_N}{\gamma_a},$$

where a_N denotes the normal dilatation of the region, and γ_a denotes a principal radius of domainal flexure of the region; and, further, the quantities $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$, $\bar{E}_4$, are proportional to the domainal direction-variables of the corresponding principal radius of flexure.

Precisely the same equations are satisfied when the magnitudes $\bar{I}$ are substituted for the magnitudes $\bar{E}$, and also when the magnitudes $\bar{K}$ are substituted for the magnitudes $\bar{E}$. We therefore associate one value of Z, say Z_1, and the corresponding principal domainal flexure (in magnitude and direction), with the ϵ-region and the magnitudes $\bar{E}$; a second value of Z, say Z_2, and the corresponding principal domainal flexure (in magnitude and direction), with the ι-region and the magnitudes $\bar{I}$; and the third value of Z, say Z_3, and the corresponding domainal flexure (in magnitude and direction), with the κ-region and the magnitudes $\bar{K}$. Also we have

$$\bar{E}_\lambda = \Omega \frac{d\epsilon}{dN_\epsilon} \frac{dx_\lambda}{d\epsilon_{N_\epsilon}},$$

$$\bar{I}_\lambda = \Omega \frac{d\iota}{dN_\iota} \frac{dx_\lambda}{d\iota_{N_\iota}},$$

$$\bar{K}_\lambda = \Omega \frac{d\kappa}{dN_\kappa} \frac{dx_\lambda}{d\kappa_{N_\kappa}},$$

the derivatives of x_λ on the right-hand sides being the domainal direction-variables of the domainal normals to the ϵ-region, the ι-region, and the κ-region, respectively. Hence we infer the theorem that, in a system of quadruply orthogonal families of regions in a domain, the curves of domainal flexure in any single region of any one of the families are the intersections of that region by the members of the other three families: a manifest extension of Dupin's theorem on the curves of curvature of a triply orthogonal system of families in a plenary triple homaloidal space.

After this association of Z_1, Z_2, Z_3, the three roots of the equation *

$$\theta_0 Z^3 - \theta_1 Z^2 + \theta_2 Z - \theta_3 = 0,$$

* The quantity θ_0 is different from zero, being equal to $\sum a a_1^2$, that is, to Ωa_N^2. We assume that the quantities θ_1, θ_2, θ_3, do not vanish simultaneously; otherwise, the a-regions would be geodesic to the domain. When all the four families do not consist of geodesic regions, we take the a-region to belong to a family of non-geodesic character. When all the four families do consist of geodesic regions, the domain is deformable to a block, thus being of an exceedingly special type.

we write

$$\Pi_\mu = \begin{vmatrix} \bar{a}_{11} - Z_\mu A_{11}, & \bar{a}_{12} - Z_\mu A_{12}, & \bar{a}_{13} - Z_\mu A_{13}, & \bar{a}_{14} - Z_\mu A_{14} \\ \bar{a}_{12} - Z_\mu A_{12}, & \bar{a}_{22} - Z_\mu A_{22}, & \bar{a}_{23} - Z_\mu A_{23}, & \bar{a}_{24} - Z_\mu A_{24} \\ \bar{a}_{13} - Z_\mu A_{13}, & \bar{a}_{23} - Z_\mu A_{23}, & \bar{a}_{33} - Z_\mu A_{33}, & \bar{a}_{34} - Z_\mu A_{34} \\ \bar{a}_{14} - Z_\mu A_{14}, & \bar{a}_{24} - Z_\mu A_{24}, & \bar{a}_{34} - Z_\mu A_{34}, & \bar{a}_{44} - Z_\mu A_{44} \end{vmatrix},$$

a quartic polynomial expression in Z_μ, the highest term in which is $\Omega Z_\mu{}^4$ and therefore is independent of the quantities $\bar{a}_{ij}$. Also, we denote by X_1, X_2, X_3, respectively, the values of X (on p. 431) to be associated with Z_1, Z_2, Z_3, respectively. Then the values of the quantities $\bar{E}$, $\bar{I}$, $\bar{K}$, are given by equations of the form

$$\begin{aligned}\Pi_1 \bar{E}_m &= X_1 \left(\alpha_1 \frac{\partial \Pi_1}{\partial \bar{a}_{1m}} + \alpha_2 \frac{\partial \Pi_1}{\partial \bar{a}_{2m}} + \alpha_3 \frac{\partial \Pi_1}{\partial \bar{a}_{3m}} + \alpha_4 \frac{\partial \Pi_1}{\partial \bar{a}_{4m}} \right) \\ &= X_1 (T_m Z_1{}^3 + U_m Z_1{}^2 + V_m Z_1 + W_m)\,;\end{aligned}$$

which, by the use of the cubic equation in Z, can be taken as

$$\Pi_1 \bar{E}_m = X_1 (P_m Z_1{}^2 + Q_m Z_1 + R_m) = X_1 \epsilon_m,$$

for $m = 1, 2, 3, 4$; and similarly

$$\begin{aligned}\Pi_2 \bar{I}_m &= X_2 (P_m Z_2{}^2 + Q_m Z_2 + R_m) = X_2 \iota_m, \\ \Pi_3 \bar{K}_m &= X_3 (P_m Z_3{}^2 + Q_m Z_3 + R_m) = X_3 \kappa_m.\end{aligned}$$

The quantities T_m, U_m, V_m, W_m, are homogeneous in the magnitudes $\bar{a}_{ij}$, of orders 0, 1, 2, 3, respectively; and therefore the quantities P_m, Q_m, R_m, are homogeneous in the magnitudes $\bar{a}_{ij}$, of orders 1, 2, 3, respectively.

The three third-order conditions, being

$$\sum \sum \bar{\bar{A}}_{\lambda\mu} \bar{I}_\lambda \bar{K}_\mu = 0, \quad \sum \sum \bar{\bar{A}}_{\lambda\mu} \bar{K}_\lambda \bar{E}_\mu = 0, \quad \sum \sum \bar{\bar{A}}_{\lambda\mu} \bar{E}_\lambda \bar{I}_\mu = 0,$$

can now be modified to the forms

$$\sum \sum \bar{\bar{A}}_{\lambda\mu} \iota_\lambda \kappa_\mu = 0, \quad \sum \sum \bar{\bar{A}}_{\lambda\mu} \kappa_\lambda \epsilon_\mu = 0, \quad \sum \sum \bar{\bar{A}}_{\lambda\mu} \epsilon_\lambda \iota_\mu = 0.$$

The first of these is an equation, symmetric in Z_2 and Z_3: the second is the like equation, symmetric in Z_3 and Z_1: and the third is also the like equation, symmetric in Z_1 and Z_2. The result of elimination, by use of the cubic equation having the roots Z_1, Z_2, Z_3, is the same in each instance: so that a single equation in the quantities $\bar{\bar{A}}_{\lambda\mu}$ and the quantities P, Q, R, would be the ultimate eliminant the same for all. But a simpler equation, linear in the quantities $\bar{\bar{A}}_{\lambda\mu}$ and therefore linear in the third-order derivatives of α, is given by

$$\sum \sum \bar{\bar{A}}_{\lambda\mu} (\iota_\lambda \kappa_\mu + \kappa_\lambda \epsilon_\mu + \epsilon_\lambda \iota_\mu) = 0,$$

the left-hand side of which involves only symmetric functions of Z_1, Z_2, Z_3, all of which are expressible in terms of $\theta_0, \theta_1, \theta_2, \theta_3$, the coefficients of the combinations

being functions of the quantities P, Q, R. In this last equation, the full coefficient of $\bar{\bar{A}}_{\lambda\mu}$

$$\begin{aligned}
&= \bar{\iota}_\lambda \bar{\kappa}_\mu + \bar{\iota}_\mu \bar{\kappa}_\lambda + \bar{\kappa}_\lambda \bar{\epsilon}_\mu + \bar{\kappa}_\mu \bar{\epsilon}_\lambda + \bar{\epsilon}_\lambda \bar{\iota}_\mu + \bar{\epsilon}_\mu \bar{\iota}_\lambda \\
&= P_\lambda P_\mu (2 \sum Z_1^2 Z_2^2) + Q_\lambda Q_\mu (2 \sum Z_1 Z_2) + 6 R_\lambda R_\mu \\
&\qquad + (P_\lambda Q_\mu + P_\mu Q_\lambda) \sum (Z_1^2 Z_2 + Z_2^2 Z_1) \\
&\qquad + (P_\lambda R_\mu + P_\mu R_\lambda)(2 \sum Z_1^2) + (Q_\lambda R_\mu + Q_\mu R_\lambda)(2 \sum Z_1) \\
&= \frac{1}{\theta_0^2} C_{\lambda\mu},
\end{aligned}$$

where

$$\begin{aligned}
C_{\lambda\mu} = 2 P_\lambda P_\mu (\theta_2^2 - 2\theta_1\theta_3) &+ 2 Q_\lambda Q_\mu \theta_0 \theta_2 + 6 R_\lambda R_\mu \theta_0^2 \\
&+ 2 (Q_\lambda R_\mu + Q_\mu R_\lambda) \theta_0 \theta_1 + 2 (P_\lambda R_\mu + P_\mu R_\lambda)(\theta_1^2 - 2\theta_0\theta_2) \\
&\qquad + (P_\lambda Q_\mu + P_\mu Q_\lambda)(\theta_1 \theta_2 - 3\theta_0\theta_3);
\end{aligned}$$

and therefore the equation of the third order, to be satisfied by the parameter of the family of α-regions if they constitute their part in a quadruply orthogonal system of regions in the domain, is

$$\sum C_{\lambda\mu} \bar{\bar{A}}_{\lambda\mu} = 0,$$

an equation which is of the first degree (but is not homogeneous) in the third-order derivatives of α with respect to the parameters.

331. Further, when once an integral α of this equation has been obtained, the other three families required to complete the system can be constructed by quadratures. For we have

$$\begin{aligned}
\Pi_1 \bar{E}_m &= X_1 \left(\sum_j a_j \frac{\partial \Pi_1}{\partial \bar{a}_{jm}} \right), \\
\Pi_2 \bar{I}_m &= X_2 \left(\sum_j a_j \frac{\partial \Pi_2}{\partial \bar{a}_{jm}} \right), \\
\Pi_3 \bar{K}_m &= X_3 \left(\sum_j a_j \frac{\partial \Pi_3}{\partial \bar{a}_{jm}} \right);
\end{aligned}$$

and it is known that the regions thus defined are orthogonal to one another and to the α-region. Also, as

$$\bar{E}_m = \sum_r a_{rm} \epsilon_r,$$

we have

$$\begin{aligned}
\epsilon_r &= \frac{1}{\Omega} \sum_i A_{ri} \bar{E}_i \\
&= \frac{X_1}{\Omega \Pi_1} \sum_i \sum_j \left(A_{ri} a_j \frac{\partial \Pi_1}{\partial \bar{a}_{ij}} \right);
\end{aligned}$$

and therefore the ϵ-regions are given by the quadrature of the exact Pfaffian equation

$$\sum_m \sum_n \left\{ a_m \frac{\partial \Pi_1}{\partial \bar{a}_{mn}} (A_{1n} dp + A_{2n} dq + A_{3n} dr + A_{4n} dt) \right\} = 0.$$

Similarly the ι-regions are given by the quadrature of the exact equation

$$\sum_m \sum_n \left\{ a_m \frac{\partial \Pi_2}{\partial \bar{a}_{mn}} (A_{1n} dp + A_{2n} dq + A_{3n} dr + A_{4n} dt) \right\} = 0,$$

and the κ-regions by the quadrature of the equation

$$\sum_m \sum_n \left\{ a_m \frac{\partial \Pi_3}{\partial \bar{a}_{mn}} (A_{1n} dp + A_{2n} dq + A_{3n} dr + A_{4n} dt) \right\} = 0.$$

The complete system of quadruply orthogonal regions in the domain is thus derivable by quadratures alone, when an integral of the central third-order partial differential equation satisfied by a family parameter is known.

NOTE. The method in § 328 is based on a method due to Cayley, and the method in § 329 is based on a method due to Darboux, for the determination of triply orthogonal systems of surfaces in homaloidal triple space *.

* For references to Cayley and to Darboux, see my *Lectures on Differential Geometry*, ch. xi.

CHAPTER XXVIII

Curvatures of Geodesics in Domainal Regions

Primary and secondary magnitudes of a domainal region and the magnitudes for the domain.

332. The curvatures of a region enclosed in a domain, both those relative to space and those relative to the enclosing domain, can be expressed in terms of domainal magnitudes, in a manner similar to that used (§§ 204, 205) for the expression of the curvatures of a surface enclosed in a region.

The parameters p, q, r, of the domainal set are retained as the regional parameters, the domainal region being represented by the parametric equation $\epsilon(p, q, r, t)=0$, so that

$$t' = -(c_1 p' + c_2 q' + c_3 r'),$$

where

$$\epsilon_1 - c_1\epsilon_4 = 0, \quad \epsilon_2 - c_2\epsilon_4 = 0, \quad \epsilon_3 - c_3\epsilon_4 = 0.$$

The arc-element of the region is also an arc-element of the domain, so that, if A_0, B_0, C_0, F_0, G_0, H_0, denote the primary magnitudes for the region,

$$(A_0 \between p', q', r')^2 = (A \between p', q', r', t')^2,$$

and therefore, on substitution for t',

$$\left.\begin{aligned} A_0 &= A - 2c_1 L + c_1^2 D, & F_0 &= F - c_2 N - c_3 M + c_2 c_3 D \\ B_0 &= B - 2c_2 M + c_2^2 D, & G_0 &= G - c_3 L - c_1 N + c_3 c_1 D \\ C_0 &= C - 2c_3 N + c_3^2 D, & H_0 &= H - c_1 M - c_2 L + c_1 c_2 D \end{aligned}\right\}.$$

Let $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, acquire for the domainal region the customary significance of u_1, u_2, u_3, for any region; then

$$\left.\begin{aligned} \bar{u}_1 &= A_0 p' + H_0 q' + G_0 r' = u_1 - c_1 u_4 \\ \bar{u}_2 &= H_0 p' + B_0 q' + F_0 r' = u_2 - c_2 u_4 \\ \bar{u}_3 &= G_0 p' + F_0 q' + C_0 r' = u_3 - c_3 u_4 \end{aligned}\right\},$$

the symbols u on the right-hand side belonging to the domain.

Also denoting by Ω_0 the determinant of A_0, B_0, C_0, F_0, G_0, H_0, we have

$$\Omega_0 = \begin{vmatrix} A_0, & H_0, & G_0 \\ H_0, & B_0, & F_0 \\ G_0, & F_0, & C_0 \end{vmatrix}$$

$$= -\frac{1}{\epsilon_4^2}\begin{vmatrix} A, & H, & G, & L, & \epsilon_1 \\ H, & B, & F, & M, & \epsilon_2 \\ G, & F, & C, & N, & \epsilon_3 \\ L, & M, & N, & D, & \epsilon_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0 \end{vmatrix}$$

$$= \frac{1}{\epsilon_4^2}\sum a\epsilon_1^2 = \frac{\epsilon_n^2}{\epsilon_4^2}\,\Omega.$$

Next, denoting the magnitude of the circular curvature and the typical direction-cosine of a regional geodesic by ρ_0 and Y_0 respectively, we have

$$\frac{Y_0}{\rho_0} = \frac{Y}{\rho} + \frac{1}{\gamma}\frac{dy}{dn}.$$

Now, for the region, we take

$$\frac{Y_0}{\rho_0} = y_0'' = \zeta_{11}p'^2 + 2\zeta_{12}p'q' + \zeta_{22}q'^2 + 2\zeta_{13}p'r' + 2\zeta_{23}q'r' + \zeta_{33}r'^2;$$

and also

$$\frac{Y}{\rho} = (\eta_{11}\between p', q', r', t')^2 = (\bar{\eta}_{11}\between p', q', r')^2,$$

$$-\frac{\epsilon_n}{\gamma} = (\bar{\epsilon}_{11}\between p', q', r', t')^2 = (E_{11}\between p', q', r')^2,$$

where, for all values $i, j, = 1, 2, 3$,

$$\bar{\eta}_{ij} = \eta_{ij} - c_i\eta_{j4} - c_j\eta_{i4} + c_ic_j\eta_{44},$$
$$E_{ij} = \bar{\epsilon}_{ij} - c_i\bar{\epsilon}_{j4} - c_j\bar{\epsilon}_{i4} + c_ic_j\bar{\epsilon}_{44}.$$

When these values are substituted in the equation for Y_0/ρ_0, and the coefficients of the powers of p', q', r', in the resulting homogeneous relation are compared, we find the general relation

$$\zeta_{ij} = \bar{\eta}_{ij} - \frac{1}{\epsilon_n}E_{ij}\frac{dy}{dn}.$$

We denote the secondary magnitudes of the domainal region by $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$, $\bar{F}_0$, $\bar{G}_0$, $\bar{H}_0$; then, as

$$\bar{A}_0 = \sum Y_0\zeta_{11}$$

by definition of those magnitudes (§ 168), it follows that

$$\frac{\bar{A}_0}{\rho_0} = \sum\left(\frac{Y}{\rho} + \frac{1}{\gamma}\frac{dy}{dn}\right)\left(\bar{\eta}_{11} - \frac{1}{\epsilon_n}E_{11}\frac{dy}{dn}\right)$$

$$= \frac{1}{\rho}\sum Y\bar{\eta}_{11} - \frac{1}{\gamma\epsilon_n}E_{11},$$

because

$$\sum Yy_k=0, \quad \sum \eta_{\lambda\mu}y_k=0,$$

for all values of k, λ, μ, the summation being over the dimensions of the plenary space. But

$$\sum Y\bar{\eta}_{11}=\sum Y(\eta_{11}-2c_1\eta_{14}+c_1^2\eta_{44})=\bar{A}-2c_1\bar{L}+c_1^2\bar{D};$$

and therefore

$$\frac{\bar{A}_0}{\rho_0}=\frac{1}{\rho}(\bar{A}-2c_1\bar{L}+c_1^2\bar{D})-\frac{1}{\gamma\epsilon_n}(\bar{\epsilon}_{11}-2c_1\bar{\epsilon}_{14}+c_1^2\bar{\epsilon}_{44}).$$

In the same way, we find

$$\frac{\bar{F}_0}{\rho_0}=\frac{1}{\rho}(\bar{F}-c_2\bar{N}-c_3\bar{M}+c_2c_3\bar{D})-\frac{1}{\gamma\epsilon_n}(\bar{\epsilon}_{23}-c_2\bar{\epsilon}_{34}-c_3\bar{\epsilon}_{24}+c_2c_3\bar{\epsilon}_{44});$$

and there are corresponding values for the remaining secondary magnitudes $\bar{B}_0$, $\bar{C}_0$, $\bar{G}_0$, $\bar{H}_0$.

Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, acquire for the domainal region the customary significance of v_1, v_2, v_3, for any region; then

$$\begin{aligned}\frac{\bar{v}_1}{\rho_0}&=\frac{1}{\rho_0}(\bar{A}_0p'+\bar{H}_0q'+\bar{G}_0r')\\&=\frac{1}{\rho}\{p'(\bar{A}-2c_1\bar{L}+c_1^2\bar{D})+q'(\bar{H}-c_1\bar{M}-c_2\bar{L}+c_1c_2\bar{D})\\&\qquad+r'(\bar{G}-c_1\bar{N}-c_3\bar{L}+c_1c_3\bar{D})\}-\frac{1}{\gamma\epsilon_n}(E_{11}p'+E_{12}q'+E_{13}r')\\&=\frac{1}{\rho}(v_1-c_1v_4)-\frac{1}{\gamma\epsilon_n}(\bar{\epsilon}_1-c_1\bar{\epsilon}_4),\end{aligned}$$

the symbols $\bar{\epsilon}_i$ (for $i=1, 2, 3, 4$) having the significance defined in § 269. Similarly we find

$$\frac{\bar{v}_2}{\rho_0}=\frac{1}{\rho}(v_2-c_2v_4)-\frac{1}{\gamma\epsilon_n}(\bar{\epsilon}_2-c_2\bar{\epsilon}_4),$$
$$\frac{\bar{v}_3}{\rho_0}=\frac{1}{\rho}(v_3-c_3v_4)-\frac{1}{\gamma\epsilon_n}(\bar{\epsilon}_3-c_3\bar{\epsilon}_4).$$

These results may also be stated in the forms

$$\left.\begin{aligned}\bar{v}_1&=(v_1-c_1v_4)\cos\psi-\frac{1}{\epsilon_n}(\bar{\epsilon}_1-c_1\bar{\epsilon}_4)\sin\psi\\\bar{v}_2&=(v_2-c_2v_4)\cos\psi-\frac{1}{\epsilon_n}(\bar{\epsilon}_2-c_2\bar{\epsilon}_4)\sin\psi\\\bar{v}_3&=(v_3-c_3v_4)\cos\psi-\frac{1}{\epsilon_n}(\bar{\epsilon}_3-c_3\bar{\epsilon}_4)\sin\psi\end{aligned}\right\},$$

where ψ denotes the angle between the prime normal of the regional geodesic and the prime normal of the domainal geodesic tangent; and they can be obtained also as follows. We have

$$\frac{1}{{\rho_0}^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2},$$

and an equivalent form is

$$\frac{1}{\rho_0}=\frac{1}{\rho}\cos\psi+\frac{1}{\gamma}\sin\psi.$$

Now

$$\begin{aligned}\frac{1}{\rho}&=v_1p'+v_2q'+v_3r'+v_4t'\\&=(v_1-c_1v_4)p'+(v_2-c_2v_4)q'+(v_3-c_3v_4)t';\end{aligned}$$

also

$$\begin{aligned}\frac{1}{\gamma}&=\frac{1}{\epsilon_n}(\bar{\epsilon}_1p'+\bar{\epsilon}_2q'+\bar{\epsilon}_3r'+\bar{\epsilon}_4t')\\&=\frac{1}{\epsilon_n}\{(\bar{\epsilon}_1-c_1\bar{\epsilon}_4)p'+(\bar{\epsilon}_2-c_2\bar{\epsilon}_4)q'+(\bar{\epsilon}_3-c_3\bar{\epsilon}_4)r'\},\end{aligned}$$

and

$$\frac{1}{\rho_0}=\bar{v}_1p'+\bar{v}_2q'+\bar{v}_3r'.$$

The quantities $\cos\psi$ and $\sin\psi$ are non-rational functions of p', q', r', t', and are homogeneous of order zero in these variables; hence, there being no linear homogeneous relation in p', q', r', alone, and as the relation now becomes a definition of quantities $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, we take

$$\begin{aligned}\bar{v}_1&=(v_1-c_1v_4)\cos\psi-\frac{1}{\epsilon_n}(\bar{\epsilon}_1-c_1\bar{\epsilon}_4)\sin\psi,\\ \bar{v}_2&=(v_2-c_2v_4)\cos\psi-\frac{1}{\epsilon_n}(\bar{\epsilon}_2-c_2\bar{\epsilon}_4)\sin\psi,\\ \bar{v}_3&=(v_3-c_3v_4)\cos\psi-\frac{1}{\epsilon_n}(\bar{\epsilon}_3-c_3\bar{\epsilon}_4)\sin\psi,\end{aligned}$$

as before.

Value of $\frac{d}{ds}\left(\frac{1}{\rho}\right)$ for a geodesic in a domainal region.

333. Corresponding expressions are required for magnitudes connected with the arc-derivative of the circular curvature of the regional geodesic. Denoting by $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$, for the geodesic in the domainal region, the quantities w_1, w_2, w_3, for a free region, so that

$$\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right)=\bar{w}_1p'+\bar{w}_2q'+\bar{w}_3r',$$

it is desirable to express $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$, in terms of domainal magnitudes connected with the domainal geodesic tangent.

When the equation

$$\frac{1}{\rho_0^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2}$$

is differentiated along the regional geodesic, we have

$$\frac{1}{\rho_0}\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right)=\frac{1}{\rho}\frac{d}{ds_0}\left(\frac{1}{\rho}\right)+\frac{1}{\gamma}\frac{d}{ds_0}\left(\frac{1}{\gamma}\right),$$

and therefore

$$\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right)=\left\{\frac{d}{ds_0}\left(\frac{1}{\rho}\right)\right\}\cos\psi+\left\{\frac{d}{ds_0}\left(\frac{1}{\gamma}\right)\right\}\sin\psi.$$

But

$$\begin{aligned}\frac{d}{ds_0}\left(\frac{1}{\rho}\right)-\frac{d}{ds}\left(\frac{1}{\rho}\right)&=\sum(p_0''-p'')\frac{\partial}{\partial p'}\left(\frac{1}{\rho}\right)\\&=\frac{2}{\gamma}\left(v_1\frac{dp}{dn}+v_2\frac{dq}{dn}+v_3\frac{dr}{dn}+v_4\frac{dt}{dn}\right)\\&=2\phi,\end{aligned}$$

where ϕ denotes the cubic in p', q', r', t', represented by

$$\phi=-\frac{1}{\epsilon_n}(\sum\sum\bar{\epsilon}_{11}p'^2)\left(\sum v_1\frac{dp}{dn}\right).$$

If the right-hand side be denoted by

$$-\sum\sum\sum\phi_{ijk}x_i'x_j'x_k',$$

with the usual convention $x_1, x_2, x_3, x_4, =p, q, r, t$, we have

$$3\epsilon_n\phi_{ijk}=\bar{\epsilon}_{jk}\left(\sum_a A_{ia}\frac{dx_a}{dn}\right)+\bar{\epsilon}_{ij}\left(\sum_a \bar{A}_{ka}\frac{dx_a}{dn}\right)+\bar{\epsilon}_{ki}\left(\sum_a \bar{A}_{ja}\frac{dx_a}{dn}\right);$$

and now

$$\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right)=\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)+2\phi\right\}\cos\psi+\left\{\frac{d}{ds_0}\left(\frac{1}{\gamma}\right)\right\}\sin\psi.$$

Now

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{\rho}\right)&=(e_{111}\between p', q', r', t')^3\\&=w_1p'+w_2q'+w_3r'+w_4t',\end{aligned}$$

with the former significance (§ 285) for w_1, w_2, w_3, w_4: that is,

$$\frac{d}{ds}\left(\frac{1}{\rho}\right)=(w_1-c_1w_4)p'+(w_2-c_2w_4)q'+(w_3-c_3w_4)r'.$$

Again, with the like significance for symbols ϕ_1, ϕ_2, ϕ_3, ϕ_4, so that

$$3\phi_1=\frac{\partial\phi}{\partial p'},\quad 3\phi_2=\frac{\partial\phi}{\partial q'},\quad 3\phi_3=\frac{\partial\phi}{\partial r'},\quad 3\phi_4=\frac{\partial\phi}{\partial t'},$$

we have

$$\phi=(\phi_1-c_1\phi_4)p'+(\phi_2-c_2\phi_4)q'+(\phi_3-c_3\phi_4)r'.$$

The value of $\dfrac{d}{ds_0}\left(\dfrac{1}{\gamma}\right)$ has been obtained in the relation in § 325; so we write

$$\frac{d}{ds_0}\left(\frac{1}{\gamma}\right)=W,$$

which is a cubic in p', q', r', t'; with a corresponding significance for W_1, W_2, W_3, W_4, we have

$$\frac{d}{ds_0}\left(\frac{1}{\gamma}\right)=(W_1-c_1W_4)p'+(W_2-c_2W_4)q'+(W_3-c_3W_4)r'.$$

Let these values be inserted in the equation for $\dfrac{d}{ds_0}\left(\dfrac{1}{\rho_0}\right)$. The quantities $\cos\psi$ and $\sin\psi$ are non-rational functions of p', q', r', t', and are homogeneous of zero order in those variables; hence

$$\left.\begin{aligned}\bar{\varpi}_1&=\{(w_1-c_1w_4)+2(\phi_1-c_1\phi_4)\}\cos\psi+(W_1-c_1W_4)\sin\psi\\ \bar{\varpi}_2&=\{(w_2-c_2w_4)+2(\phi_2-c_2\phi_4)\}\cos\psi+(W_2-c_2W_4)\sin\psi\\ \bar{\varpi}_3&=\{(w_3-c_3w_4)+2(\phi_3-c_3\phi_4)\}\cos\psi+(W_3-c_3W_4)\sin\psi\end{aligned}\right\}.$$

Ex. These results can also be obtained by a process, of which the following is an outline, the calculations being omitted.

The fundamental equation connecting the circular curvatures of the regional geodesic and the domainal geodesic can be taken as

$$y_0''=y''+\frac{1}{\gamma}\frac{dy}{dn}.$$

Differentiation along the regional geodesic gives the relation

$$\begin{aligned}y_0'''&=\frac{d}{ds_0}(y'')+\frac{1}{\gamma}\frac{d}{ds_0}\left(\frac{dy}{dn}\right)+\frac{dy}{dn}\frac{d}{ds_0}\left(\frac{1}{\gamma}\right)\\ &=y'''+\frac{2}{\gamma}\left(\eta_1\frac{dp}{dn}+\eta_2\frac{dq}{dn}+\eta_3\frac{dr}{dn}+\eta_4\frac{dt}{dn}\right)\\ &\quad+\frac{1}{\gamma}\frac{d}{ds_0}\left(\frac{dy}{dn}\right)+\frac{dy}{dn}\frac{d}{ds_0}\left(\frac{1}{\gamma}\right).\end{aligned}$$

Take quantities ξ_1, ξ_2, ξ_3, ξ_4, such that

$$\xi_\iota=\eta_{1\iota}\frac{dp}{dn}+\eta_{2\iota}\frac{dq}{dn}+\eta_{3\iota}\frac{dr}{dn}+\eta_{4\iota}\frac{dt}{dn},$$

for the four values of i; then, as the formulæ of § 323 give

$$\frac{d}{ds_0}\left(\frac{dy}{dn}\right)=\xi_1p'+\xi_2q'+\xi_3r'+\xi_4t'-\frac{dy}{dn}\frac{1}{\epsilon_n}\frac{d\epsilon_n}{ds}+\frac{1}{\Omega\epsilon_n}\cdot\sum_i\sum_j y_\iota a_{ij}\bar{\epsilon}_j,$$

we find

$$y_0''' = y''' + \frac{3}{\gamma}(\xi_1 p' + \xi_2 q' + \xi_3 r' + \xi_4 t')$$
$$+ \frac{dy}{dn}\left\{\frac{d}{ds_0}\left(\frac{1}{\gamma}\right) - \frac{1}{\gamma\epsilon_n}\frac{d\epsilon_n}{ds}\right\} + \frac{1}{\Omega\epsilon_n\gamma}\sum_i\sum_j y_i a_{ij}\bar{\epsilon}_j,$$

all the complete terms on the right-hand side being quantities, homogeneous in p', q', r', t', of the third order. We have

$$y''' = \sum\sum\sum \eta_{ijk} x_i' x_j' x_k',$$

and we write

$$\frac{3}{\gamma}(\xi_1 p' + \xi_2 q' + \xi_3 r' + \xi_4 t') = \sum\sum\sum f_{ijk} x_i' x_j' x_k',$$

$$\frac{d}{ds_0}\left(\frac{1}{\gamma}\right) - \frac{1}{\gamma\epsilon_n}\frac{d\epsilon_n}{ds} = \sum\sum\sum g_{ijk} x_i' x_j' x_k',$$

$$\frac{1}{\Omega\epsilon_n\gamma}\sum_\lambda\sum_\mu y_\lambda a_{\lambda\mu}\bar{\epsilon}_\mu = \sum\sum\sum h_{ijk} x_i' x_j' x_k'$$

so that, if

$$P_{ijk} = \eta_{ijk} + f_{ijk} + \frac{dy}{dn} g_{ijk} + h_{ijk},$$

we have

$$y_0''' = \sum\sum\sum P_{ijk} x_i' x_j' x_k',$$

where the direction-variables on the right-hand side still are p', q', r', t'. Let

$$y_0''' = \sum\sum\sum \zeta_{ijk} x_i' x_j' x_k',$$

the triple summation referring only to the direction-variables p', q', r', for the region; then in the former expression for y_0''', when we substitute $-(c_1 p' + c_2 q' + c_3 r')$ for t', and compare coefficients, we find

$$\zeta_{ijk} = P_{ijk} - (c_i P_{jk4} + c_j P_{ki4} + c_k P_{ij4})$$
$$+ (c_j c_k P_{i44} + c_k c_i P_{j44} + c_i c_j P_{k44}) - c_i c_j c_k P_{444} = \bar{P}_{ijk},$$

as the law for the coefficients ζ in the expression of y_0'''.

Next, when we take

$$\frac{d}{ds_0}\left(\frac{1}{\rho}\right) = \sum\sum\sum E_{ijk} x_i' x_j' x_k',$$

the summation being for the three variables, we have (§ 174) the quantities E_{ijk} given by the definition

$$E_{ijk} = \sum Y_0 \zeta_{ijk};$$

and therefore

$$E_{ijk} = (\sum Y\bar{P}_{ijk})\cos\psi + \left(\sum \frac{dy}{dn}\bar{P}_{ijk}\right)\sin\psi.$$

In the coefficient of $\cos\psi$, we have

$$\sum YP_{\lambda\mu\nu} = \sum\{Y(\eta_{\lambda\mu\nu} + f_{\lambda\mu\nu})\},$$

the other terms vanishing; that is, as

$$\sum Y\eta_{\lambda\mu\nu} = e_{\lambda\mu\nu},$$

$$\sum Yf_{\lambda\mu\nu} = -\frac{3}{\epsilon_n}\left[\bar{\epsilon}_{\mu\nu}\left(\sum_a \bar{A}_{\lambda a}\frac{dx_a}{dn}\right) + \bar{\epsilon}_{\nu\lambda}\left(\sum_a \bar{A}_{\mu a}\frac{dx_a}{dn}\right) + \bar{\epsilon}_{\lambda\mu}\left(\sum_a \bar{A}_{\nu a}\frac{dx_a}{dn}\right)\right],$$

we have the means of obtaining the full coefficient of $\cos\psi$.

In the coefficient of $\sin\psi$, we have

$$\sum \frac{dy}{dn} P_{\lambda\mu\nu} = \sum\left(\frac{dy}{dn}\eta_{\lambda\mu\nu}\right) + g_{\lambda\mu\nu} + \sum\left(\frac{dy}{dn}h_{\lambda\mu\nu}\right),$$

the other term vanishing. But (§ 280, *Ex.* 1)

$$\sum (y_\kappa \eta_{\lambda\mu\nu}) = -\tfrac{1}{3}(\eta_{\kappa\lambda}\eta_{\mu\nu} + \eta_{\kappa\mu}\eta_{\nu\lambda} + \eta_{\kappa\nu}\eta_{\lambda\mu}),$$

so that

$$\sum\left(\frac{dy}{dn}\eta_{\lambda\mu\nu}\right) = -\tfrac{1}{3}\sum_k\left\{(\eta_{k\lambda}\eta_{\mu\nu} + \eta_{k\mu}\eta_{\nu\lambda} + \eta_{k\nu}\eta_{\lambda\mu})\frac{dx_k}{dn}\right\};$$

and similarly

$$\sum\left(\frac{dy}{dn}h_{\lambda\mu\nu}\right) = -\tfrac{1}{3}\sum_a\left[(\bar{\epsilon}_{\mu\nu}\bar{\epsilon}_{a\lambda} + \bar{\epsilon}_{\nu\lambda}\bar{\epsilon}_{a\mu} + \bar{\epsilon}_{\lambda\mu}\bar{\epsilon}_{a\nu})\frac{dx_a}{dn}\right],$$

so that we have the means of obtaining the full coefficient of $\sin\psi$. We therefore may regard E_{ijk} as expressible in terms of the domainal quantities.

Then the values of ϖ_1, ϖ_2, ϖ_3, are obtainable, by the relations

$$\varpi_1 = \sum_\lambda\sum_\mu E_{1\lambda\mu}x_\lambda' x_\mu',$$

$$\varpi_2 = \sum_\lambda\sum_\mu E_{2\lambda\mu}x_\lambda' x_\mu',$$

$$\varpi_3 = \sum_\lambda\sum_\mu E_{3\lambda\mu}x_\lambda' x_\mu',$$

the summations with regard to λ and μ being for the values $\lambda, \mu, = 1, 2, 3$. The final results are as stated in the text.

Spatial and domainal curvatures for a region.

334. As for a surface in a region (§ 199), so for a region in a domain, there are two species of curvature to consider. The region, contained in the domain, still is a configuration in the plenary space of that domain; and its geodesics have their successive curvatures relative to that plenary space alone. For convenience, these curvatures are called *spatial*; and there is a corresponding orthogonal frame for the geodesic, the first four lines of which are the tangent to the geodesic, the prime normal, the binormal, and the trinormal, all but the prime normal being gremial to the region (that is, lying in the tangent flat of the region). The typical direction-cosines of these lines are denoted by y', for the tangent; by Y_0, for the prime normal; by λ_3 (instead of l_3, as the region is domainal, not free), for the

binormal; and by λ_4 (instead of l_4, for the same reason), for the trinormal. The spatial curvatures are denoted by $1/\rho_0$, $1/\sigma_0$, $1/\tau_0$, respectively; and differentiation along the regional geodesic being indicated by $\frac{d}{ds_0}$, we have

$$\frac{dy'}{ds_0} = \frac{Y_0}{\rho_0},$$

$$\frac{dY_0}{ds_0} = \frac{\lambda_3}{\sigma_0} - \frac{y'}{\rho_0},$$

$$\frac{d\lambda_3}{ds_0} = \frac{\lambda_4}{\tau_0} - \frac{Y_0}{\sigma_0},$$

as equations in the Frenet system for the spatial curvatures.

There are also the curvatures of the regional geodesic relative solely to the enclosing domain. The geodesic is still a geodesic in the region, and its gremial lines are unaltered; but its prime normal is the domainal normal, being the direction of the radius of domainal flexure which now takes the place of the spatial circular curvature. Thus, within the tangent block of the domain, there is the beginning of an orthogonal frame for the regional geodesic, the first four lines of which are the tangent, with a typical direction-cosine y'; the domainal normal, with a typical direction-cosine $\frac{dy}{dn}$; the binormal, with the foregoing typical direction-cosine λ_3; and the trinormal, with the foregoing typical direction-cosine λ_4. The domainal flexure of the regional geodesic is denoted by $1/\gamma$, as before; the domainal torsion, the domainal tilt, and the domainal coil, by $1/\sigma_\epsilon$, $1/\tau_\epsilon$, $1/\kappa_\epsilon$, respectively, the region being represented parametrically by

$$\epsilon(p, q, r, t) = 0.$$

The complete orthogonal frame of the regional geodesic, for domainal curvatures subsequent in rank to the coil, is constituted by a part of the frame for domainal curves. The principal lines, thus far retained for the regional geodesic in its domainal relations, have typical direction-cosines

$$y', \frac{dy}{dn}, \lambda_3, \lambda_4;$$

the lines are perpendicular in pairs, and they constitute the same block as

$$y', l_3, l_4, l_5,$$

for a domainal geodesic; hence the succeeding principal lines for the regional geodesic can be taken to be the prime normal of the domainal geodesic and its normals that rank subsequent to the quartinormal. Accordingly, the first

equation of the Frenet system for the domainal curvatures of the regional geodesic is obtained as in § 199 for superficial geodesics in a region. Let $d\mu_1$ denote the angular deviation between the regional geodesic and the domainal geodesic; then

$$\frac{dy}{dn} d\mu_1 = \text{component of the deviation in an orbicular representation}$$

$$= (y' + y_0''\delta + \ldots) - (y' + y''\delta + \ldots) = (y_0'' - y'')\delta + \ldots .$$

We have $1/\gamma$ as the limiting value of $d\mu_1/\delta$; and therefore

$$\frac{1}{\gamma}\frac{dy}{dn} = y_0'' - y'' = \frac{Y_0}{\rho_0} - \frac{Y}{\rho}.$$

For the remaining equations, we have (as in § 200)

$$\frac{d}{ds_0}\left(\frac{dy}{dn}\right) = \frac{\lambda_3}{\sigma_\epsilon} - \frac{y'}{\gamma},$$

$$\frac{d\lambda_3}{ds_0} = \frac{\lambda_4}{\tau_\epsilon} - \frac{1}{\sigma_\epsilon}\frac{dy}{dn},$$

$$\frac{d\lambda_4}{ds_0} = \frac{Y}{\kappa_\epsilon} - \frac{1}{\tau_\epsilon}\lambda_3,$$

and so on.

The values of λ_3 and λ_4 for a regional geodesic are known (§§ 172, 178) in terms of the magnitudes belonging to the region. We proceed to use these values, for the determination of the initial spatial curvatures and the initial domainal curvatures of the ϵ-region enclosed in the domain.

Spatial torsion of a regional geodesic.

335. For the binormal of a regional geodesic (§ 172), the typical direction-cosine λ_3, when the region is domainal, becomes

$$\lambda_3 = \bar{y}_1 l + \bar{y}_2 m + \bar{y}_3 n$$
$$= (y_1 - c_1 y_4) l + (y_2 - c_2 y_4) m + (y_3 - c_3 y_4) n,$$

where

$$\frac{l}{\sigma_0} = \frac{p'}{\rho_0} - \frac{\bar{V}_1}{\Omega_0}, \quad \frac{m}{\sigma_0} = \frac{q'}{\rho_0} - \frac{\bar{V}_2}{\Omega_0}, \quad \frac{n}{\sigma_0} = \frac{r'}{\rho_0} - \frac{\bar{V}_3}{\rho_0},$$

and

$$\bar{V}_1 = a_0\bar{v}_1 + h_0\bar{v}_2 + g_0\bar{v}_3,$$
$$\bar{V}_2 = h_0\bar{v}_1 + b_0\bar{v}_2 + f_0\bar{v}_3,$$
$$\bar{V}_3 = g_0\bar{v}_1 + f_0\bar{v}_2 + c_0\bar{v}_3.$$

Thus with the values of $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, on p. 438, we have expressions of the form

$$\bar{V}_\mu = Y_\mu \cos\psi - J_\mu \sin\psi,$$

for $\mu = 1, 2, 3$; and therefore, as

$$\bar{y}_1 p' + \bar{y}_2 q' + \bar{y}_3 r' = (y_1 - c_1 y_4) p' + (y_2 - c_2 y_4) q' + (y_3 - c_3 y_4) r'$$
$$= y_1 p' + y_2 p' + y_3 q' + y_4 r' = y',$$

we have

$$-\left(\frac{\lambda_3}{\sigma_0} - \frac{y'}{\rho_0}\right) \Omega_0 = (\bar{y}_1 Y_1 + \bar{y}_2 Y_2 + \bar{y}_3 Y_3) \cos\psi - (\bar{y}_1 J_1 + \bar{y}_2 J_2 + \bar{y}_3 J_3) \sin\psi.$$

As regards the quantities Y, we have

$$Y_1 = a_0 (v_1 - c_1 v_4) + h_0 (v_2 - c_2 v_4) + g_0 (v_3 - c_3 v_4)$$
$$= \begin{vmatrix} v_1 - c_1 v_4, & v_2 - c_2 v_4, & v_3 - c_3 v_4 \\ H_0\ , & B_0\ , & F_0 \\ G_0\ , & F_0\ , & C_0 \end{vmatrix}$$
$$= -\frac{1}{\epsilon_4^2} \begin{vmatrix} v_1, & v_2, & v_3, & v_4, & 0 \\ H, & B, & F, & M, & \epsilon_2 \\ G, & F, & C, & N, & \epsilon_3 \\ L, & M, & N, & D, & \epsilon_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0 \end{vmatrix},$$

with similar expressions for Y_2, Y_3: and therefore

$$\bar{y}_1 Y_1 + y_2 \bar{Y}_2 + y_3 \bar{Y}_3 = (y_1 - c_1 y_4) Y_1 + (y_2 - c_2 y_4) Y_2 + (y_3 - c_3 y_4) Y_3$$
$$= \frac{1}{\epsilon_4^2} \begin{vmatrix} 0, & v_1, & v_2, & v_3, & v_4, & 0 \\ y_1, & A, & H, & G, & L, & \epsilon_1 \\ y_2, & H, & B, & F, & M, & \epsilon_2 \\ y_3, & G, & F, & C, & N, & \epsilon_3 \\ y_4, & L, & M, & N, & D, & \epsilon_4 \\ 0, & \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0 \end{vmatrix} = \frac{1}{\epsilon_4^2} \nabla_v,$$

using the symbol ∇_v for brevity. In the same way, we find

$$\bar{y}_1 J_1 + \bar{y}_2 J_2 + \bar{y}_3 J_3 = \frac{1}{\epsilon_4^2 \epsilon_n} \begin{vmatrix} 0, & \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4, & 0 \\ y_1, & A, & H, & G; & L, & \epsilon_1 \\ y_2, & H, & B, & F, & M, & \epsilon_2 \\ y_3, & G, & F, & C, & N, & \epsilon_3 \\ y_4, & L, & M, & N, & D, & \epsilon_4 \\ 0, & \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0 \end{vmatrix} = \frac{1}{\epsilon_4^2 \epsilon_n} \nabla_{\bar{\epsilon}}.$$

Also, we have

$$\Omega_0 = \frac{\epsilon_n^2}{\epsilon_4^2} \Omega;$$

and therefore

$$\frac{dY_0}{ds_0}=\frac{\lambda_3}{\sigma_0}-\frac{y'}{\rho_0}=-\frac{1}{\Omega\epsilon_n^2}\left(\nabla_v\cos\psi-\frac{1}{\epsilon_n}\nabla_{\bar{\epsilon}}\sin\psi\right).$$

Now let

$$\nabla=\nabla_v\cos\psi-\frac{1}{\epsilon_n}\nabla_{\bar{\epsilon}}\sin\psi=\frac{\rho_0}{\rho}\nabla_v-\frac{\rho_0}{\gamma\epsilon_n}\nabla_{\bar{\epsilon}},$$

so that ∇ can be derived from the six-row determinants ∇_v and $\nabla_{\bar{\epsilon}}$ merely by changing their first lines of constituents into

$$0,\quad \frac{\rho_0}{\rho}v_1-\frac{\rho_0}{\gamma\epsilon_n}\bar{\epsilon}_1,\quad \frac{\rho_0}{\rho}v_2-\frac{\rho_0}{\gamma\epsilon_n}\bar{\epsilon}_2,\quad \frac{\rho_0}{\rho}v_3-\frac{\rho_0}{\gamma\epsilon_n}\bar{\epsilon}_3,\quad \frac{\rho_0}{\rho}v_4-\frac{\rho_0}{\gamma\epsilon_n}\bar{\epsilon}_n,\quad 0\,;$$

and we have

$$\frac{dY_0}{ds_0}=\frac{\lambda_3}{\sigma_0}-\frac{y'}{\rho_0}=-\frac{1}{\Omega\epsilon_n^2}\nabla,$$

a typical equation expressing λ_3 in terms of domainal magnitudes when σ_0 is known.

To determine σ_0, we have

$$\begin{aligned}\frac{1}{\sigma_0^2}+\frac{1}{\rho_0^2}&=\sum\left(\frac{dY_0}{ds_0}\right)^2\\&=-\sum\frac{dY_0}{ds_0}\left(\frac{1}{\Omega\epsilon_n^2}\nabla\right)=-\frac{1}{\Omega\epsilon_n^2}\sum\left(\frac{dY_0}{ds_0}\nabla\right),\end{aligned}$$

the summation being taken over the range of the plenary space of the domain. This summation is most simply effected in connection with the constituents of the first column in ∇, so that we must evaluate quantities

$$\sum y_\mu\frac{dY_0}{ds_0},\quad (\mu=1,2,3,4),$$

the summation extending over that range. Now

$$\sum y_1Y_0=\sum y_1\left(\frac{\rho_0}{\rho}Y+\frac{\rho_0}{\gamma}\frac{dy}{dn}\right)=\frac{\rho_0}{\gamma\epsilon_n}\epsilon_1,$$

and therefore

$$\sum y_1\frac{dY_0}{ds_0}=-\sum y_1'Y_0+\frac{\rho_0}{\gamma\epsilon_n}(\epsilon_{11}p'+\epsilon_{12}q'+\epsilon_{13}r'+\epsilon_{14}t')+\epsilon_1\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma\epsilon_n}\right).$$

Also

$$\begin{aligned}\sum y_1'Y_0&=\sum(y_{11}p'+y_{12}q'+y_{13}r'+y_{14}t')\left(\frac{\rho_0}{\rho}Y+\frac{\rho_0}{\gamma}\frac{dy}{dn}\right)\\&=\frac{\rho_0}{\rho}v_1+\frac{\rho_0}{\gamma}\sum\left\{\frac{dy}{dn}(y_{11}p'+y_{12}q'+y_{13}r'+y_{14}t')\right\};\end{aligned}$$

or, as

$$\sum y_{11}\frac{dy}{dn}=\sum y_{11}\left(y_1\frac{dp}{dn}+y_2\frac{dq}{dn}+y_3\frac{dr}{dn}+y_4\frac{dt}{dn}\right)$$
$$=\frac{1}{\epsilon_n}(\epsilon_1\Gamma_{11}+\epsilon_2\Delta_{11}+\epsilon_3\Theta_{11}+\epsilon_4\Phi_{11}),$$

and so for the other like sums, we find, after re-arrangement,

$$\sum y_1\frac{dY_0}{ds_0}=-\frac{\rho_0}{\rho}v_1+\frac{\rho_0}{\gamma\epsilon_n}\bar{\epsilon}_1+\epsilon_1\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma\epsilon_n}\right).$$

The full result, for $\mu=1, 2, 3, 4$, is

$$\sum y_\mu\frac{dY_0}{ds_0}=-\frac{\rho_0}{\rho}v_\mu+\frac{\rho_0}{\gamma\epsilon_n}\bar{\epsilon}_\mu+\epsilon_\mu\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma\epsilon_n}\right).$$

When these values are inserted in the first column of the six-row determinant

$$\sum\left(\frac{dY_0}{ds_0}\nabla\right),$$

all the terms in $\dfrac{d}{ds_0}\left(\dfrac{\rho_0}{\gamma\epsilon_n}\right)$ can be cancelled because the constituents of the sixth column are $0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, 0$. Hence, writing

$$\omega_\mu=\frac{\rho_0}{\rho}v_\mu-\frac{\rho_0}{\gamma\epsilon_n}\bar{\epsilon}_\mu,$$

for $\mu=1, 2, 3, 4$, we have

$$\Omega\left(\frac{1}{\sigma_0^2}+\frac{1}{\rho_0^2}\right)=\frac{1}{\epsilon_n^2}\square(\omega),$$

where

$$\square(\omega)=\begin{vmatrix} 0, & \omega_1, & \omega_2, & \omega_3, & \omega_4, & 0 \\ \omega_1, & A, & H, & G, & L, & \epsilon_1 \\ \omega_2, & H, & B, & F, & M, & \epsilon_2 \\ \omega_3, & G, & F, & C, & N, & \epsilon_3 \\ \omega_4, & L, & M, & N, & D, & \epsilon_4 \\ 0, & \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0 \end{vmatrix},$$

so that the torsion of the regional geodesic is thus expressible, in terms of domainal magnitudes and of quantities connected with the parametric equation of the region.

Let $\square(v, \bar{\epsilon})$ denote the determinant $\square(\omega)$, when the constituents of the first row are changed to $0, v_1, v_2, v_3, v_4, 0$, and simultaneously the constituents of the first column are changed to $0, \bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3, \bar{\epsilon}_4$; then

$$\square(\omega)=\frac{\rho_0^2}{\rho^2}\square(v)-2\frac{\rho_0^2}{\rho\gamma\epsilon_n}\square(v, \bar{\epsilon})+\frac{\rho_0^2}{\gamma^2\epsilon_n^2}\square(\bar{\epsilon}),$$

and therefore

$$\frac{\Omega\epsilon_n^2}{\rho_0^2}\left(\frac{1}{\sigma_0^2}+\frac{1}{\rho_0^2}\right)=\frac{1}{\rho^2}\Box(v)-\frac{2}{\rho\gamma\epsilon_n}\Box(v,\bar{\epsilon})+\frac{1}{\gamma^2\epsilon_n^2}\Box(\bar{\epsilon}).$$

To this result, we shall return later. Meanwhile, a modified expression can be given to $\Box(v)$; for

$$\Box(v)=\begin{vmatrix} A, & H, & G, & L, & v_1, & \epsilon_1 \\ H, & B, & F, & M, & v_2, & \epsilon_2 \\ G, & F, & C, & N & v_3, & \epsilon_3 \\ L, & M, & N, & D, & v_4, & \epsilon_4 \\ v_1, & v_2, & v_3, & v_4, & 0, & 0 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0, & 0 \end{vmatrix}$$

$$=\sum(AB-H^2)(v_3\epsilon_4-v_4\epsilon_3)^2$$

$$=\frac{1}{\Omega}\sum(cd-n^2)(v_3\epsilon_4-v_4\epsilon_3)^2$$

$$=\frac{1}{\Omega}\{(\sum a\epsilon_1^2)(\sum av_1^2)-(\sum av_1\epsilon_1)^2\}$$

$$=\Omega\epsilon_n^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-\frac{1}{\Omega}(\sum av_1\epsilon_1)^2,$$

where the quantities $1/\rho$ and $1/\sigma$ are the circular curvature and the torsion of the domainal geodesic tangent. Let χ_3 denote (as in § 318) the inclination of the binormal of that domainal geodesic tangent to the domainal normal of the region, then

$$\cos\chi_3=\sum l_3\frac{dy}{dn}$$

$$=\sum(y_1\lambda+y_2\mu+y_3\nu+y_4\varpi)\frac{dy}{dn},$$

where (§ 284)

$$\frac{\lambda}{\sigma}=\frac{p'}{\rho}-\frac{1}{\Omega}(av_1+hv_2+gv_3+lv_4),$$

with corresponding values for μ, ν, ϖ. Now

$$\sum y_\iota\frac{dy}{dn}=\frac{\epsilon_\iota}{\epsilon_n},$$

for $i=1, 2, 3, 4$; and $\epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t'=0$; and therefore

$$\frac{1}{\sigma}\cos\chi_3=-\frac{1}{\Omega\epsilon_n}\sum av_1\epsilon_1.$$

Consequently

$$\Box(v)=\Omega\epsilon_n^2\left(\frac{1}{\rho^2}+\frac{\sin^2\chi_3}{\sigma^2}\right).$$

Domainal torsion of a regional geodesic.

336. From the foregoing investigation, we have

$$\frac{\lambda_3}{\sigma_0}-\frac{y'}{\rho_0}=-\frac{1}{\Omega\epsilon_n^2}\nabla;$$

and the Frenet equation for the domainal torsion of the regional geodesic is

$$\frac{\lambda_3}{\sigma_\epsilon}-\frac{y'}{\gamma}=\frac{d}{ds}\left(\frac{dy}{dn}\right).$$

At first sight, it would appear advantageous to eliminate λ_3 and thus obtain a simpler relation

$$y'\left(\frac{1}{\rho_0\sigma_\epsilon}-\frac{1}{\gamma\sigma_0}\right)=\frac{1}{\sigma_0}\frac{d}{ds}\left(\frac{dy}{dn}\right)+\frac{1}{\sigma_\epsilon}\frac{1}{\Omega\epsilon_n^2}\nabla.$$

But the relation serves mainly as an alternative expression for ∇. We have

$$\sum y'\nabla=(\sum y'\nabla_v)\cos\psi-(\sum y'\nabla_\epsilon)\sin\psi;$$

and

$$\sum y'\nabla_v=\begin{vmatrix} 0, & v_1, & v_2, & v_3, & v_4, & 0 \\ u_1, & A, & H, & G, & L, & \epsilon_1 \\ u_2, & H, & B, & F, & M, & \epsilon_2 \\ u_3, & G, & F, & C, & N, & \epsilon_3 \\ u_4, & L, & M, & N, & D, & \epsilon_4 \\ 0, & \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0 \end{vmatrix}$$

$$=\sum\{(AB-H^2)(\epsilon_3v_4-\epsilon_4v_3)(\epsilon_3u_4-\epsilon_4u_3)\}$$

$$=\frac{1}{\Omega}\sum\{(cd-n^2)(\epsilon_3v_4-\epsilon_4v_3)(\epsilon_3u_4-\epsilon_4u_3)\}$$

$$=\frac{1}{\Omega}\{(\sum a\epsilon_1^2)(\sum au_1v_1)-(\sum av_1\epsilon_1)(\sum au_1\epsilon_1)\}:$$

also

$$\sum au_1v_1=\Omega\sum v_1p'=\frac{\Omega}{\rho},\quad \sum au_1\epsilon_1=\Omega\sum\epsilon_1p'=0;$$

and therefore

$$\sum y'\nabla_v=\frac{\Omega\epsilon_n^2}{\rho}.$$

Similarly

$$\sum y'\nabla_\epsilon=-\frac{\Omega\epsilon_n^3}{\gamma}.$$

Also

$$\sum y'\frac{d}{ds}\left(\frac{dy}{dn}\right)=-\frac{1}{\gamma}.$$

Hence multiplying the simpler relation by y', adding for the range of the plenary space, and inserting the values of these sums just obtained, the result is an identity: thus the simpler relation is not useful for the determination of the domainal torsion of the regional geodesic.

Accordingly, we multiply the left-hand sides of the two equations involving λ_3, also the right-hand sides, and add for the range of the plenary homaloidal space. Because

$$\sum y'\lambda_3 = 0,$$

the result is

$$\begin{aligned}\frac{1}{\sigma_0\sigma_\epsilon}+\frac{1}{\rho_0\gamma} &= -\frac{1}{\Omega\epsilon_n{}^2}\sum\left\{\nabla\frac{d}{ds}\left(\frac{dy}{dn}\right)\right\}\\ &= -\frac{1}{\Omega\epsilon_n{}^2}\left[\frac{\rho_0}{\rho}\sum\left\{\nabla_v\frac{d}{ds}\left(\frac{dy}{dn}\right)\right\}-\frac{\rho_0}{\gamma\epsilon_n}\sum\left\{\nabla_{\bar\epsilon}\frac{d}{ds}\left(\frac{dy}{dn}\right)\right\}\right].\end{aligned}$$

To evaluate these summations, we need the quantities

$$\sum y_\mu\frac{d}{ds}\left(\frac{dy}{dn}\right).$$

We have

$$\sum y_\mu\frac{dy}{dn}=\frac{\epsilon_\mu}{\epsilon_n},$$

and therefore

$$\begin{aligned}\sum y_\mu\frac{d}{ds}\left(\frac{dy}{dn}\right) &= \epsilon_\mu\frac{d}{ds}\left(\frac{1}{\epsilon_n}\right)+\frac{1}{\epsilon_n}(\epsilon_{\mu1}p'+\epsilon_{\mu2}q'+\epsilon_{\mu3}r'+\epsilon_{\mu4}t')\\ &\qquad -\sum\left\{\frac{dy}{dn}(y_{1\mu}p'+y_{2\mu}q'+y_{3\mu}r'+y_{4\mu}t')\right\}\\ &= \epsilon_\mu\frac{d}{ds}\left(\frac{1}{\epsilon_n}\right)+\frac{\bar\epsilon_\mu}{\epsilon_n},\end{aligned}$$

for all values of μ. When these values are inserted in the two summations, the terms involving $\frac{d}{ds}\left(\frac{1}{\epsilon_n}\right)$ disappear because the constituents in the six columns of ∇_v and ∇_ϵ are $0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, 0$. Thus

$$\epsilon_n\sum\left\{\nabla_v\frac{d}{ds}\left(\frac{dy}{dn}\right)\right\}=\begin{vmatrix} 0, & v_1, & v_2, & v_3, & v_4, & 0\\ \bar\epsilon_1, & A, & H, & G, & L, & \epsilon_1\\ \bar\epsilon_2, & H, & B, & F, & M, & \epsilon_2\\ \bar\epsilon_3, & G, & F, & C, & N, & \epsilon_3\\ \bar\epsilon_4, & L, & M, & N, & D, & \epsilon_4\\ 0, & \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0\end{vmatrix}=\square(v,\bar\epsilon).$$

Similarly, we have

$$\epsilon_n \sum \left\{ \nabla_{\bar{\epsilon}} \frac{d}{ds}\left(\frac{dy}{dn}\right) \right\} = \square(\bar{\epsilon}).$$

Hence

$$\Omega \epsilon_n{}^3 \left(\frac{1}{\sigma_0 \sigma_\epsilon} + \frac{1}{\rho_0 \gamma}\right) = -\frac{\rho_0}{\rho} \square(v, \bar{\epsilon}) + \frac{\rho_0}{\gamma \epsilon_n} \square(\bar{\epsilon}),$$

thus giving a covariantive expression for the domainal torsion.

Again, returning to the equation

$$\frac{\lambda_3}{\sigma_\epsilon} - \frac{y'}{\gamma} = \frac{d}{ds}\left(\frac{dy}{dn}\right),$$

we develop the right-hand side. As

$$\frac{dy}{dn} = y_1 \frac{dp}{dn} + y_2 \frac{dq}{dn} + y_3 \frac{dr}{dn} + y_4 \frac{dt}{dn},$$

and as the arc-derivatives of $\frac{dp}{dn}$ and the other direction-variables of the domainal normal to the region have been obtained (§ 324), it follows that

$$\frac{d}{ds}\left(\frac{dy}{dn}\right) = \left(\eta_1 \frac{dp}{dn} + \eta_2 \frac{dq}{dn} + \eta_3 \frac{dr}{dn} + \eta_4 \frac{dt}{dn}\right) - \frac{1}{\epsilon_n} \frac{d\epsilon_n}{ds} \frac{dy}{dn} + \frac{1}{\Omega \epsilon_n} \sum a \bar{\epsilon}_1 y_1,$$

after reduction, that is,

$$\frac{d}{ds}\left(\frac{dy}{dn}\right) + \frac{1}{\epsilon_n} \frac{d\epsilon_n}{ds} \frac{dy}{dn} = \sum \eta_1 \frac{dp}{dn} + \frac{1}{\Omega \epsilon_n} \sum a \bar{\epsilon}_1 y_1.$$

Let the equation be squared, and the squares of the two sides be added for the plenary dimensions. The new left-hand side

$$= \sum \left\{ \frac{d}{ds}\left(\frac{dy}{dn}\right) \right\}^2 + \frac{1}{\epsilon_n{}^2}\left(\frac{d\epsilon_n}{ds}\right)^2 = \frac{1}{\sigma_\epsilon{}^2} + \frac{1}{\gamma^2} + \frac{1}{\epsilon_n{}^2}\left(\frac{d\epsilon_n}{ds}\right)^2.$$

The quantities

$$\sum \eta_i{}^2 = c_{ii}, \quad \sum \eta_i \eta_j = c_{ij},$$

arising out of the circular curvature of the domainal geodesic tangent, have already occurred (§§ 298-300); and, in fact,

$$\frac{1}{\rho^2} = \sum c_{11} p'^4.$$

All the magnitudes of the type $\sum y_\lambda \eta_\mu$ vanish; and

$$\frac{1}{\Omega^2 \epsilon_n{}^2} (\sum a \bar{\epsilon}_1 y_1)^2 = \frac{1}{\Omega \epsilon_n{}^2} \sum a \bar{\epsilon}_1{}^2,$$

the covariant $\sum a\bar{\epsilon}_1^2$ being analogous in form to $\sum av_1^2$. Also

$$\sum a\epsilon_1\bar{\epsilon}_1 = \Omega\epsilon_n\left(\bar{\epsilon}_1\frac{dp}{dn} + \bar{\epsilon}_2\frac{dq}{dn} + \bar{\epsilon}_3\frac{dr}{dn} + \bar{\epsilon}_4\frac{dt}{dn}\right)$$
$$= \Omega\epsilon_n\frac{d\epsilon_n}{ds},$$

by § 269 ; and

$$\begin{aligned}(\textstyle\sum a\epsilon_1^2)(\sum a\bar{\epsilon}_1^2) - (\sum a\epsilon_1\bar{\epsilon}_1)^2 \\ = \textstyle\sum\{(cd - n^2)(\epsilon_3\bar{\epsilon}_4 - \epsilon_4\bar{\epsilon}_3)^2\} \\ = \Omega\textstyle\sum\{(AB - H^2)(\epsilon_3\bar{\epsilon}_4 - \epsilon_4\bar{\epsilon}_3)^2\} = \Omega\,\square(\bar{\epsilon}),\end{aligned}$$

so that

$$\sum a\bar{\epsilon}_1^2 = \Omega\left(\frac{d\epsilon_n}{ds}\right)^2 + \frac{1}{\epsilon_n^2}\square(\bar{\epsilon}),$$

a relation connecting three concomitants of the region. When these values are substituted, we find

$$\frac{1}{\sigma_\epsilon^2} + \frac{1}{\gamma^2} = \sum\left\{c_{11}\left(\frac{dp}{dn}\right)^2\right\} + \frac{1}{\Omega\epsilon_n^4}\square(\bar{\epsilon}).$$

337. In the next place, let ϕ_3 denote the angle between the binormal of the regional geodesic and the binormal of the domainal geodesic tangent, the typical direction-cosine l_3 of the latter being given by

$$\frac{l_3}{\sigma} - \frac{y'}{\rho} = Y'.$$

We had the equation

$$\frac{\lambda_3}{\sigma_0} - \frac{y'}{\rho_0} = -\frac{1}{\Omega\epsilon_n^2}\nabla.$$

Let the two left-hand sides of these equations be multiplied, and likewise the two right-hand sides ; and let these products be summed over the range of the plenary space. Then, as

$$\sum l_3 y' = 0, \quad \sum \lambda_3 y' = 0,$$

we have

$$\frac{\cos\phi_3}{\sigma\sigma_0} + \frac{1}{\rho\rho_0} = -\frac{1}{\Omega\epsilon_n^2}\sum Y'\nabla.$$

To evaluate the last sum, we take the quantities Y' with the constituents y_1, y_2, y_3, y_4, in ∇_v and $\nabla_{\bar{\epsilon}}$, using the relation

$$\sum y_\mu Y' = -v_\mu.$$

Hence

$$\sum Y'\nabla_v = -\square(v), \quad \sum Y'\nabla_{\bar{\epsilon}} = -\square(v, \bar{\epsilon});$$

and therefore

$$\Omega\epsilon_n^2\left(\frac{\cos\phi_3}{\sigma\sigma_0}+\frac{1}{\rho\rho_0}\right)=\square(v)\cos\psi-\frac{1}{\epsilon_n}\square(v,\bar{\epsilon})\sin\psi.$$

Thus we have the set of relations

$$\Omega\epsilon_n^2\left(\frac{1}{\sigma_0^2}+\frac{1}{\rho_0^2}\right)=\square(v)\cos^2\psi-\frac{2}{\epsilon_n}\square(v,\bar{\epsilon})\cos\psi\sin\psi+\frac{1}{\epsilon_n^2}\square(\bar{\epsilon})\sin^2\psi,$$

$$\Omega\epsilon_n^2\left(\frac{1}{\sigma_0\sigma_\epsilon}+\frac{1}{\rho_0\gamma}\right)=-\frac{1}{\epsilon_n}\square(v,\bar{\epsilon})\cos\psi+\frac{1}{\epsilon_n^2}\square(\bar{\epsilon})\sin\psi,$$

$$\Omega\epsilon_n^2\left(\frac{1}{\sigma_\epsilon^2}+\frac{1}{\gamma^2}\right)=\frac{1}{\epsilon_n^2}\square(\bar{\epsilon})+\epsilon_n^2\sum\left\{c_{11}\left(\frac{dp}{dn}\right)^2\right\},$$

$$\Omega\epsilon_n^2\left(\frac{\cos\phi_3}{\sigma\sigma_0}+\frac{1}{\rho\rho_0}\right)=\square(v)\cos\psi-\frac{1}{\epsilon_n}\square(v,\bar{\epsilon})\sin\psi;$$

and there was the relation

$$\Omega\epsilon_n^2\left(\frac{\sin^2\chi_3}{\sigma^2}+\frac{1}{\rho^2}\right)=\square(v),$$

where χ_3 denotes the inclination of the domainal normal of the region to the binormal of the domainal geodesic tangent.

From the first, the second, and the fourth, of the foregoing set of relations, we have

$$\frac{1}{\sigma_0^2}+\frac{1}{\rho_0^2}=\left(\frac{1}{\sigma_0\sigma_\epsilon}+\frac{1}{\rho_0\gamma}\right)\sin\psi+\left(\frac{\cos\phi_3}{\sigma\sigma_0}+\frac{1}{\rho\rho_0}\right)\cos\psi,$$

or, as

$$\frac{1}{\rho_0}=\frac{\cos\psi}{\rho}+\frac{\sin\psi}{\gamma},$$

we have

$$\frac{1}{\sigma_0}=\frac{1}{\sigma_\epsilon}\sin\psi+\frac{1}{\sigma}\cos\phi_3\cos\psi,$$

a purely geometrical relation between the torsion of the domainal geodesic tangent, and the torsions (spatial and domainal) of the regional geodesic.

Moreover, the relations provide geometrical interpretations of the covariants $\square(v)$, $\square(v,\bar{\epsilon})$, $\square(\bar{\epsilon})$, $\sum\left\{c_{11}\left(\frac{dp}{dn}\right)^2\right\}$.

As a last inference of this class, consider the two equations

$$\frac{l_3}{\sigma}-\frac{y'}{\rho}=Y', \quad \frac{\lambda_3}{\sigma_\epsilon}-\frac{y'}{\gamma}=\frac{d}{ds}\left(\frac{dy}{dn}\right),$$

the former relating to the domainal geodesic. As before, we have

$$\sum l_3\lambda_3=\cos\phi_3, \quad \sum y'\lambda_3=0, \quad \sum y'l_3=0;$$

and therefore multiplying the left-hand sides together, also the right-hand sides, and adding for the range of the plenary space, we have

$$\frac{\cos\phi_3}{\sigma\sigma_\epsilon}+\frac{1}{\rho\gamma}=\sum Y'\frac{d}{ds}\left(\frac{dy}{dn}\right),$$

where, on the right-hand side, we use the value of $\frac{d}{ds}\left(\frac{dy}{dn}\right)$ on p. 411. Now (§ 297)

$$\eta_\mu=Yv_\mu+l_6\xi_\mu,$$

where l_6 is the typical direction-cosine of the quintinormal of the domainal geodesic; and therefore

$$\sum Y'\eta_\mu=\sum\left(\frac{l_3}{\sigma}-\frac{y'}{\rho}\right)(Yv_\mu+l_6\xi_\mu)=0,$$

because of the orthogonality of the directions typified by y', Y, l_3, l_6. Also

$$\begin{aligned}\sum Y'\frac{dy}{dn}&=-\left(v_1\frac{dp}{dn}+v_2\frac{dq}{dn}+v_3\frac{dr}{dn}+v_4\frac{dt}{dn}\right)\\&=-\frac{1}{\Omega\epsilon_n}\sum av_1\epsilon_1=\frac{\cos\chi_3}{\sigma};\end{aligned}$$

and

$$\sum Y'y_\mu=-v_\mu.$$

Consequently

$$\frac{\cos\phi_3}{\sigma\sigma_\epsilon}+\frac{1}{\rho\gamma}=-\frac{1}{\epsilon_n}\frac{d\epsilon_n}{ds}\frac{\cos\chi_3}{\sigma}-\frac{1}{\Omega\epsilon_n}\sum a\bar{\epsilon}_1v_1,$$

thus providing another concomitant $\sum a\bar{\epsilon}_1v_1$, and assigning its significance in terms of geometrical magnitudes.

Spatial torsion and spatial tilt of a regional geodesic.

338. By using the typical equation (§ 178) for the regional trinormal which, for the region now enclosed in a domain, becomes

$$\frac{\Omega_0^{\frac{1}{2}}}{\sigma_0}\lambda_4=\begin{vmatrix}\bar{y}_1, & \bar{y}_2, & \bar{y}_3\\ \bar{u}_1, & \bar{u}_2, & \bar{u}_3\\ \bar{v}_1, & \bar{v}_2, & \bar{v}_3\end{vmatrix},$$

we can obtain, by merely squaring the equation and adding, for the whole range of the plenary space, an expression for the spatial torsion of a geodesic in the domainal region. With the notation of § 332, let quantities R be introduced under the definitions

$$R_i=v_i\cos\psi-\frac{\bar{\epsilon}_i}{\epsilon_n}\sin\psi,$$

for $i=1, 2, 3, 4$. Then using the values of $\sum \bar{y}_1^2 = A_0$, and like combinations, we easily find

$$-\frac{\Omega\epsilon_n^2}{\sigma_0^2} = \begin{vmatrix} A, & H, & G, & L, & u_1, & R_1, & \epsilon_1 \\ H, & B, & F, & M, & u_2, & R_2, & \epsilon_2 \\ G, & F, & C, & N, & u_3, & R_3, & \epsilon_3 \\ L, & M, & N, & D, & u_4, & R_4, & \epsilon_4 \\ u_1, & u_2, & u_3, & u_4, & 0, & 0, & 0 \\ R_1, & R_2, & R_3, & R_4, & 0, & 0, & 0 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0, & 0, & 0 \end{vmatrix}.$$

It is not difficult to identify the value of $1/\sigma_0^2$ thus obtained with the value that has been obtained in § 335.

Similarly, we can obtain an expression for the spatial tilt of a geodesic in the domainal region, by adapting the equation obtained in § 179 which, in present circumstances, has the form

$$-\frac{\Omega_0^{\frac{1}{2}}}{\sigma_0^2\tau_0} = \begin{vmatrix} \bar{u}_1, & \bar{u}_2, & \bar{u}_3 \\ \bar{v}_1, & \bar{v}_2, & \bar{v}_3 \\ \bar{w}_1, & \bar{w}_2, & \bar{w}_3 \end{vmatrix}.$$

In connection with the magnitudes $\bar{w}$ of § 333, we introduce symbols S, which are analogous to the symbols R introduced in connection with the magnitudes $\bar{v}$, and are defined by the relations

$$w_i + 2\phi_i \cos\psi + W_i \sin\psi = S_i\,;$$

and now the cited formula leads to the equation

$$-\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\sigma_0^2\tau_0} = \begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ R_1, & R_2, & R_3, & R_4 \\ S_1, & S_2, & S_3, & S_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \end{vmatrix},$$

which, in association with the preceding result, provides an expression for the tilt of the geodesic in the domainal region.

Domainal orientation of a regional configuration.

339. The orientation of the regional configuration relative to the enclosing domain can be indicated by the positions of the principal lines of a regional geodesic in the frame of its domainal geodesic tangent.

The tangent direction of both geodesics being common, the remaining gremial directions of the domainal geodesic are the binormal, the trinormal, and the quartinormal, with typical direction-cosines l_3, l_4, l_5, respectively. The

remaining principal directions, belonging to the regional geodesic, have typical direction-cosines $\frac{dy}{dn}$, λ_3, λ_4, being the directions of the radius of domainal flexure (the domainal normal to the region), the binormal, and the trinormal, of the regional geodesic. In each set, the three lines are perpendicular in pairs; each set determines the same flat in the tangent block of the domain, the flat being orthogonal to the common tangent of the two geodesics. In accordance with § 317, we denote the inclinations of the domainal normal to the three lines of the domainal geodesic in this flat by χ_3, χ_4, χ_5, so that

$$\frac{\Omega\epsilon_n}{\sigma}\cos\chi_3 = -\sum a\epsilon_1 v_1,$$

$$\frac{\Omega\epsilon_n}{\sigma\tau}\cos\chi_4 = \sigma\frac{d}{ds}\left(\frac{1}{\sigma}\right)\sum a\epsilon_1 v_1 - \sum a\epsilon_1 w_1,$$

$$\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\sigma^2\tau}\cos\chi_5 = \begin{vmatrix} \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

Again, we denote the inclinations of the binormal of the regional geodesic to the same three lines of the domainal geodesic by ϕ_3, ϕ_4, ϕ_5. The value of ϕ_3 has been obtained (§ 337) in the form

$$\frac{\Omega\epsilon_n}{\sigma\sigma_\epsilon}\cos\phi_3 = -\frac{\Omega\epsilon_n}{\rho\gamma} + \frac{1}{\epsilon_n}\frac{d\epsilon_n}{ds}\sum a\epsilon_1 v_1 - \sum a\bar{\epsilon}_1 v_1.$$

The value of ϕ_4 is given by

$$\cos\phi_4 = \sum \lambda_3 l_4.$$

Now (§ 286)

$$\frac{l_4}{\tau} - y'\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) = -\frac{1}{\Omega}\{\sigma'\sum av_1y_1 + \sigma\sum aw_1y_1\} = \Theta,$$

where Θ is used for brevity; and

$$\frac{\lambda_3}{\sigma_0} - \frac{y'}{\rho_0} = -\frac{1}{\Omega\epsilon_n{}^2}\nabla = -\frac{1}{\Omega\epsilon_n{}^2}\left(\nabla_v\cos\psi - \frac{1}{\epsilon_n}\nabla_\epsilon\sin\psi\right).$$

Hence multiplying the left-hand sides together, likewise the right-hand sides, and adding the products for the plenary range, we have

$$\frac{1}{\sigma_0\tau}\cos\phi_4 + \frac{1}{\rho_0}\frac{d}{ds}\left(\frac{\sigma}{\rho}\right) = -\frac{1}{\Omega\epsilon_n{}^2}[(\sum\Theta\nabla_v)\cos\psi - \frac{1}{\epsilon_n}(\sum\Theta\nabla_\epsilon)\sin\psi].$$

To evaluate the right-hand side, because

$$\sum\{y_\mu(\sum av_1y_1)\} = \Omega v_\mu, \quad \sum\{y_\mu(\sum aw_1y_1)\} = \Omega w_\mu,$$

for $\mu=1, 2, 3, 4$, it follows that

$$\sum\{\nabla_v(\sum av_1y_1)\}=\Omega\square(v), \quad \sum\{\nabla_{\bar\epsilon}(\sum av_1y_1)\}=\Omega\square(v, \bar\epsilon),$$

and therefore

$$\frac{1}{\Omega}\sum\{\nabla(\sum av_1y_1)\}=\square(v)\cos\psi-\frac{1}{\epsilon_n}\square(v, \bar\epsilon)\sin\psi$$
$$=\Omega\epsilon_n{}^2\left(\frac{\cos\phi_3}{\sigma\sigma_0}+\frac{1}{\rho\rho_0}\right).$$

In the same way, we have

$$\frac{1}{\Omega}\sum\{\nabla(\sum aw_1y_1)\}=\square(v, w)\cos\psi-\frac{1}{\epsilon_n}\square(\bar\epsilon, w)\sin\psi.$$

When these values are inserted, and the expression is reduced, we find

$$\frac{1}{\sigma\sigma_0\tau}\cos\phi_4+\frac{1}{\rho_0}\frac{d}{ds}\left(\frac{1}{\rho}\right)+\frac{1}{\sigma_0}\frac{d}{ds}\left(\frac{1}{\sigma}\right)\cos\phi_3$$
$$=\square(v, w)\cos\psi-\frac{1}{\epsilon_n}\square(\bar\epsilon, w)\sin\psi.$$

The typical direction-cosine l_5 of the quartinormal of the domainal geodesic tangent to the regional geodesic is (§ 288)

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau}l_5=\begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}=\Phi;$$

and

$$\frac{\lambda_3}{\sigma_0}-\frac{y'}{\rho_0}=-\frac{1}{\Omega\epsilon_n{}^2}\nabla.$$

Hence, as $\sum y'l_5=0$, the inclination ϕ_5 of the regional binormal to the quartinormal of the domainal geodesic tangent is given by

$$\frac{\Omega^{\frac{1}{2}}}{\sigma_0\sigma^2\tau}\cos\phi_5=-\frac{1}{\Omega\epsilon_n{}^2}\sum\Phi\nabla,$$

the right-hand sum extending over the plenary range. Now

$$\sum y_\mu\nabla_v=\Omega\epsilon_n{}^2v_\mu, \quad \sum y_\mu\nabla_{\bar\epsilon}=\Omega\epsilon_n{}^2\bar\epsilon_\mu,$$

for $\mu=1, 2, 3, 4$, and therefore

$$\sum y_\mu\nabla=\Omega\epsilon_n{}^2\left(v_\mu\cos\psi-\frac{1}{\epsilon_n}\bar\epsilon_\mu\sin\psi\right).$$

Also $\gamma \sin\psi = \rho_0$. Hence

$$\frac{\Omega^{\frac{1}{2}}}{\rho_0\sigma_0\sigma^2\tau}\cos\phi_5 = \frac{1}{\gamma\epsilon_n}\begin{vmatrix} \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

We thus have expressions for $\cos\phi_3$, $\cos\phi_4$, $\cos\phi_5$, connected with the inclinations of the regional binormal to the three selected principal lines of the domainal geodesic.

To determine the inclinations θ_3, θ_4, θ_5, of the trinormal of the regional geodesic to the binormal, the trinormal, and the quartinormal of the domainal geodesic tangent, we use the value of the typical direction-cosine λ_4 of the regional trinormal, as obtained in § 178 in the form

$$\frac{\Omega_0^{\frac{1}{2}}}{\sigma_0}\lambda_4 = \begin{vmatrix} \bar{y}_1, & \bar{y}_2, & \bar{y}_3 \\ \bar{u}_1, & \bar{u}_2, & \bar{u}_3 \\ \bar{v}_1, & \bar{v}_2, & \bar{v}_3 \end{vmatrix}$$

$$= \frac{1}{\epsilon_4}\left(Z_v \cos\psi - \frac{1}{\epsilon_n} Z_{\bar{\epsilon}} \sin\psi\right),$$

when the values of $\bar{y}_\mu$, $\bar{u}_\mu$, $\bar{v}_\mu$, are inserted, where

$$Z_v = \begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \end{vmatrix}, \qquad Z_{\bar{\epsilon}} = \begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ u_1, & u_2, & u_3, & u_4 \\ \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \end{vmatrix}.$$

Also we have

$$\Omega_0\epsilon_4^2 = \Omega\epsilon_n^2;$$

and therefore

$$\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\sigma_0}\lambda_4 = Z_v\cos\psi - \frac{1}{\epsilon_n}Z_{\bar{\epsilon}}\sin\psi.$$

To obtain the inclination θ_3 to the binormal of the domainal geodesic, with the typical direction-cosine l_3, we take

$$\frac{l_3}{\sigma} - \frac{y'}{\rho} = Y';$$

and so, as $\sum y'\lambda_4 = 0$,

$$\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\sigma\sigma_0}\cos\theta_3 = (\sum Y'Z_v)\cos\psi - \frac{1}{\epsilon_n}(\sum Y'Z_{\bar{\epsilon}})\sin\psi.$$

Now

$$\sum y_\mu Y' = -v_\mu;$$

and therefore

$$\sum Y'Z_v=0, \quad \sum Y'Z_{\bar{\epsilon}}=\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \end{vmatrix};$$

consequently

$$\frac{\Omega^{\frac{1}{2}}\epsilon_n^2}{\rho_0\sigma_0\sigma}\cos\theta_3=-\frac{1}{\gamma}\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \end{vmatrix}.$$

To obtain the inclination θ_4, to the trinormal of the domainal geodesic with the typical direction-cosine l_4, we take (§ 8)

$$\frac{l_4}{\sigma\tau}+l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right)-Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-y'\frac{d}{ds}\left(\frac{1}{\rho}\right)=Y''.$$

Now

$$\sum l_4\lambda_4=\cos\theta_4, \quad \sum l_3\lambda_4=\cos\theta_3, \quad \sum Y\lambda_4=0, \quad \sum y'\lambda_4=0:$$

hence multiplying the left-hand sides of this l_4-equation and the λ_4-equation, likewise their right-hand sides, and adding for the plenary space, we find

$$\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\sigma_0\sigma\tau}\cos\theta_4+\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\sigma_0}\frac{d}{ds}\left(\frac{1}{\sigma}\right)\cos\theta_3=\sum\left\{Y''\left(Z_v\cos\psi-\frac{1}{\epsilon_n}Z_{\bar{\epsilon}}\sin\psi\right)\right\}.$$

To evaluate the right-hand side, we use the formula (§ 285, *Ex.* 1)

$$\sum y_\mu Y''=-w_\mu,$$

for $\mu=1, 2, 3, 4$; and we find

$$\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\rho_0\sigma_0\sigma\tau}\cos\theta_4+\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\rho_0\sigma_0}\frac{d}{ds}\left(\frac{1}{\sigma}\right)\cos\theta_3=-\frac{1}{\rho}\Xi_v+\frac{1}{\gamma\epsilon_n}\Xi_{\bar{\epsilon}},$$

where

$$\Xi_v=\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \end{vmatrix}, \quad \Xi_{\bar{\epsilon}}=\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ w_1, & w_2, & w_3, & w_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \end{vmatrix}.$$

To obtain the inclination θ_5 to the quartinormal of the domainal geodesic with the typical direction-cosine l_4, we have

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau}l_5=\Phi,$$

as before (p. 458); and therefore

$$\frac{\Omega\epsilon_n}{\sigma_0\sigma^2\tau}\cos\theta_5=\sum\left\{\Phi\left(Z_v\cos\psi-\frac{1}{\epsilon_n}Z_{\bar{\epsilon}}\sin\psi\right)\right\}$$

To evaluate the quantities $\sum \Phi Z_v$, $\sum \Phi Z_{\bar{\epsilon}}$, the customary properties of determinants may be employed. With the definitions of Φ and Z_v, we have

$$\sum \Phi Z_v = \sum A\,(u_2 v_3 w_4)(u_2 v_3 \epsilon_4),$$

and therefore

$$\Omega^2 \sum \Phi Z_v = \sum (bcd)(u_2 v_3 w_4)(u_2 v_3 \epsilon_4)$$
$$= \begin{vmatrix} \sum a u_1^2, & \sum a u_1 v_1, & \sum a u_1 \epsilon_1 \\ \sum a u_1 v_1, & \sum a v_1^2, & \sum a v_1 \epsilon_1 \\ \sum a u_1 w_1, & \sum a v_1 w_1, & \sum a w_1 \epsilon_1 \end{vmatrix},$$

all the constituents in the last determinant being concomitants of the domain. In particular, the values of

$$\sum a u_1^2, \quad \sum a u_1 v_1, \quad \sum a u_1 w_1, \quad \sum a v_1^2, \quad \sum a v_1 w_1,$$

are known (p. 306); also

$$\sum a u_1 \epsilon_1 = \Omega (\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t') = 0,$$

while $\sum a v_1 \epsilon_1$ and $\sum a w_1 \epsilon_1$ occur in the expressions for $\cos \chi_3$ and $\cos \chi_4$.

Similarly we find

$$\Omega^2 \sum \Phi Z_{\bar{\epsilon}} = \begin{vmatrix} \sum a u_1^2, & \sum a u_1 \bar{\epsilon}_1, & \sum a u_1 \epsilon_1 \\ \sum a u_1 v_1, & \sum a v_1 \bar{\epsilon}_1, & \sum a v_1 \epsilon_1 \\ \sum a u_1 w_1, & \sum a w_1 \bar{\epsilon}_1, & \sum a w_1 \epsilon_1 \end{vmatrix},$$

where again all the constituents are known concomitants of the domain.

As the three directions, typified by the direction-cosines l_3, l_4, l_5, are an orthogonal set within the flat which, in the tangent block of the domain, is at right angles to the common tangent of the regional geodesic and the domainal geodesic, and as the three lines typified by the direction-cosines $\frac{dy}{dn}$, λ_3, λ_4, are another orthogonal set within the same flat, the determinant

$$\begin{vmatrix} \cos \chi_3, & \cos \chi_4, & \cos \chi_5 \\ \cos \phi_3, & \cos \phi_4, & \cos \phi_5 \\ \cos \theta_3, & \cos \theta_4, & \cos \theta_5 \end{vmatrix}$$

is equal to unity: every constituent in the determinant is equal to its own co-factor; and relations, of the type

$$\cos^2 \chi_3 + \cos^2 \chi_4 + \cos^2 \chi_5 = 1,$$
$$\cos^2 \chi_3 + \cos^2 \phi_3 + \cos^2 \theta_3 = 1,$$

lead to expressions for concomitants in terms of geometrical magnitudes.

CHAPTER XXIX

Domainal Surfaces: Surfaces Geodesic to the Domain

Parametric surfaces: superficial geodesics.

340. When a surface is contained wholly within a domain, one analytical method of representing the surface provides it as the intersection of two parametric regions

$$\epsilon(p, q, r, t)=0, \quad \omega(p, q, r, t)=0,$$

each lying wholly within the domain. Another method, formally different and ultimately the same in effect, expresses the four parameters of the domain as functions of two new superficial parameters u and v, as in § 117. The former mode of expression will be adopted, mainly because intrinsic properties of the surface are brought into relation with intrinsic properties of the two regions as well as with intrinsic properties of the domain. But some results formally associated with the bi-parametric representation will be given later (§§ 352, 353).

For the discussion of such a surface, we therefore assume the established properties of the region $\epsilon=0$ and the analogous properties of the region $\omega=0$. The elementary geometrical properties of each region and its domainal normal, as well as some of their properties relative to their superficial intersection, have already been obtained in the preceding chapters; and the notation there adopted will be continued.

As usual in past investigations, two fundamental preliminary results must be obtained. One of these is the formation of the intrinsic equations of superficial geodesics; the other is the determination of the relation between the tangent plane to the surface at a point O and a small range of the surface in the immediate vicinity of O. These are discussed in succession.

341. The intrinsic equations of a geodesic on a domainal surface are, initially, the four critical equations exacted if an integral

$$\int\left\{\sum A\left(\frac{dp}{du}\right)^2\right\}^{\frac{1}{2}} du$$

is to be made a minimum among the values arising from all values of the four domainal parameters satisfying the two equations $\epsilon=0$, $\omega=0$, the quantity u being a passing independent variable. These critical equations are of the customary form and, after calculations, similar to those in preceding instances, can be expressed in the modified forms

$$\left.\begin{aligned} p_0'' + \sum \Gamma_{11} p'^2 &= \lambda \sum a\epsilon_1 + \mu \sum a\omega_1 \\ q_0'' + \sum \Delta_{11} p'^2 &= \lambda \sum h\epsilon_1 + \mu \sum h\omega_1 \\ r_0'' + \sum \Theta_{11} p'^2 &= \lambda \sum g\epsilon_1 + \mu \sum g\omega_1 \\ t_0'' + \sum \Phi_{11} p'^2 &= \lambda \sum l\epsilon_1 + \mu \sum l\omega_1 \end{aligned}\right\},$$

where, for $\psi = \epsilon$ or ω,

$$\begin{aligned} \sum a\psi_1 &= a\psi_1 + h\psi_2 + g\psi_3 + l\psi_4, \\ \sum h\psi_1 &= h\psi_1 + b\psi_2 + f\psi_3 + m\psi_4, \\ \sum g\psi_1 &= g\psi_1 + f\psi_2 + c\psi_3 + n\psi_4, \\ \sum l\psi_1 &= l\psi_1 + m\psi_2 + n\psi_3 + d\psi_4. \end{aligned}$$

The quantities λ and μ are multipliers undetermined in the construction of the critical equations; and p_0'', q_0'', r_0'', t_0'', are second derivatives along the arc of the superficial geodesic, being distinct from second derivatives along the domainal geodesic and along the respective regional geodesics in the same initial direction p', q', r', t'. Also, by § 269, we have

$$\sum a\epsilon_1 = \Omega\epsilon_n \frac{dp}{dn}, \quad \sum h\epsilon_1 = \Omega\epsilon_n \frac{dq}{dn}, \quad \sum g\epsilon_1 = \Omega\epsilon_n \frac{dr}{dn}, \quad \sum l\epsilon_1 = \Omega\epsilon_n \frac{dt}{dn},$$

$$\sum a\omega_1 = \Omega\omega_\nu \frac{dp}{d\nu}, \quad \sum h\omega_1 = \Omega\omega_\nu \frac{dq}{d\nu}, \quad \sum g\omega_1 = \Omega\omega_\nu \frac{dr}{d\nu}, \quad \sum l\omega_1 = \Omega\omega_\nu \frac{dt}{d\nu},$$

where

$$\sum a\epsilon_1^2 = \Omega\epsilon_n^2, \quad \sum a\omega_1^2 = \Omega\omega_\nu^2, \quad \sum a\epsilon_1\omega_1 = \Omega\epsilon_n\omega_\nu \cos\iota,$$

ι denoting the inclination of the domainal normals of the intersecting regions at O and therefore the inclination of those regions themselves.

The multipliers λ and μ are to be determined. We first multiply the four modified critical equations by ϵ_1, ϵ_2, ϵ_3, ϵ_4, respectively, and add the products. The sum of the right-hand sides

$$\begin{aligned} &= \lambda \sum a\epsilon_1^2 + \mu \sum a\epsilon_1\omega_1 \\ &= \Omega\epsilon_n(\lambda\epsilon_n + \mu\omega_\nu \cos\iota); \end{aligned}$$

and the sum of the left-hand sides

$$\begin{aligned} &= \epsilon_1 p_0'' + \epsilon_2 q_0'' + \epsilon_3 r_0'' + \epsilon_4 t_0'' \\ &\qquad + \epsilon_1 \sum \Gamma_{11} p'^2 + \epsilon_2 \sum \Delta_{11} p'^2 + \epsilon_3 \sum \Theta_{11} p'^2 + \epsilon_4 \sum \Phi_{11} p'^2. \end{aligned}$$

For all variations lying in the region $\epsilon = 0$ (and therefore for the superficial variations in question), we have

$$\epsilon_1 p_0'' + \epsilon_2 q_0'' + \epsilon_3 r_0'' + \epsilon_4 t_0'' + \sum \epsilon_{11} p'^2 = 0;$$

and therefore the foregoing sum of the left-hand sides

$$= -\sum \epsilon_{11} p'^2 + \epsilon_1 \sum \Gamma_{11} p'^2 + \epsilon_2 \sum \Delta_{11} p'^2 + \epsilon_3 \sum \Theta_{11} p'^2 + \epsilon_4 \sum \Phi_{11} p'^2 = -\sum \bar{\epsilon}_{11} p'^2,$$

with the notation of § 269, viz.

$$\bar{\epsilon}_{ij}=\epsilon_{ij}-\epsilon_1\Gamma_{ij}-\epsilon_2\Delta_{ij}-\epsilon_3\Theta_{ij}-\epsilon_4\Phi_{ij}.$$

As before (§ 315), we write

$$-\sum\bar{\epsilon}_{11}p'^2=\frac{\epsilon_n}{\gamma_\epsilon},$$

and the result then becomes

$$\frac{1}{\gamma_\epsilon}=\Omega(\lambda\epsilon_n+\mu\omega_\nu\cos\iota).$$

Next, we multiply the same four modified equations by ω_1, ω_2, ω_3, ω_4, respectively, and proceed in precisely the same fashion; and we find

$$\frac{1}{\gamma_\omega}=\Omega(\lambda\epsilon_n\cos\iota+\mu\omega_\nu),$$

where

$$\frac{\omega_\nu}{\gamma_\omega}=-\sum\bar{\omega}_{11}p'^2,$$

with the similar notation

$$\bar{\omega}_{ij}=\omega_{ij}-\omega_1\Gamma_{ij}-\omega_2\Delta_{ij}-\omega_3\Theta_{ij}-\omega_4\Phi_{ij}.$$

Now take two quantities g_ϵ and g_ω, such that

$$g_\epsilon=\lambda\Omega\epsilon_n,\quad g_\omega=\mu\Omega\omega_\nu,$$

so that

$$g_\epsilon+g_\omega\cos\iota=\frac{1}{\gamma_\epsilon},\quad g_\omega+g_\epsilon\cos\iota=\frac{1}{\gamma_\omega}.$$

Then the first of the modified critical equations is

$$\begin{aligned}p_0''+\sum\Gamma_{11}p'^2&=\lambda\sum a\epsilon_1+\mu\sum a\omega_1\\&=\lambda\Omega\epsilon_n\frac{dp}{dn}+\mu\Omega\omega_\nu\frac{dp}{d\nu}\\&=g_\epsilon\frac{dp}{dn}+g_\omega\frac{dp}{d\nu};\end{aligned}$$

and similarly for the other three equations of the same kind.

Accordingly, the (parametric) intrinsic equations of the geodesic on the surface, drawn in the superficial direction p', q', r', t', through O, are

$$\left.\begin{aligned}p_0''+\sum\Gamma_{11}p'^2&=g_\epsilon\frac{dp}{dn}+g_\omega\frac{dp}{d\nu}\\q_0''+\sum\Delta_{11}p'^2&=g_\epsilon\frac{dq}{dn}+g_\omega\frac{dq}{d\nu}\\r_0''+\sum\Theta_{11}p'^2&=g_\epsilon\frac{dr}{dn}+g_\omega\frac{dr}{d\nu}\\t_0''+\sum\Phi_{11}p'^2&=g_\epsilon\frac{dt}{dn}+g_\omega\frac{dt}{d\nu}\end{aligned}\right\},$$

where g_ϵ and g_ω are given by the equations

$$g_\epsilon + g_\omega \cos\iota = \frac{1}{\gamma_\epsilon}, \quad g_\omega + g_\epsilon \cos\iota = \frac{1}{\gamma_\omega}.$$

The quantities γ_ϵ and γ_ω are interpreted, exactly as in § 316; and we note that, for the domainal geodesic through O in the same direction p', q', r', t', as the superficial geodesic,

$$\sum \Gamma_{11} p'^2 = -p'', \quad \sum \Delta_{11} p'^2 = -q'', \quad \sum \Theta_{11} p'^2 = -r'', \quad \sum \Phi_{11} p'^2 = -t'',$$

so that the equations of the superficial geodesics also have the form

$$\left.\begin{aligned} p_0'' - p'' &= g_\epsilon \frac{dp}{dn} + g_\omega \frac{dp}{d\nu} \\ q_0'' - q'' &= g_\epsilon \frac{dq}{dn} + g_\omega \frac{dq}{d\nu} \\ r_0'' - r'' &= g_\epsilon \frac{dr}{dn} + g_\omega \frac{dr}{d\nu} \\ t_0'' - t'' &= g_\epsilon \frac{dt}{dn} + g_\omega \frac{dt}{d\nu} \end{aligned}\right\}.$$

Circular curvature and domainal flexure of a superficial geodesic.

342. We can derive a first expression for the circular curvature of the superficial geodesic and for the direction of its prime normal. Denoting the radius of curvature by ρ_0 and a typical direction-cosine of the prime normal by Y_0, we have

$$Y_0 = \rho_0 y_0'',$$

where y_0'' denotes the second arc-derivative of the typical coordinate y estimated along the superficial geodesic.

Now, along the superficial geodesic,

$$y_0'' = y_1 p_0'' + y_2 q_0'' + y_3 r_0'' + y_4 t_0'' + \sum y_{11} p'^2,$$

and, along the domainal geodesic,

$$y'' = y_1 p'' + y_2 q'' + y_3 r'' + y_4 t'' + \sum y_{11} p'^2,$$

the variables p', q', r', t', being the same in the two relations because the geodesics have the same tangent at O. Hence

$$y_0'' - y'' = \sum y_1 (p_0'' - p'').$$

When the foregoing equations, characteristic of the superficial geodesic, are used to modify the expression on the right-hand side, this typical equation becomes

$$\begin{aligned} y_0'' - y'' &= g_\epsilon \sum y_1 \frac{dp}{dn} + g_\omega \sum y_1 \frac{dp}{d\nu} \\ &= g_\epsilon \frac{dy}{dn} + g_\omega \frac{dy}{d\nu}, \end{aligned}$$

that is,

$$\frac{Y_0}{\rho_0} - \frac{Y}{\rho} = g_\epsilon \frac{dy}{dn} + g_\omega \frac{dy}{d\nu},$$

the quantities $\frac{dy}{dn}$ and $\frac{dy}{d\nu}$ being the typical direction-cosines, in the plenary space, of the domainal normals to the regions $\epsilon=0$ and $\omega=0$ respectively.

Next, the significance of the expression on the right-hand side of this equation has to be determined. We consider the deviation, in magnitude and in direction, of the superficial geodesic from the domainal geodesic in the same initial direction p', q', r', t'. We take a small arc-length δ, the same for the two geodesics, measured along them from O; the points thus obtained are denoted by Q_S on the surface and by Q_D in the domain. The typical y-coordinate of Q_D

$$=y+y'\delta+\tfrac{1}{2}y''\delta^2+\dots,$$

and that of Q_S

$$=y+y'\delta+\tfrac{1}{2}y_0''\delta^2+\dots,$$

the unexpressed terms being of the third and higher orders in δ. Let the (small) domainal length Q_SQ_D, being the deviation of the superficial geodesic from the domainal geodesic, be denoted by D; and let the typical spatial direction-cosine of this length in the direction from Q_D to Q_S be denoted by l, so that

$$\begin{aligned} lD &= (y+y'\delta+\tfrac{1}{2}y_0''\delta^2+\dots)-(y+y'\delta+\tfrac{1}{2}y''\delta^2+\dots) \\ &= \tfrac{1}{2}(y_0''-y'')\delta^2+\dots. \end{aligned}$$

We define the radius of domainal flexure of the superficial geodesic to be given by

$$\frac{1}{\gamma}=\operatorname*{Lim}_{\delta\to 0}\frac{2D}{\delta^2},$$

and $1/\gamma$ is called the domainal flexure; hence we have

$$\begin{aligned} \frac{l}{\gamma} &= y_0''-y'' \\ &= g_\epsilon\frac{dy}{dn}+g_\omega\frac{dy}{d\nu}. \end{aligned}$$

Hence also

$$\frac{Y_0}{\rho_0}-\frac{Y}{\rho}=\frac{l}{\gamma},$$

the typical equation connecting the circular curvature and the domainal flexure of the superficial geodesic with the circular curvature of the domainal geodesic tangent.

The analytical magnitude denoted by $1/\gamma_\epsilon$ has already appeared in the discussion of the domainal flexure of a geodesic of the region $\epsilon=0$; and it was shewn (§ 316) that, when this regional geodesic in the direction p', q', r', t', is drawn, thus also touching the superficial geodesic under discussion, the radius of domainal flexure of this regional geodesic is equal to γ_ϵ, while the equations

$$\frac{Y_\epsilon}{\rho_\epsilon}-\frac{Y}{\rho}=\frac{1}{\gamma_\epsilon}\frac{dy}{dn}$$

hold, where $\frac{dy}{dn}$ is the typical spatial direction-cosine of the radius of geodesic flexure (so that the radius is in the direction of the domainal normal of the region), and Y_ϵ is the typical spatial direction-cosine of the prime normal of the geodesic of the region $\epsilon=0$. Also, the four intrinsic equations of that regional geodesic are of the form

$$p_\epsilon'' - p'' = \frac{1}{\gamma_\epsilon}\frac{dp}{dn}.$$

Similarly for the regional geodesic in the domain $\omega=0$, drawn in that same direction p', q', r', t', the analytical quantity denoted by γ_ω is the radius of domainal flexure of that geodesic: that radius is in the direction of the domainal normal to the region $\omega=0$ with a typical direction-cosine $\frac{dy}{d\nu}$. The radius of circular curvature ρ_ω and its typical spatial direction-cosine Y_ω are connected with the circular curvature of the domainal geodesic tangent by the typical equation

$$\frac{Y_\omega}{\rho_\omega} - \frac{Y}{\rho} = \frac{1}{\gamma_\omega}\frac{dy}{d\nu},$$

while the four intrinsic equations of this regional geodesic are of the form

$$p_\omega'' - p'' = \frac{1}{\gamma_\omega}\frac{dp}{d\nu}.$$

The foregoing equations imply a number of geometrical relations among the positions of the various radii of circular curvature and the various radii of domainal flexure.

We have already seen (§ 316), from the equations

$$\frac{Y_\epsilon}{\rho_\epsilon} - \frac{Y}{\rho} = \frac{1}{\gamma_\epsilon}\frac{dy}{dn},$$

that the prime normal of the ϵ-regional geodesic is complanar with the domainal normal of the ϵ-region and the prime normal of the domainal geodesic tangent. Similarly, from the equations

$$\frac{Y_\omega}{\rho_\omega} - \frac{Y}{\rho} = \frac{1}{\gamma_\omega}\frac{dy}{d\nu},$$

it follows that the prime normal of the ω-regional geodesic is complanar with the domainal normal of the ω-region and the prime normal of the domainal geodesic tangent.

From the equations

$$\frac{l}{\gamma} = g_\epsilon\frac{dy}{dn} + g_\omega\frac{dy}{d\nu},$$

it follows that the direction of the radius of domainal flexure of the superficial geodesic is complanar with the domainal normals to the two regions: that is, it lies in the domainal orientation which is orthogonal to the surface.

From the equations

$$\frac{Y_0}{\rho_0}-\frac{Y}{\rho}=\frac{l}{\gamma},$$

it follows that the direction of the prime normal of the superficial geodesic, the direction of its radius of domainal flexure, and the prime normal of the domainal geodesic tangent, lie in one plane.

Finally, from the equations

$$\frac{Y_0}{\rho_0}-\frac{Y}{\rho}=g_\epsilon\frac{dy}{dn}+g_\omega\frac{dy}{d\nu},$$

it follows that the direction of the prime normal of the superficial geodesic lies in the same flat as the domainal normals to the two regions and the prime normal of the domainal geodesic tangent. Moreover, this flat obviously contains each of the four preceding planes; and, in the configuration thus formed, there are three leading lines, being the domainal normals to the two regions and the prime normal of the domainal geodesic, itself normal to the domain, so that

$$\sum Y\frac{dy}{dn}=0,\quad \sum Y\frac{dy}{d\nu}=0.$$

In the diagram, OY represents the prime normal of the domainal geodesic tangent, OF_ϵ the radius of domainal flexure of the ϵ-regional geodesic, OF_ω the radius of domainal flexure of the ω-regional geodesic: in each instance, in magnitude and direction. The line OY is at right angles to the plane $F_\epsilon OF_\omega$.

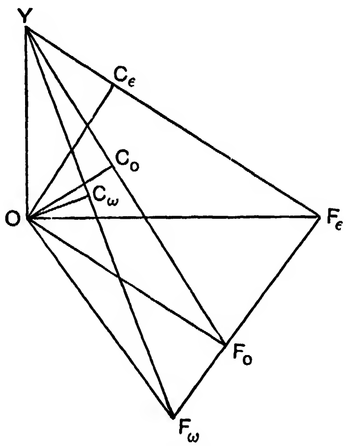

FIG. 33.

Draw OF_0 at right angles to $F_\epsilon F_\omega$ in the latter plane: then OF_0 represents the radius of domainal flexure of the superficial geodesic, in magnitude and direction.

Draw perpendiculars OC_ϵ on YF_ϵ, OC_ω on YF_ω, OC_0 on YF_0; then OC_ϵ represents, in magnitude and direction, the radius of circular curvature of the ϵ-regional geodesic; OC_ω represents, in magnitude and direction, the radius of circular curvature of the ω-regional geodesic; and OC_0 represents, in magnitude and direction, the radius of circular curvature of the superficial geodesic.

Also, as the angles at C_ϵ, C_ω, C_0, are right angles, the points C_ϵ, C_ω, C_0, lie on a sphere in the flat, OY being the diameter; hence the locus of the centres of circular curvature of geodesics, which belong to sub-amplitudes of a domain and are drawn through O to touch a domainal geodesic through O, is a sphere.

Let ψ_0, ψ_ϵ, ψ_ω, denote the angles YOC_0, YOC_ϵ, YOC_ω, respectively. The angle $F_\epsilon OF_\omega$, the angle between the domainal normals, has already been denoted by ι; let α, β, denote the angles $F_\epsilon OF_0$ and F_0OF_ω respectively, so that $\alpha+\beta=\iota$. Then the following results are easily obtained, when account is taken of the typical direction-cosines of the lines OY; OC_0, OC_ϵ, OC_ω; OF_0, OF_ϵ, OF_ω; as relating to angles and magnitudes:

$$\left.\begin{aligned}
\frac{\sin^2\iota}{\gamma^2}&=\frac{1}{\gamma_\epsilon^2}-\frac{2\cos\iota}{\gamma_\epsilon\gamma_\omega}+\frac{1}{\gamma_\omega^2}\\
\frac{1}{\rho_0^2}&=\frac{1}{\rho^2}+\frac{1}{\sin^2\iota}\left(\frac{1}{\gamma_\epsilon^2}-\frac{2\cos\iota}{\gamma_\epsilon\gamma_\omega}+\frac{1}{\gamma_\omega^2}\right)\\
\frac{1}{\rho_\epsilon^2}&=\frac{1}{\rho^2}+\frac{1}{\gamma_\epsilon^2}\\
\frac{1}{\rho_\omega^2}&=\frac{1}{\rho^2}+\frac{1}{\gamma_\omega^2}
\end{aligned}\right\};$$

$$\left.\begin{aligned}
\frac{\sin\alpha}{\dfrac{1}{\gamma_\epsilon}-\dfrac{\cos\iota}{\gamma_\omega}}&=\frac{\cos\alpha}{\dfrac{\sin\iota}{\gamma_\omega}}=\frac{1}{\dfrac{\sin\iota}{\gamma}}\\
\frac{\sin\beta}{\dfrac{1}{\gamma_\omega}-\dfrac{\cos\iota}{\gamma_\epsilon}}&=\frac{\cos\beta}{\dfrac{\sin\iota}{\gamma_\epsilon}}=\frac{1}{\dfrac{\sin\iota}{\gamma}}
\end{aligned}\right\};$$

$$\left.\begin{aligned}
&\frac{\cos\psi_0}{\rho_0}=\frac{\cos\psi_\epsilon}{\rho_\epsilon}=\frac{\cos\psi_\omega}{\rho_\omega}=\frac{1}{\rho}\\
&\frac{\sin\psi_0}{\rho_0}=\frac{1}{\gamma},\quad \frac{\sin\psi_\epsilon}{\rho_\epsilon}=\frac{1}{\gamma_\epsilon},\quad \frac{\sin\psi_\omega}{\rho_\omega}=\frac{1}{\gamma_\omega}
\end{aligned}\right\};$$

together with many others.

Ex. 1. Denoting by $\bar{\iota}$ the angle between the prime normals of the two regional geodesics, prove the relations

$$\frac{\cos\bar{\iota}}{\rho_\epsilon\rho_\omega}-\frac{\cos\iota}{\gamma_\epsilon\gamma_\omega}=\frac{1}{\rho^2},\quad \frac{\sin^2\iota}{\gamma^2}=\frac{1}{\rho_\epsilon^2}+\frac{1}{\rho_\omega^2}-\frac{2\cos\bar{\iota}}{\rho_\epsilon\rho_\omega}.$$

Ex. 2. Obtain the relation

$$\frac{2}{\rho_0^2}-\frac{1}{\rho_\epsilon^2}-\frac{1}{\rho_\omega^2}-2\left\{\left(\frac{1}{\rho_0^2}-\frac{1}{\rho_\epsilon^2}\right)\left(\frac{1}{\rho_0^2}-\frac{1}{\rho_\omega^2}\right)\right\}^{\frac{1}{2}}\cos\iota=\frac{1}{\rho_0^2}-\frac{1}{\rho^2},$$

connecting the radii of circular curvature of the four geodesics drawn in the direction p', q', r', t'.

Tangent plane of a surface and the prime normal of a geodesic.

343. It is desirable to establish, for a domainal surface, the customary relation (§ 20) between the tangent homaloid of an amplitude at a point and a range of the configuration in the near vicinity of the point.

Any line, tangent to the surface, is given by the typical equation

$$\frac{\bar{y}-y}{y'}=\mu_0,$$

where μ_0 is a parameter along the line: that is,

$$\begin{aligned}\bar{y}-y&=\mu_0 y'\\ &=\kappa y_1+\lambda y_2+\mu y_3+\varpi y_4,\end{aligned}$$

where

$$\kappa=p'\mu_0,\quad \lambda=q'\mu_0,\quad \mu=r'\mu_0,\quad \varpi=r'\mu_0,$$

these symbols κ, λ, μ, ϖ, now merely denoting current parameters along the line. As p', q', r', t', are direction-variables of a tangent to the surface, they must satisfy the two equations

$$\epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t'=0,\quad \omega_1 p'+\omega_2 q'+\omega_3 r'+\omega_4 t'=0\;;$$

and therefore the foregoing quantities κ, λ, μ, ϖ, must be subject to the two conditions

$$\begin{aligned}\epsilon_1\kappa+\epsilon_2\lambda+\epsilon_3\mu+\epsilon_4\varpi&=0,\\ \omega_1\kappa+\omega_2\lambda+\omega_3\mu+\omega_4\varpi&=0.\end{aligned}$$

When these quantities are regarded as parameters, their range is two-fold; and therefore every point on the foregoing tangent lines for different directions ds, the quantities y_1', y_2', ... no longer appearing in the parametric equations, lies in the two-fold (and necessarily homaloidal) range. Thus the tangent plane of the surface at O, being the locus of these tangent lines, is represented by the equations

$$\bar{y}-y=\kappa y_1+\lambda y_2+\mu y_3+\varpi y_4,$$

the parameters κ, λ, μ, ϖ, being subject to the two specified conditions: or by the form

$$\left\|\begin{matrix}\bar{y}-y, & y_1, & y_2, & y_3, & y_4\\ 0\;, & \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4\\ 0\;, & \omega_1, & \omega_2, & \omega_3, & \omega_4\end{matrix}\right\|=0,$$

having the equivalent interpretation.

Let Q be a point on the surface near O, such that the length OQ in the surface is a small quantity δ; and let the space-coordinates of Q be $\eta_1, \eta_2, \ldots$, typified by η. Let a perpendicular be drawn from Q upon the tangent plane of the surface at O: denote its length by Π and its typical spatial direction-cosine by l_0, so that,

if the foot of the perpendicular be the point in the plane with the typical coordinate $\bar{y}=y+\kappa y_1+\lambda y_2+\mu y_3+\varpi y_4$, we have

$$l_0 \Pi = \eta - \bar{y}.$$

To secure this perpendicularity, we make the quantity Π^2, that is, $\sum(\eta-\bar{y})^2$, a minimum among all the values of the parameters κ, λ, μ, ϖ, that are subject to the two prescribed conditions; and we therefore have four critical equations

$$\sum[y_\iota\{\eta-(y+\kappa y_1+\lambda y_2+\mu y_3+\varpi y_4)\}]=I\epsilon_\iota+J\omega_i,$$

for $i=1, 2, 3, 4$, where I and J are two multipliers left undetermined in forming the critical equations.

In the first place, these equations can be written in the form

$$\Pi\sum(y_\iota l_0)=I\epsilon_i+J\omega_i;$$

and therefore, for all parameters κ', λ', μ', ϖ', of the plane,

$$\Pi\sum\{l_0(\kappa' y_1+\lambda' y_2+\mu' y_3+\varpi' y_4)\}=I(\sum\kappa'\epsilon_1)+J\sum(\kappa'\omega_1)=0,$$

that is,

$$\sum l_0(\kappa' y_1+\lambda' y_2+\mu' y_3+\varpi' y_4)=0:$$

or the perpendicular Π is at right angles to every direction in the plane.

Next, the four critical equations and the two prescribed conditions are potentially sufficient for the determination of the six quantities I, J, κ, λ, μ, ϖ. We proceed as follows. The first equation (for $i=1$) is

$$A\kappa+H\lambda+G\mu+L\varpi+I\epsilon_1+J\omega_1=\sum y_1(\eta-y).$$

Now on the surface, as Q and P are near one another, we represent the direction PQ by variables typified in y'; and, without specifying any special curve $PQ\ldots$, we denote continued variation of y by y_0''; hence

$$\begin{aligned}\sum\{y_1(\eta-y)\}&=\sum[y_1\{y'\delta+\tfrac{1}{2}y_0''\delta^2+\ldots\}]\\ &=(Ap'+Hq'+Gr'+Lt')\delta+\tfrac{1}{2}(\sum y_1 y_0'')\delta^2+\ldots,\end{aligned}$$

the unstated terms containing third and higher powers of δ. Again, whatever be the curve $PQ\ldots$, let p_0'', q_0'', r_0'', t_0'', denote the second variations at O of the parameters along that curve, so that

$$y_0''=y_1 p_0''+y_2 q_0''+y_3 r_0''+y_4 t_0''+\sum y_{11}p'^2,$$

and therefore

$$\sum y_1 y_0''=Ap_0''+Hq_0''+Gr_0''+Lt_0''+\sum y_1\{\sum y_{11}p'^2\}.$$

But (§ 267)

$$\sum y_1 y_{\iota j}=A\Gamma_{\iota j}+H\Delta_{\iota j}+G\Theta_{\iota j}+L\Phi_{\iota j};$$

consequently

$$\begin{aligned}\sum y_1 y_0''=\;&A(p_0''+\sum\Gamma_{11}p'^2)+H(q_0''+\sum\Delta_{11}p'^2)\\ &+G(r_0''+\sum\Theta_{11}p'^2)+L(t_0''+\sum\Phi_{11}p'^2).\end{aligned}$$

Let

$$\begin{aligned}
\bar{\kappa} &= \kappa - p'\delta - \tfrac{1}{2}(p_0'' + \textstyle\sum \Gamma_{11} p'^2)\delta^2, \\
\bar{\lambda} &= \lambda - q'\delta - \tfrac{1}{2}(q_0'' + \textstyle\sum \Delta_{11} p'^2)\delta^2, \\
\bar{\mu} &= \mu - r'\delta - \tfrac{1}{2}(r_0'' + \textstyle\sum \Theta_{11} p'^2)\delta^2, \\
\bar{\varpi} &= \varpi - t'\delta - \tfrac{1}{2}(t_0'' + \textstyle\sum \Phi_{11} p'^2)\delta^2 ;
\end{aligned}$$

then the first of the critical equations becomes

$$A\bar{\kappa} + H\bar{\lambda} + G\bar{\mu} + L\bar{\varpi} = -I\epsilon_1 - J\omega_1 + [3]_1,$$

where $[3]_1$ represents an aggregate of terms involving δ^3 and higher powers of δ linearly.

The other three critical equations, after similar treatment, are

$$\begin{aligned}
H\bar{\kappa} + B\bar{\lambda} + F\bar{\mu} + M\bar{\varpi} &= -I\epsilon_2 - J\omega_2 + [3]_2, \\
G\bar{\kappa} + F\bar{\lambda} + C\bar{\mu} + N\bar{\varpi} &= -I\epsilon_3 - J\omega_3 + [3]_3, \\
L\bar{\kappa} + M\bar{\lambda} + N\bar{\mu} + D\bar{\varpi} &= -I\epsilon_4 - J\omega_4 + [3]_4,
\end{aligned}$$

with like significance for $[3]_\iota$, for $i=2, 3, 4$. When the four equations are resolved, and the aggregates of terms of order higher than the second in the small quantity δ are omitted, we have

$$\begin{aligned}
\bar{\kappa} &= -\frac{1}{\Omega}\{I(\textstyle\sum a\epsilon_1) + J(\textstyle\sum a\omega_1)\}, \\
\bar{\lambda} &= -\frac{1}{\Omega}\{I(\textstyle\sum h\epsilon_1) + J(\textstyle\sum h\omega_1)\}, \\
\bar{\mu} &= -\frac{1}{\Omega}\{I(\textstyle\sum g\epsilon_1) + J(\textstyle\sum g\omega_1)\}, \\
\bar{\varpi} &= -\frac{1}{\Omega}\{I(\textstyle\sum l\epsilon_1) + J(\textstyle\sum l\omega_1)\},
\end{aligned}$$

all up to the second order included. Consequently

$$\begin{aligned}
\epsilon_1\bar{\kappa} + \epsilon_2\bar{\lambda} + \epsilon_3\bar{\mu} + \epsilon_4\bar{\varpi} &= -I\epsilon_n{}^2 - J\epsilon_n\omega_\nu \cos\iota, \\
\omega_1\bar{\kappa} + \omega_2\bar{\lambda} + \omega_3\bar{\mu} + \omega_4\bar{\varpi} &= -I\epsilon_n\omega_\nu \cos\iota - J\omega_\nu{}^2.
\end{aligned}$$

Reverting to the definitions of $\bar{\kappa}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\varpi}$, which are complete and not approximate in their values, we have

$$\begin{aligned}
\epsilon_1\bar{\kappa} + \epsilon_2\bar{\lambda} + \epsilon_3\bar{\mu} + \epsilon_4\bar{\varpi} = \; & \epsilon_1\kappa + \epsilon_2\lambda + \epsilon_3\mu + \epsilon_4\varpi \\
& - (\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t')\delta \\
& - \tfrac{1}{2}\{\textstyle\sum \epsilon_1(p_0'' + \textstyle\sum \Gamma_{11} p'^2)\}\delta^2.
\end{aligned}$$

On the right-hand side, the first line vanishes because of one of the prescribed conditions, and the second line vanishes because the superficial direction p', q', r', t', necessarily lies also in the ϵ-region. Further, the continued direction p_0'', q_0'',

r_0'', t_0'', in the unspecified curve PQ ... necessarily lies in the ϵ-region, so that, for second variations, we have

$$\epsilon_1 p_0'' + \epsilon_2 q_0'' + \epsilon_3 r_0'' + \epsilon_4 t_0'' + \sum \epsilon_{11} p'^2 = 0,$$

and therefore

$$\begin{aligned}\sum \{\epsilon_1 (p_0'' + \Gamma_{11} p'^2)\} &= -\sum \{(\epsilon_{11} - \epsilon_1 \Gamma_{11} - \epsilon_2 \Delta_{11} - \epsilon_3 \Theta_{11} - \epsilon_4 \Phi_{11}) p'^2\} \\ &= -\sum \bar{\epsilon}_{11} p'^2 \\ &= \frac{\epsilon_n}{\gamma_\epsilon},\end{aligned}$$

with the former notation. Hence

$$\epsilon_1 \bar{\kappa} + \epsilon_2 \bar{\lambda} + \epsilon_3 \bar{\mu} + \epsilon_4 \bar{\varpi} = -\tfrac{1}{2} \frac{\epsilon_n}{\gamma_\epsilon} \delta^2.$$

Similarly, we find

$$\omega_1 \bar{\kappa} + \omega_2 \bar{\lambda} + \omega_3 \bar{\mu} + \omega_4 \bar{\varpi} = -\tfrac{1}{2} \frac{\omega_\nu}{\gamma_\omega} \delta^2.$$

Thus the preceding equations for I and J become

$$I\epsilon_n + J\omega_\nu \cos\iota = \tfrac{1}{2} \frac{\delta^2}{\gamma_\epsilon} = \tfrac{1}{2}(g_\epsilon + g_\omega \cos\iota)\,\delta^2,$$

$$I\epsilon_n \cos\iota + J\omega_\nu = \tfrac{1}{2} \frac{\delta^2}{\gamma_\omega} = \tfrac{1}{2}(g_\epsilon \cos\iota + g_\omega)\,\delta^2,$$

and therefore

$$I\epsilon_n = \tfrac{1}{2} g_\epsilon \delta^2, \quad J\omega_\nu = \tfrac{1}{2} g_\omega \delta^2,$$

thus giving values of the quantities I and J up to the second order of the small quantity δ inclusive.

When these values of I and J are used, we have

$$\begin{aligned}-\bar{\kappa} &= \tfrac{1}{2} g_\epsilon \delta^2 \left\{ \frac{1}{\Omega \epsilon_n} \sum (a\epsilon_1) \right\} + \tfrac{1}{2} g_\omega \delta^2 \left\{ \frac{1}{\Omega \omega_\nu} (\sum a\omega_1) \right\} \\ &= \tfrac{1}{2} \left(g_\epsilon \frac{dp}{dn} + g_\omega \frac{dp}{d\nu} \right) \delta^2,\end{aligned}$$

$$-\bar{\lambda} = \tfrac{1}{2} \left(g_\epsilon \frac{dq}{dn} + g_\omega \frac{dq}{d\nu} \right) \delta^2,$$

$$-\bar{\mu} = \tfrac{1}{2} \left(g_\epsilon \frac{dr}{dn} + g_\omega \frac{dr}{d\nu} \right) \delta^2,$$

$$-\bar{\varpi} = \tfrac{1}{2} \left(g_\epsilon \frac{dt}{dn} + g_\omega \frac{dt}{d\nu} \right) \delta^2.$$

With these results, the values of κ, λ, μ, ϖ, are known, also up to the second order of the small quantity δ inclusive.

We now can infer the length and the direction of the perpendicular Π, drawn from Q upon the tangent plane to the surface at O. A typical equation for its length and its direction-cosines is

$$\begin{aligned} l_0\Pi &= \eta - (y + \kappa y_1 + \lambda y_2 + \mu y_3 + \varpi y_4) \\ &= y'\delta + \tfrac{1}{2}y_0''\delta^2 - (\kappa y_1 + \lambda y_1 + \mu y_3 + \varpi y_4), \end{aligned}$$

up to the second order inclusive. Let the values

$$\kappa = p'\delta + \tfrac{1}{2}(p_0'' + \sum \Gamma_{11}p'^2)\delta^2 + \bar{\kappa},$$

with similar expressions for λ, μ, ϖ, be inserted in this expression for Π. The total terms of the first order

$$= y'\delta - (y_1p' + y_2q' + y_3r' + y_4t')\delta = 0.$$

In terms of the second order, not arising through $\bar{\kappa}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\varpi}$, the coefficient of $\frac{1}{2}\delta^2$ is

$$y_0'' - \sum\{y_1(p_0'' + \sum \Gamma_{11}p'^2)\}:$$

but along the curve QP on the surface,

$$y_0'' = \sum y_1 p_0'' + \sum y_{11}p'^2,$$

and therefore the coefficient of $\frac{1}{2}\delta^2$

$$= \sum\{(y_{11} - y_1\Gamma_{11} - y_2\Delta_{11} - y_3\Theta_{11} - y_4\Phi_{11})p'^2\} = \sum \eta_{11}p'^2 = \frac{Y}{\rho},$$

the magnitude appertaining to the domainal geodesic. Hence, in all, we have

$$\begin{aligned} l_0\Pi &= \tfrac{1}{2}\frac{Y}{\rho}\delta^2 - (\bar{\kappa}y_1 + \bar{\lambda}y_2 + \bar{\mu}y_3 + \bar{\varpi}y_4) \\ &= \tfrac{1}{2}\delta^2\left(\frac{Y}{\rho} + g_\epsilon\frac{dy}{dn} + g_\omega\frac{dy}{dv}\right) \\ &= \tfrac{1}{2}\delta^2\frac{Y_0}{\rho_0}, \end{aligned}$$

where $1/\rho_0$ is the circular curvature of the superficial geodesic in the direction PQ and Y_0 is the typical direction-cosine of its prime normal (§ 341). Consequently,

$$\Pi = \frac{\delta^2}{2\rho_0},$$

giving an approximation to the length of the perpendicular, accurate up to δ^2 inclusive: and

$$l_0 = Y_0,$$

shewing that the direction of the perpendicular from Q on the tangent plane at O coincides with the direction of the prime normal of the superficial geodesic in the direction OP.

Surfaces geodesic to the domain.

344. Among the surfaces within a domain (or within any amplitude), a special geometrical significance attaches to those which are styled geodesic. Through any point O in the domain, let two domainal geodesics OA and OB be drawn, so that therefore their directions are two different tangents to the domain and define a superficial orientation. Through O, let domainal geodesics be drawn having their initiating directions in this orientation; these geodesics generate a surface which is said to be geodesic to the domain at O. After the corresponding properties of geodesic surfaces in a region, it is not to be expected (and it is not the fact) that the surface is everywhere geodesic to the domain; it is geodesic for all directions through O that is, all superficial geodesics through O are domainal geodesics: but at any point U on OA, no superficial geodesic through U other than OUA is a domainal geodesic: and at any point V on OB, no superficial geodesic through V other than OVB is a domainal geodesic. It thus is of analytical importance to obtain the tests, necessary and sufficient to secure that a parametric surface $\epsilon=0$ and $\omega=0$, through O, should be either (i) geodesic to the domain at O: or (ii), if not geodesic, have contact of ascertainable order with the surface which is geodesic in the same orientation at O. The completely geodesic property will be established only if every domainal geodesic through O, in directions which lie within the orientation of the surface at O, lies wholly within the surface.

In order that a superficial geodesic through O shall be a domainal geodesic, it is necessary that the domainal flexure of that geodesic shall be zero; and if the surface is to be geodesic to the domain at O, the domainal flexure of all superficial geodesics through O must vanish. Consequently, if the surface represented by the parametric equations $\epsilon=0$ and $\omega=0$ be geodesic, the quantity

$$\frac{\sin^2 \iota}{\gamma^2}, \quad =\frac{1}{\gamma_\epsilon^2}+\frac{2\cos\iota}{\gamma_\epsilon\gamma_\omega}+\frac{1}{\gamma_\omega^2}$$
$$=\left(\frac{\cos\iota}{\gamma_\epsilon}+\frac{1}{\gamma_\omega}\right)^2+\frac{\sin^2\iota}{\gamma_\epsilon^2}=\left(\frac{1}{\gamma_\epsilon}+\frac{\cos\iota}{\gamma_\omega}\right)^2+\frac{\sin^2\iota}{\gamma_\omega^2},$$

must vanish for all superficial directions; and therefore

$$\frac{1}{\gamma_\epsilon}=0, \quad \frac{1}{\gamma_\omega}=0,$$

for all directions in the surface. (It follows that all the ϵ-regional geodesics in such directions are domainal geodesics, though the ϵ-region may not be completely geodesic to the domain: and likewise that all the ω-regional geodesics in those same superficial directions are domainal geodesics, though the ω-region may not be completely geodesic to the domain.) These conditions are

$$\sum \bar{\epsilon}_{11} p'^2=0, \quad \sum \bar{\omega}_{11} p'^2=0.$$

But while these conditions are necessary, they are not sufficient to constitute the geodesic character of the surface: they only secure, at O, contact of the second order—it may be called osculation—between the parametric surface and the geodesic surface. Sufficiency will be provided if the domainal geodesic in any direction through O on the surface lies wholly in the surface. The parametric coordinates P, Q, R, T, at any point O', distant δ from O along a domainal geodesic drawn in the superficial direction p', q', r', t', are given by four expressions of the form

$$P=p+p'\delta+\tfrac{1}{2}p''\delta^2+\tfrac{1}{6}p'''\delta^3+\dots;$$

and the adequate requisite will be supplied if the equations

$$\epsilon(P, Q, R, T)=0, \quad \omega(P, Q, R, T)=0,$$

are satisfied for all values of δ. Hence all the domainal geodesics in these superficial directions must lie wholly within the region $\epsilon=0$ and also must lie wholly within the region $\omega=0$.

By § 326, we know that a domainal geodesic in a direction p', q', r', t', at O lies wholly within a region $\epsilon=0$ if the set of conditions

$$\epsilon=0, \quad E_1=\sum \epsilon_1 p'=0, \quad E_2=\sum \bar{\epsilon}_{11}p'^2=0, \quad E_3=\sum \bar{\epsilon}_{111}p'^3=0, \dots,$$

is satisfied; and it will lie wholly within a region $\omega=0$ if the set of conditions (with similar notation)

$$\omega=0, \quad \sum \omega_1 p'=0, \quad \sum \bar{\omega}_{11}p'^2=0, \quad \sum \bar{\omega}_{111}p'^3=0, \dots,$$

is satisfied. Moreover, the directions p', q', r', t', are to be superficial, so that

$$\epsilon_1 p'+\epsilon_2 q'+\epsilon_3 r'+\epsilon_4 t'=0, \quad \omega_1 p'+\omega_2 q'+\omega_3 r'+\omega_4 t'=0.$$

When these conditions are satisfied for all the values of p', q', r', t', we have conditions sufficient to constitute the geodesic character of the surface at O in relation to the domain.

But this geodesic surface can be approached otherwise. The parameters P, Q, R, T, at a point O' on the domainal geodesic which originates in the superficial orientation, are given by the foregoing four equations which involve also the current variable δ of length along the geodesic. The direction-variables at O satisfy the equations

$$\sum Ap'^2=1, \quad \sum \epsilon_1 p'=0, \quad \sum \omega_1 p'=0.$$

The geodesic surface can be obtained by eliminating the five quantities p', q', r', t', δ, among the seven equations; it therefore can be represented by two eliminant equations

$$G(P, Q, R, T)=0, \quad \bar{G}(P, Q, R, T)=0,$$

these being expressible in a variety of forms, each equivalent to two equations. Moreover, when the expressions for P, Q, R, T, are substituted in $G=0$, $\bar{G}=0$, and the equations are arranged in powers of δ, the coefficient of every power of δ in each of them must vanish.

The eliminant-pair $G=0$, $\bar{G}=0$, and the parametric-pair $\epsilon=0$, $\omega=0$, may be functionally equivalent: in that event, the parametric surface is geodesic to the domain at O. When the two pairs are not functionally equivalent, there still can be a partial agreement between them, in the form of successive sets of vanishing terms of the same order when the equations are arranged in powers of δ.

The simultaneous vanishing of the terms in the parametric-pair, which are independent of δ, merely expresses the fact that O lies on the surface.

The simultaneous vanishing of the terms, which involve the first power of δ, requires the conditions

$$\sum \epsilon_1 p'=0, \quad \sum \omega_1 p'=0.$$

These relations are the parametric equations of the tangent plane of the parametric surface; and the geodesic surface at O has this plane orientation. The implied geometrical property can be called *contact of the first order* between the parametric surface and the geodesic surface. Also when the orientation-variables of the plane are denoted (§ 270) by s_{23}, s_{31}, s_{12}, s_{14}, s_{24}, s_{34}, the relations

$$x_i' s_{jk} + x_j' s_{ki} + x_k' s_{ij} = 0,$$

$$\epsilon_i s_{il} + \epsilon_j s_{jl} + \epsilon_k s_{kl} = 0, \quad \omega_i s_{il} + \omega_j s_{jl} + \omega_k s_{kl} = 0,$$

(with the usual convention for the variables x') are satisfied.

The simultaneous vanishing of the terms, involving the second power of δ in the parametric equations, gives the conditions

$$\sum \bar{\epsilon}_{11} p'^2=0, \quad \sum \bar{\omega}_{11} p'^2=0.$$

When both conditions hold for all values of p', q', r', t', in the superficial orientation, every domainal geodesic originating in the orientation osculates the surface. The property can be called *contact of the second order* between the two surfaces. If only one relation holds, as for the ϵ-region, the parametric surface is inflexional in that region; the other relation then determines two directions which give domainal geodesics inflexional to the ω-region.

Similarly, the simultaneous vanishing of the terms in the third power of δ in the parametric equations gives

$$\sum \bar{\epsilon}_{111} p'^3=0, \quad \sum \bar{\omega}_{111} p'^3=0.$$

When both relations hold for all values of p', q', r', t', in the superficial orientation, the implied geometrical property can be called *contact of the third order* between the parametric surface and the geodesic surface at O.

As for the region (§ 210), so here for the domain, the process establishes the geodesic surface in any orientation at O as the uniquely determinate surface of reference for all surfaces in the orientation.

It remains to consider the analytical import of the conditions, for the successive orders of contact.

I. As the geodesic surface as O is drawn in the orientation of the tangent plane at O to the parametric surface, the first-order conditions are satisfied.

II. The conditions for contact of the second order, being

$$\sum \bar{\epsilon}_{11} p'^2 = 0, \quad \sum \bar{\omega}_{11} p'^2 = 0,$$

are to be satisfied for all values of p', q', r', t', in the orientation: that is, for all values of p', q', r', t', subject to two equations such as

$$s_{12} R_0 = s_{12} r' + s_{23} p' + s_{31} q' = s_{12}(r' - \alpha p' - \beta q') = 0,$$
$$s_{12} T_0 = s_{12} t' + s_{24} p' + s_{41} q' = s_{12}(t' - \kappa p' - \lambda q') = 0,$$

with obvious significance for α, β, κ, λ. Introducing sets of umbral symbols m and μ, under the definitions

$$\bar{\epsilon}_{ij} = m_i m_j, \quad \bar{\omega}_{ij} = \mu_i \mu_j,$$

we have

$$\sum \bar{\epsilon}_{11} p'^2 = m_{p'}^2 = (m_1 p' + m_2 q' + m_3 r' + m_4 t')^2 = (M_1 p' + M_2 q' + m_3 R_0 + m_4 T_0)^2,$$
$$\sum \bar{\omega}_{11} p'^2 = \mu_{p'}^2 = (\mu_1 p' + \mu_2 q' + \mu_3 r' + \mu_4 t')^2 = (N_1 p' + N_2 q' + \mu_3 R_0 + \mu_4 T_0)^2,$$

where

$$\left.\begin{aligned} M_1 &= m_1 + m_3 \alpha + m_4 \kappa \\ M_2 &= m_2 + m_3 \beta + m_4 \lambda \end{aligned}\right\}, \quad \left.\begin{aligned} N_1 &= \mu_1 + \mu_3 \alpha + \mu_4 \kappa \\ N_2 &= \mu_2 + \mu_3 \beta + \mu_4 \lambda \end{aligned}\right\}.$$

When $R_0 = 0$, $T_0 = 0$, that is, for values of p', q', r', t', in the orientation, the conditions become

$$(M_1 p' + M_2 q')^2 = 0, \quad (N_1 p' + N_2 q')^2 = 0;$$

and, as p', q', are independent, these modified conditions must be evanescent. Hence there are three relations

$$\begin{aligned} 0 = M_1^2 &= (m_1 + m_3 \alpha + m_4 \kappa)^2 \\ &= \bar{\epsilon}_{11} + 2\bar{\epsilon}_{13} \alpha + 2\bar{\epsilon}_{14} \kappa + \bar{\epsilon}_{33} \alpha^2 + 2\bar{\epsilon}_{34} \alpha\kappa + \bar{\epsilon}_{44} \kappa^2, \\ 0 = M_1 M_2 &= (m_1 + m_3 \alpha + m_4 \kappa)(m_2 + m_3 \beta + m_4 \lambda) \\ &= \bar{\epsilon}_{12} + \bar{\epsilon}_{13} \beta + \bar{\epsilon}_{14} \lambda + \bar{\epsilon}_{23} \alpha + \bar{\epsilon}_{24} \kappa + \bar{\epsilon}_{33} \alpha\beta + \bar{\epsilon}_{34}(\alpha\lambda + \beta\kappa) + \bar{\epsilon}_{44} \kappa\lambda, \\ 0 = M_2^2 &= (m_2 + m_3 \beta + m_4 \lambda)^2 \\ &= \bar{\epsilon}_{22} + 2\bar{\epsilon}_{23} \beta + 2\bar{\epsilon}_{24} \lambda + \bar{\epsilon}_{33} \beta^2 + 2\bar{\epsilon}_{34} \beta\lambda + \bar{\epsilon}_{44} \lambda^2, \end{aligned}$$

from the first of the modified conditions; and, from the second, three other similar relations, with $\bar{\omega}_{ij}$ in place of $\bar{\epsilon}_{ij}$.

These six relations are necessary. If we write

$$\bar{\epsilon}_i = \bar{\epsilon}_{i1} p' + \bar{\epsilon}_{i2} p' + \bar{\epsilon}_{i3} q' + \bar{\epsilon}_{i4} t', \quad \bar{\omega}_i = \bar{\omega}_{i1} p' + \bar{\omega}_{i2} q' + \bar{\omega}_{i3} r' + \bar{\omega}_{i4} t',$$

and assume the relations satisfied, we find

$$\sum \bar{\epsilon}_{11} p'^2 = 2(R_0 \bar{\epsilon}_3 + T_0 \bar{\epsilon}_4) + (\bar{\epsilon}_{33}, \bar{\epsilon}_{34}, \bar{\epsilon}_{44} \between R_0, T_0)^2,$$
$$\sum \bar{\omega}_{11} p'^2 = 2(R_0 \bar{\omega}_3 + T_0 \bar{\omega}_4) + (\bar{\omega}_{33}, \bar{\omega}_{34}, \bar{\omega}_{44} \between R_0, T_0)^2,$$

identically; and therefore the six necessary relations are sufficient to ensure that the second-order conditions are satisfied.

Equivalent (but not independent) forms of the relations are obtained by taking any pair, other than p' and q', from p', q', r', t', as variables of reference. The whole set of relations, when α, β, κ, λ, are expressed in terms of the first derivatives of ϵ and ω (as in § 270), are included in the statement: When ∇ denotes the determinant

$$\begin{vmatrix} \xi\bar{\epsilon}_{11}+\zeta\bar{\omega}_{11}, & \xi\bar{\epsilon}_{12}+\zeta\bar{\omega}_{12}, & \xi\bar{\epsilon}_{13}+\zeta\bar{\omega}_{13}, & \xi\bar{\epsilon}_{14}+\zeta\bar{\omega}_{14}, & \epsilon_1, & \omega_1 \\ \xi\bar{\epsilon}_{21}+\zeta\bar{\omega}_{21}, & \xi\bar{\epsilon}_{22}+\zeta\omega_{22}, & \xi\bar{\epsilon}_{23}+\zeta\bar{\omega}_{23}, & \xi\bar{\epsilon}_{24}+\zeta\bar{\omega}_{24}, & \epsilon_2, & \omega_2 \\ \xi\bar{\epsilon}_{31}+\zeta\bar{\omega}_{31}, & \xi\bar{\epsilon}_{32}+\zeta\bar{\omega}_{32}, & \xi\bar{\epsilon}_{33}+\zeta\bar{\omega}_{33}, & \xi\bar{\epsilon}_{34}+\zeta\bar{\omega}_{34}, & \epsilon_3, & \omega_3 \\ \xi\bar{\epsilon}_{41}+\zeta\bar{\omega}_{41}, & \xi\bar{\epsilon}_{42}+\zeta\bar{\omega}_{42}, & \xi\bar{\epsilon}_{43}+\zeta\bar{\omega}_{43}, & \xi\bar{\epsilon}_{44}+\zeta\bar{\omega}_{44}, & \epsilon_4, & \omega_4 \\ \epsilon_1 \quad , & \epsilon_2 \quad , & \epsilon_3 \quad , & \epsilon_4 \quad , & 0, & 0 \\ \omega_1 \quad , & \omega_2 \quad , & \omega_3 \quad , & \omega_4 \quad , & 0, & 0 \end{vmatrix},$$

where ξ and ζ are arbitrary constants, the co-factor of every constituent $\xi\bar{\epsilon}_{ij}+\zeta\bar{\omega}_{ij}$ in ∇ must vanish for all values of ξ and ζ.

Ex. 1. Assuming the relations satisfied, shew that

$$\begin{Vmatrix} \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{Vmatrix}=0, \quad \begin{vmatrix} \bar{\omega}_1, & \bar{\omega}_2, & \bar{\omega}_3, & \bar{\omega}_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix}=0.$$

Ex. 2. Verify that the six relations, for contact of the second order between the two surfaces, are invariantive in character for all transformations $\bar{\epsilon}=f(\epsilon, \omega)=$ constant, $\bar{\omega}=g(\epsilon, \omega)=$ constant, where f and g denote any two independent functional forms.

NOTE. The analytical results can be interpreted in terms of the geometry of skew curves in homaloidal triple space. We take p', q', r', t', as homogeneous coordinates of a point in that space. The two equations $\sum \epsilon_1 p'=0$, $\sum \omega_1 p'=0$, are the equations of a line L in that space. The two equations $\sum \bar{\epsilon}_{11} p'^2=0$, $\sum \bar{\omega}_{11} p'^2=0$, when taken separately, represent two quadrics; when taken together, they represent a twisted quartic curve. When the six relations are satisfied, the line L lies on each of the quadrics, and the quartic curve degenerates into the line L and a twisted cubic *.

III. The conditions for contact of the third order can be discussed in the same manner. We introduce umbral symbols g and γ, defined by

$$\bar{\epsilon}_{ijk}=g_i g_j g_k, \quad \bar{\omega}_{ijk}=\gamma_i\gamma_j\gamma_k,$$

so that

$$\sum \bar{\epsilon}_{111} p'^3=g_{p'}^3=(g_1p'+g_2q'+g_3r'+g_4t')^3=(g_3R_0+g_4T_0+G_1p'+G_2q')^3,$$
$$\sum \bar{\omega}_{111} p'^3=\gamma_{p'}^3=(\gamma_1p'+\gamma_2q'+\gamma_3r'+\gamma_4t')^3=(\gamma_3R_0+\gamma_4T_0+\Gamma_1p'+\Gamma_2q')^3,$$

when
$$\left.\begin{aligned} G_1&=g_1+g_3\alpha+g_4\kappa \\ G_2&=g_2+g_3\beta+g_4\lambda \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_1&=\gamma_1+\gamma_3\alpha+\gamma_4\kappa \\ \Gamma_2&=\gamma_2+\gamma_3\beta+\gamma_4\lambda \end{aligned}\right\}.$$

* Salmon's *Analytical Geometry of Three Dimensions* (5th edn., by R. A. P. Rogers, 1912), vol. i, §§ 342, 347-351.

Then the following results can be established :

(*a*) The relations, necessary and sufficient to ensure that the conditions for third-order contact are satisfied, are

$$G_1^3=0,\quad G_1^2G_2=0,\quad G_1G_2^2=0,\quad G_2^3=0,$$
$$\Gamma_1^3=0,\quad \Gamma_1^2\Gamma_2=0,\quad \Gamma_1\Gamma_2^2=0,\quad \Gamma_2^3=0,$$

in umbral forms, the literal expressions being immediate :

(*b*) The preceding eight relations, in (*a*), are invariantive in character for all functional transformations $\bar{\epsilon}=\phi(\epsilon, \omega)$, $\bar{\omega}=\psi(\epsilon, \omega)$:

(*c*) If $\bar{E}_i$ and $\bar{\Omega}_i$, for all values of i, be defined by the equations

$$\bar{E}_i=\tfrac{1}{3}\frac{\partial}{\partial x_i'}\sum\sum\sum\bar{\epsilon}_{ijk}x_i'x_j'x_k',\quad \bar{\Omega}_i=\tfrac{1}{3}\frac{\partial}{\partial x_i'}\bar{\omega}_{ijk}x_i'x_j'x_k',$$

then, for all directions in the orientation, the equations

$$\left\|\begin{matrix} \xi\bar{E}_1+\zeta\bar{\Omega}_1, & \xi\bar{E}_2+\zeta\bar{\Omega}_2, & \xi\bar{E}_3+\zeta\bar{\Omega}_3, & \xi\bar{E}_4+\zeta\bar{\Omega}_4 \\ \epsilon_1\quad , & \epsilon_2\quad , & \epsilon_3\quad , & \epsilon_4 \\ \omega_1\quad , & \omega_2\quad , & \omega_3\quad , & \omega_4 \end{matrix}\right\|=0$$

hold for arbitrary values for the constants ξ and ζ, when the eight relations of third-order contact are satisfied.

Primary and secondary magnitudes of a domainal surface.

345. We return now to the properties of a general (non-geodesic) surface in a domain. As any surface requires only a couple of parameters for parametric expression, we may represent a domainal surface by the two parameters p and q, the other two r and t (and all their derivatives) being supposed to be eliminated by means of the equations $\epsilon=0$ and $\omega=0$ of the surface. As the arc of the surface is an arc in the domain, the value of ds is unaltered. With the orientation-variables s_{ij}, we have

$$-r's_{12}=p's_{23}+q's_{31},\quad -t's_{12}=p's_{24}+q's_{41};$$

on the elimination of r' and t' from the permanent arc-relation $\sum Ap'^2=1$ of the domain, the arc-relation of the surface is

$$A_0p'^2+2H_0p'q'+B_0q'^2=1,$$

where

$$s_{12}^2A_0=As_{12}^2-2(Gs_{23}+Ls_{24})s_{12}+Cs_{23}^2+2Ns_{23}s_{24}+Ds_{24}^2,$$
$$s_{12}^2H_0=Hs_{12}^2-(Gs_{31}+Fs_{23}+Ls_{41}+Ms_{24})s_{12}+Cs_{23}s_{31}+N(s_{23}s_{41}+s_{31}s_{24})+Ds_{24}s_{41},$$
$$s_{12}^2B_0=Bs_{12}^2-2(Fs_{31}+Ms_{41})s_{12}+Cs_{31}^2+2Ns_{31}s_{41}+Ds_{41}^2.$$

In the first place, we have

$$s_{12}^2(A_0B_0-H_0^2)=\sum\{(AB-H^2)s_{12}^2\},$$

on reduction; and therefore (p. 255) we have (as is to be expected)

$$(A_0B_0-H_0^2)^{\frac{1}{2}}(p_1'q_2'-q_1'p_2')=\sin\widehat{12},$$

where $\widehat{12}$ denotes the angle between the two directions p_1', q_1', r_1', t_1', and p_2', q_2', r_2', t_2'. But by § 270,

$$\frac{p_1'q_2'-q_1'p_2'}{\sin\widehat{12}}=\begin{vmatrix}\epsilon_3, & \epsilon_4\\ \omega_3, & \omega_4\end{vmatrix}\frac{1}{\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota};$$

and therefore

$$(\epsilon_3\omega_4-\epsilon_4\omega_3)(A_0B_0-H_0^2)^{\frac{1}{2}}=\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota.$$

In the next place, we form the magnitudes $A_0p'+H_0q'$, $H_0p'+B_0q'$. By direct substitution and reduction, and with the customary significance of u_1, u_2, u_3, u_4, we obtain the formulæ

$$\left.\begin{aligned}s_{12}(A_0p'+H_0q')&=s_{12}u_1-s_{23}u_3-s_{24}u_4\\ s_{12}(H_0p'+B_0q')&=s_{12}u_2-s_{31}u_3-s_{41}u_4\end{aligned}\right\}.$$

Similarly, if we take quantities $\overline{A}_0$, $\overline{H}_0$, $\overline{B}_0$, bearing to the secondary magnitudes of the domain the same formal relations as A_0, H_0, B_0, bear to its primary magnitudes, so that

$$\begin{aligned}s_{12}^2\overline{A}_0&=\overline{A}s_{12}^2-2(\overline{G}s_{23}+\overline{L}s_{24})s_{12}+\overline{C}s_{23}^2+2\overline{N}s_{23}s_{24}+\overline{D}s_{24}^2,\\ s_{12}^2\overline{H}_0&=\overline{H}s_{12}^2-(\overline{G}s_{31}+\overline{F}s_{23}+\overline{L}s_{41}+\overline{M}s_{24})s_{12}+\overline{C}s_{23}s_{31}+\overline{N}(s_{23}s_{41}+s_{31}s_{24})+\overline{D}s_{24}s_{41},\\ s_{12}^2\overline{B}_0&=\overline{B}s_{12}^2-2(\overline{F}s_{31}+\overline{M}s_{41})s_{12}+\overline{C}s_{31}^2+2\overline{N}s_{31}s_{41}+\overline{D}s_{41}^2,\end{aligned}$$

the expression for the circular curvature of the domainal geodesic, in the direction tangential at O to the superficial geodesic, becomes

$$\frac{1}{\rho}=\overline{A}_0p'^2+2\overline{H}_0p'q'+\overline{B}_0q'^2.$$

We require magnitudes $\overline{A}_0p'+\overline{H}_0q'$ and $\overline{H}_0p'+\overline{B}_0q'$; using the symbols v_1, v_2, v_3, v_4, as defined in § 281, we obtain the formulæ

$$\left.\begin{aligned}s_{12}(\overline{A}_0p'+\overline{H}_0q')&=s_{12}v_1-s_{23}v_3-s_{24}v_4\\ s_{12}(\overline{H}_0p'+\overline{B}_0q')&=s_{12}v_2-s_{31}v_3-s_{41}v_4\end{aligned}\right\}.$$

Again, we have

$$-\frac{\epsilon_n}{\gamma_\epsilon}=\sum\bar{\epsilon}_{11}p'^2,\quad -\frac{\omega_\nu}{\gamma_\omega}=\sum\bar{\omega}_{11}p'^2.$$

When we take quantities $\overline{E}_{11}$, $\overline{E}_{12}$, $\overline{E}_{22}$, bearing to the quantities $\bar{\epsilon}_{ij}$ the same relations as A_0, H_0, B_0, bear to the primary magnitudes of the domain, and

quantities $\bar{\Omega}_{11}$, $\bar{\Omega}_{12}$, $\bar{\Omega}_{22}$, bearing a like relation to the quantities ω_{ij}, and if, as in § 269, we write

$$\bar{\epsilon}_i = \tfrac{1}{2}\frac{\partial}{\partial x_i'}\left(\sum \bar{\epsilon}_{ij} x_i' x_j'\right) = \bar{\epsilon}_{i1} p' + \bar{\epsilon}_{i2} q' + \bar{\epsilon}_{i3} r' + \bar{\epsilon}_{i4} t',$$

$$\bar{\omega}_i = \tfrac{1}{2}\frac{\partial}{\partial x_i'}\left(\sum \bar{\omega}_{ij} x_i' x_j'\right) = \bar{\omega}_{i1} p' + \bar{\omega}_{i2} q' + \bar{\omega}_{i3} r' + \bar{\omega}_{i4} t',$$

we find

$$\left.\begin{aligned} s_{12}(\bar{E}_{11}p' + \bar{E}_{12}q') &= s_{12}\bar{\epsilon}_1 - s_{23}\bar{\epsilon}_3 - s_{24}\bar{\epsilon}_4 \\ s_{12}(\bar{E}_{12}p' + \bar{E}_{22}q') &= s_{12}\bar{\epsilon}_2 - s_{31}\bar{\epsilon}_3 - s_{41}\bar{\epsilon}_4 \end{aligned}\right\},$$

$$\left.\begin{aligned} s_{12}(\bar{\Omega}_{11}p' + \bar{\Omega}_{12}q') &= s_{12}\bar{\omega}_1 - s_{23}\bar{\omega}_3 - s_{24}\bar{\omega}_4 \\ s_{12}(\bar{\Omega}_{12}p' + \bar{\Omega}_{22}q') &= s_{12}\bar{\omega}_2 - s_{31}\bar{\omega}_3 - s_{41}\bar{\omega}_4 \end{aligned}\right\}.$$

Similarly, in connection with the equation of the domainal geodesic

$$\frac{Y}{\rho} = \sum \eta_{11} p'^2,$$

we take quantities $[\eta_{11}]$, $[\eta_{12}]$, $[\eta_{22}]$, bearing to the quantities η_{ij} the same relations as the magnitudes A_0, H_0, B_0, bear to the primary magnitudes of the domain; and the equation becomes

$$\frac{Y}{\rho} = [\eta_{11}]p'^2 + 2[\eta_{12}]p'q' + [\eta_{22}]q'^2,$$

giving the circular curvature and the direction of the prime normal of the domainal geodesic touched by the superficial geodesic.

The typical equation for the circular curvature of the superficial geodesic is

$$\frac{Y_0}{\rho_0} = \frac{Y}{\rho} + g_\epsilon \frac{dy}{dn} + g_\omega \frac{dy}{d\nu},$$

where

$$g_\epsilon + g_\omega \cos\iota = \frac{1}{\gamma_\epsilon}, \quad g_\omega + g_\epsilon \cos\iota = \frac{1}{\gamma_\omega}.$$

Thus

$$\begin{aligned} g_\epsilon \sin^2\iota &= \frac{1}{\gamma_\epsilon} - \frac{\cos\iota}{\gamma_\omega} \\ &= -\frac{1}{\epsilon_n}(\bar{E}_{11}p'^2 + 2\bar{E}_{12}p'q' + \bar{E}_{22}q'^2) + \frac{\cos\iota}{\omega_\nu}(\bar{\Omega}_{11}p'^2 + 2\bar{\Omega}_{12}p'q' + \bar{\Omega}_{22}q'^2), \\ g_\omega \sin^2\iota &= -\frac{1}{\omega_\nu}(\bar{\Omega}_{11}p'^2 + 2\bar{\Omega}_{12}p'q' + \bar{\Omega}_{22}q'^2) + \frac{\cos\iota}{\epsilon_n}(\bar{E}_{11}p'^2 + 2\bar{E}_{12}p'q' + \bar{E}_{22}q'^2); \end{aligned}$$

and therefore

$$g_\epsilon \frac{dy}{dn} + g_\omega \frac{dy}{d\nu} = \xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2,$$

where, for $\lambda\mu = 11,\ 12,\ 22$,

$$\xi_{\lambda\mu}\sin^2\iota = \left(\frac{\overline{\Omega}_{\lambda\mu}}{\omega_\nu}\cos\iota - \frac{\overline{E}_{\lambda\mu}}{\epsilon_n}\right)\frac{dy}{dn} + \left(\frac{\overline{E}_{\lambda\mu}}{\epsilon_n}\cos\iota - \frac{\overline{\Omega}_{\lambda\mu}}{\omega_\nu}\right)\frac{dy}{d\nu}.$$

Now the typical equation for the circular curvature of a geodesic on the surface, considered solely in relation to the surface, is

$$\frac{Y_0}{\rho_0} = \bar{\eta}_{11}p'^2 + 2\bar{\eta}_{12}p'q' + \bar{\eta}_{22}q'^2,$$

with the customary expressions (§ 93) for $\bar{\eta}_{11}$, $\bar{\eta}_{12}$, $\bar{\eta}_{22}$. Hence we have

$$\bar{\eta}_{11} = [\eta_{11}] + \xi_{11}, \quad \bar{\eta}_{12} = [\eta_{12}] + \xi_{12}, \quad \bar{\eta}_{22} = [\eta_{22}] + \xi_{22},$$

relations which connect the superficial magnitudes $\bar{\eta}_{ij}$ with the domainal magnitudes η_{ij} and the direction-variables of the domainal normals to the two regions intersecting in the surface.

The primary magnitudes of the surface are A_0, H_0, B_0; let L_0, M_0, N_0, denote its secondary magnitudes, so that (§ 104)

$$L_0 = \sum Y_0\bar{\eta}_{11}, \quad M_0 = \sum Y_0\bar{\eta}_{12}, \quad N_0 = \sum Y_0\bar{\eta}_{22}.$$

Accordingly,

$$\begin{aligned}\frac{L_0}{\rho_0} &= \sum\left(\frac{Y_0}{\rho_0}\bar{\eta}_{11}\right)\\ &= \sum\left\{\frac{Y}{\rho} + g_\epsilon\frac{dy}{dn} + g_\omega\frac{dy}{d\nu}\right\}\{[\eta_{11}] + \xi_{11}\}.\end{aligned}$$

As the quantities ξ_{ij} are linear in the direction-variables $\dfrac{dy}{dn}$, $\dfrac{dy}{d\nu}$, belonging to directions in the domain, while Y is the typical direction-cosine of the normal to the domain, we have

$$\sum Y\xi_{ij} = 0,$$

for all the combinations $ij = 11,\ 12,\ 22$. Also, as $\dfrac{dy}{dn}$ and $\dfrac{dy}{d\nu}$ are linear in the magnitudes y_1, y_2, y_3, y_4, while the quantities $[\eta_{ij}]$ are linear in the quantities η_{ij}, and because

$$\sum \eta_{ij}y_k = 0,$$

for all combinations of i, j, k, we have

$$\sum\left\{\frac{dy}{dn}[\eta_{ij}]\right\} = 0, \quad \sum\left\{\frac{dy}{d\nu}[\eta_{ij}]\right\} = 0,$$

for the combinations $ij=11, 12, 22$. Consequently

$$\frac{L_0}{\rho_0}=\frac{1}{\rho}\sum\{Y[\eta_{11}]\}+\sum\left\{\xi_{11}\left(g_\epsilon\frac{dy}{dn}+g_\omega\frac{dy}{dv}\right)\right\}.$$

As regards the right-hand side, we have

$$\begin{aligned}\sum\{Y[\eta_{11}]\}&=\sum\left[Y\left\{\eta_{11}-\frac{2}{s_{12}}(\eta_{13}s_{23}+\eta_{14}s_{24})+\frac{1}{s_{12}^2}(\eta_{33}s_{23}^2+2\eta_{34}s_{23}s_{24}+\eta_{44}s_{24}^2)\right\}\right]\\&=\bar{A}-\frac{2}{s_{12}}(\bar{G}s_{23}+\bar{L}s_{24})+\frac{1}{s_{12}^2}(\bar{C}s_{23}^2+2\bar{N}s_{23}s_{24}+\bar{D}s_{24}^2)\\&=\bar{A}_0\,;\end{aligned}$$

also

$$\sum\xi_{11}\frac{dy}{dn}=-\frac{1}{\epsilon_n}\bar{E}_{11},\quad \sum\xi_{11}\frac{dy}{dv}=-\frac{1}{\omega_\nu}\bar{\Omega}_{11}\,;$$

and therefore

$$\frac{L_0}{\rho_0}=\frac{\bar{A}_0}{\rho}-\frac{1}{\epsilon_n}g_\epsilon\bar{E}_{11}-\frac{1}{\omega_\nu}g_\nu\bar{\Omega}_{11}\,;$$

and similarly

$$\begin{aligned}\frac{M_0}{\rho_0}&=\frac{\bar{H}_0}{\rho}-\frac{1}{\epsilon_n}g_\epsilon\bar{E}_{12}-\frac{1}{\omega_\nu}g_\nu\bar{\Omega}_{12},\\ \frac{N_0}{\rho_0}&=\frac{\bar{B}_0}{\rho}-\frac{1}{\epsilon_n}g_\epsilon\bar{E}_{22}-\frac{1}{\omega_\nu}g_\nu\bar{\Omega}_{22}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{s_{12}}{\rho_0}(L_0p'+M_0q')&=\frac{1}{\rho}(s_{12}v_1-s_{23}v_3-s_{24}v_4)\\&\quad-\frac{g_\epsilon}{\epsilon_n}(s_{12}\bar{\epsilon}_1-s_{23}\bar{\epsilon}_3-s_{24}\bar{\epsilon}_4)-\frac{g_\nu}{\omega_\nu}(s_{12}\bar{\omega}_1-s_{23}\bar{\omega}_2-s_{24}\bar{\omega}_4),\\ \frac{s_{12}}{\rho_0}(M_0p'+N_0q')&=\frac{1}{\rho}(s_{12}v_2-s_{31}v_3-s_{41}v_4)\\&\quad-\frac{g_\epsilon}{\epsilon_n}(s_{12}\bar{\epsilon}_2-s_{31}\bar{\epsilon}_3-s_{41}\bar{\epsilon}_4)-\frac{g_\nu}{\omega_\nu}(s_{12}\bar{\omega}_2-s_{31}\bar{\omega}_3-s_{41}\bar{\omega}_4).\end{aligned}$$

Further,

$$V_0^2=A_0B_0-H_0^2=\frac{1}{s_{12}^2}\{\sum(AB-H^2)s_{12}^2\}=\frac{1}{s_{12}^2},$$

with the definitions of the variables s_{ij} in § 270.

The (spatial) torsion of a geodesic on a surface is given (§ 106) by

$$\frac{V_0}{\sigma_0}=\begin{vmatrix}L_0p'+M_0q', & M_0p'+N_0q'\\ A_0p'+H_0q', & H_0p'+B_0q'\end{vmatrix},$$

and therefore

$$\frac{V_0}{\rho_0\sigma_0}=\frac{1}{\rho}R-\frac{g_\epsilon}{\epsilon_n}R_\epsilon-\frac{g_\nu}{\omega_\nu}R_\omega,$$

where

$$R=\begin{vmatrix} \bar{A}_0p'+\bar{H}_0q', & \bar{H}_0p'+\bar{B}_0q' \\ A_0p'+H_0q', & H_0p'+B_0q' \end{vmatrix},$$

$$R_\epsilon=\begin{vmatrix} \bar{E}_{11}p'+\bar{E}_{12}q', & \bar{E}_{12}p'+\bar{E}_{22}q' \\ A_0p'+H_0q', & H_0p'+B_0q' \end{vmatrix},$$

$$R_\omega=\begin{vmatrix} \bar{\Omega}_{11}p'+\bar{\Omega}_{12}q', & \bar{\Omega}_{12}p'+\bar{\Omega}_{22}q' \\ A_0p'+H_0q', & H_0p'+B_0q' \end{vmatrix}.$$

Now

$$R=\begin{vmatrix} v_1-\dfrac{s_{23}}{s_{12}}v_3-\dfrac{s_{24}}{s_{12}}v_4, & v_2-\dfrac{s_{31}}{s_{12}}v_3-\dfrac{s_{41}}{s_{12}}v_4 \\ u_1-\dfrac{s_{23}}{s_{12}}u_3-\dfrac{s_{24}}{s_{12}}u_4, & u_2-\dfrac{s_{31}}{s_{12}}u_3-\dfrac{s_{41}}{s_{12}}u_4 \end{vmatrix}$$

$$=\frac{1}{s_{12}}\left\{\begin{vmatrix} v_1, & v_2 \\ u_1, & u_2 \end{vmatrix}s_{12}+\dots+\begin{vmatrix} v_3, & v_4 \\ u_3, & u_4 \end{vmatrix}s_{34}\right\}$$

$$=\frac{1}{\begin{vmatrix} \epsilon_3, & \epsilon_4 \\ \omega_3, & \omega_4 \end{vmatrix}}\begin{vmatrix} v_1, & v_2, & v_3, & v_4 \\ u_1, & u_2, & u_3, & u_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix}=\frac{1}{s_{12}\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota}\Delta_v,$$

when the orientation-variables of p. 257 are used, and Δ_v denotes the determinant $|\,v_1\,u_2\,\epsilon_3\,\omega_4\,|$. Similarly

$$R_\epsilon=\frac{1}{s_{12}\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota}\Delta_{\bar{\epsilon}},$$

$$R_\omega=\frac{1}{s_{12}\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota}\Delta_{\bar{\omega}},$$

where $\Delta_{\bar{\epsilon}}$ denotes a determinant $|\,\bar{\epsilon}_1\,u_2\,\epsilon_3\,\omega_4\,|$ and $\Delta_{\bar{\omega}}$ denotes a determinant $|\,\bar{\omega}_1\,u_2\,\epsilon_3\,\omega_4\,|$. Also $V_0=s_{12}$; and therefore

$$\frac{\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota}{\rho_0\sigma_0}=\frac{1}{\rho}\Delta_v-\frac{g_\epsilon}{\epsilon_n}\Delta_{\bar{\epsilon}}-\frac{g_\nu}{\omega_\nu}\Delta_{\bar{\omega}}.$$

Domainal curvatures of a geodesic on a surface.

346. As for a surface in a region, and for a region in a domain, we take account of the domainal curvatures of a superficial geodesic, as well as of its spatial curvatures such as $1/\rho_0$ and $1/\sigma_0$. The two directions, with the typical direction-cosines

$\frac{dy}{dn}$ and $\frac{dy}{d\nu}$, are at right angles to the tangent plane of the surface; and the typical direction-cosine of the radius of domainal flexure is the magnitude denoted by l, where

$$\frac{l}{\gamma}=g_\epsilon\frac{dy}{dn}+g_\omega\frac{dy}{d\nu}.$$

Let λ_3 denote the typical direction-cosine of the binormal of the superficial geodesic, so that

$$y',\quad \lambda_3,\quad \frac{dy}{dn},\quad \frac{dy}{d\nu},$$

are the typical direction-cosines of four lines in the tangent block of the domain; thus λ_3, $\frac{dy}{dn}$, $\frac{dy}{d\nu}$, are three non-complanar lines in the domainal flat to which the tangent common to the geodesics is orthogonal. When differentiation along the superficial geodesic arc is denoted by ds', the initial equations for the domainal curvatures are

$$\frac{dy'}{ds'}=\frac{l}{\gamma},$$
$$\frac{dl}{ds'}=\frac{\lambda_3}{\sigma_D}-\frac{y'}{\gamma},$$

where $1/\sigma_D$ denotes the domainal torsion.

As the foregoing equation for l implies a combination of the domainal flexures of the ϵ-regional geodesic and the ω-regional geodesic, the combination being similar to that of the combination of vectorial magnitudes along the domainal normals to the regions, the directional relations of the superficial geodesic to the ϵ-regional geodesic and the ω-regional geodesic have to be considered.

In connection with the geodesic of the ϵ-region, we now use ρ_ϵ, γ_ϵ, σ_ϵ, $\bar{\sigma}_\epsilon$, to denote its radius of circular curvature, its radius of domainal flexure, its radius of spatial torsion, and its radius of domainal torsion respectively; and we use Y_ϵ, $\bar{\lambda}_\epsilon$, $\bar{\mu}_\epsilon$, to denote the typical direction-cosines of its prime normal, its binormal, and its trinormal respectively. Thus, after §§ 172, 334, we have relations

$$\frac{\bar{\lambda}_\epsilon}{\sigma_\epsilon}-\frac{y'}{\rho_\epsilon}=\frac{dY_\epsilon}{ds_\epsilon},\quad \frac{\bar{\lambda}_\epsilon}{\bar{\sigma}_\epsilon}-\frac{y'}{\gamma_\epsilon}=\frac{d}{ds_\epsilon}\left(\frac{dy}{dn}\right),$$

together with

$$\frac{Y_\epsilon}{\rho_\epsilon}=\frac{Y}{\rho}+\frac{1}{\gamma_\epsilon}\frac{dy}{dn},\quad \frac{l_3}{\sigma}-\frac{y'}{\rho}=\frac{dY}{ds},$$

the last equation of which has relation to the domainal geodesic. Similarly we

use symbols ρ_ω, γ_ω, σ_ω, $\bar{\sigma}_\omega$, Y_ω, $\bar{\lambda}_\omega$, $\bar{\mu}_\omega$, to denote the corresponding magnitudes connected with the geodesic of the ω-region, in the equations

$$\frac{\bar{\lambda}_\omega}{\sigma_\omega}-\frac{y'}{\rho_\omega}=\frac{dY_\omega}{ds_\omega},\quad \frac{\bar{\lambda}_\omega}{\bar{\sigma}_\omega}-\frac{y'}{\gamma_\omega}=\frac{d}{ds_\omega}\left(\frac{dy}{d\nu}\right),$$

together with

$$\frac{Y_\omega}{\rho_\omega}=\frac{Y}{\rho}+\frac{1}{\gamma_\omega}\frac{dy}{d\nu},$$

the fourth equation being the same as before.

The quantities ρ_ϵ and ρ_ω are given by the equations

$$\frac{1}{\rho_\epsilon^2}=\frac{1}{\rho^2}+\frac{1}{\gamma_\epsilon^2},\quad \frac{1}{\rho_\omega^2}=\frac{1}{\rho^2}+\frac{1}{\gamma_\omega^2}.$$

The magnitudes σ_ϵ and σ_ω are given by the equations

$$\frac{\Omega\epsilon_n^2}{\rho_\epsilon^2}\left(\frac{1}{\sigma_\epsilon^2}+\frac{1}{\rho_\epsilon^2}\right)=\frac{1}{\rho^2}\Box_\epsilon(v)-\frac{2}{\rho\gamma_\epsilon\epsilon_n}\Box_\epsilon(v,\bar{\epsilon})+\frac{1}{\gamma_\epsilon^2\epsilon_n^2}\Box_\epsilon(\bar{\epsilon}),$$

$$\frac{\Omega\omega_\nu^2}{\rho_\omega^2}\left(\frac{1}{\sigma_\omega^2}+\frac{1}{\rho_\omega^2}\right)=\frac{1}{\rho^2}\Box_\omega(v)-\frac{2}{\rho\gamma_\omega\omega_\nu}\Box_\omega(v,\bar{\omega})+\frac{1}{\gamma_\omega^2\omega_\nu^2}\Box_\omega(\bar{\omega}),$$

where the quantities $\Box$ have the significance defined (§ 335) by

$$\Box_\theta(v,\bar{\theta})=\begin{vmatrix} A, & H, & G, & L, & v_1, & \theta_1 \\ H, & B, & F, & M, & v_2, & \theta_2 \\ G, & F, & C, & N, & v_3, & \theta_3 \\ L, & M, & N, & D, & v_4, & \theta_4 \\ \bar{\theta}_1, & \bar{\theta}_2, & \bar{\theta}_3, & \bar{\theta}_4, & 0, & 0 \\ \theta_1, & \theta_2, & \theta_3, & \theta_4, & 0, & 0 \end{vmatrix}.$$

for $\theta=\epsilon$ and $\theta=\omega$, while $\Box_\theta(v)$ denotes the same determinant $\Box_\theta(v,\bar{\theta})$ when the constituents v_1, v_2, v_3, v_4, are substituted for $\bar{\theta}_1$, $\bar{\theta}_2$, $\bar{\theta}_3$, $\bar{\theta}_4$, and $\Box_\theta(\bar{\theta})$ denotes that determinant when the constituents $\bar{\theta}_1$, $\bar{\theta}_2$, $\bar{\theta}_3$, $\bar{\theta}_4$, are substituted for v_1, v_2, v_3, v_4.

Similarly, the quantities $\bar{\sigma}_\epsilon$ and $\bar{\sigma}_\omega$ are given by equations

$$\frac{1}{\bar{\sigma}_\epsilon^2}+\frac{1}{\gamma_\epsilon^2}=\sum\left\{c_{11}\left(\frac{dp}{dn}\right)^2\right\}+\frac{1}{\Omega\epsilon_n^4}\Box_\epsilon(\bar{\epsilon}),$$

$$\frac{1}{\bar{\sigma}_\omega^2}+\frac{1}{\gamma_\omega^2}=\left\{\sum c_{11}\left(\frac{dp}{d\nu}\right)^2\right\}+\frac{1}{\Omega\omega_\nu^4}\Box_\omega(\bar{\omega}),$$

and so for other magnitudes belonging to the ϵ-region alone, and to the ω-region alone.

Again, let β denote the inclination of the respective binormals of the ϵ-regional geodesic and the ω-regional geodesic, so that

$$\cos\beta = \sum \bar{\lambda}_\epsilon \bar{\lambda}_\omega.$$

Now

$$\frac{\bar{\lambda}_\epsilon}{\sigma_\epsilon} - \frac{y'}{\rho_\epsilon} = \frac{dY_\epsilon}{ds_\epsilon} = -\frac{1}{\Omega\epsilon_n^2}\square_\epsilon(\bar{\epsilon}, y),$$

$$\frac{\bar{\lambda}_\omega}{\sigma_\omega} - \frac{y'}{\rho_\omega} = \frac{dY_\omega}{ds_\omega} = -\frac{1}{\Omega\omega_\nu^2}\square_\omega(\bar{\omega}, y),$$

where, for $\theta=\epsilon$ and $\theta=\omega$, the symbols $\square$ now are defined by the relation

$$\square_\theta(\bar{\theta}, y) = \begin{vmatrix} 0, & [\theta]_1, & [\theta]_2, & [\theta]_3, & [\theta]_4, & 0 \\ y_1, & A, & H, & G, & L, & \theta_1 \\ y_2, & H, & B, & F, & M, & \theta_2 \\ y_3, & G, & F, & C, & N, & \theta_3 \\ y_4, & L, & M, & N, & D, & \theta_4 \\ 0, & \theta_1, & \theta_2, & \theta_3, & \theta_4, & 0 \end{vmatrix},$$

while, for $i=1, 2, 3, 4$,

$$[\theta]_i = \frac{\rho_\theta}{\rho} v_i - \frac{\rho_\theta}{\gamma_\theta \theta_N}\bar{\theta}_i,$$

and θ_N denotes ϵ_n when $\theta=\epsilon$, and denotes ω_ν when $\theta=\omega$. We thus have

$$\frac{\cos\beta}{\sigma_\epsilon\sigma_\omega} + \frac{1}{\rho_\epsilon\rho_\omega} = -\frac{1}{\Omega\omega_\nu^2}\sum \frac{dY_\epsilon}{ds_\epsilon}\square_\epsilon(\bar{\epsilon}, y)$$

$$= -\frac{1}{\Omega\epsilon_n^2}\sum \frac{dY_\omega}{ds_\omega}\square_\omega(\bar{\omega}, y).$$

Expressions can be obtained for the sums on the right-hand sides by using the relations, similar to those on p. 448, of the form

$$\sum y_m \frac{dY_\epsilon}{ds_\epsilon} = -\left(\frac{\rho_\epsilon}{\rho} v_m - \frac{\rho_\epsilon}{\gamma_\epsilon \epsilon_n}\bar{\epsilon}_m\right) + \epsilon_m \frac{d}{ds_\epsilon}\left(\frac{\rho_\epsilon}{\gamma_\epsilon \epsilon_n}\right)$$

$$= -[\epsilon]_m + \epsilon_m \frac{d}{ds_\epsilon}\left(\frac{\rho_\epsilon}{\gamma_\epsilon \epsilon_n}\right),$$

$$\sum y_m \frac{dY_\omega}{ds_\omega} = -[\omega]_m + \omega_m \frac{d}{ds_\omega}\left(\frac{\rho_\omega}{\gamma_\omega \omega_\nu}\right).$$

Other forms can also be obtained; for, if we write

$$T_\epsilon = \sum a\epsilon_1^2, \quad T_\omega = \sum a\omega_1^2,$$

regarding these as concomitants of the domainal system (their values being $\Omega\epsilon_n{}^2$, $\Omega\omega_\nu{}^2$), we can write

$$-\Omega\epsilon_n{}^2\frac{dY_\epsilon}{ds_\epsilon}=\frac{\rho_\epsilon}{\rho}\left(\sum_i\sum_j y_i v_j\frac{\partial T_\epsilon}{\partial A_{ij}}\right)-\frac{\rho_\epsilon}{\gamma_\epsilon\epsilon_n}\left(\sum_i\sum_j y_i\bar{\epsilon}_j\frac{\partial T_\epsilon}{\partial A_{ij}}\right),$$

$$-\Omega\omega_\nu{}^2\frac{dY_\omega}{ds_\omega}=\frac{\rho_\omega}{\rho}\left(\sum_i\sum_j y_i v_j\frac{\partial T_\omega}{\partial A_{ij}}\right)-\frac{\rho_\omega}{\gamma_\omega\omega_\nu}\left(\sum_i\sum_j y_i\bar{\omega}_j\frac{\partial T_\omega}{\partial A_{ij}}\right),$$

and we have

$$\frac{\cos\beta}{\sigma_\epsilon\sigma_\omega}+\frac{1}{\rho_\epsilon\rho_\omega}=\sum\left(\frac{dY_\epsilon}{ds_\epsilon}\frac{dY_\omega}{ds_\omega}\right),$$

the right-hand summation extending over the range of the plenary homaloidal space.

We also require the inclination of the binormal of a geodesic of either region to the domainal normal of the other region. Let the inclination of the ϵ-regional binormal to the domainal normal of the ω-region be denoted by β_ν, so that

$$\cos\beta_\nu=\sum\bar{\lambda}_\epsilon\frac{dy}{d\nu};$$

then, as $\sum y'\frac{dy}{d\nu}=0$, we have

$$\begin{aligned}\frac{1}{\sigma_\epsilon}\cos\beta_\nu&=\sum\left(\frac{\bar{\lambda}_\epsilon}{\sigma_\epsilon}-\frac{y'}{\rho_\epsilon}\right)\frac{dy}{d\nu}\\&=-\frac{1}{\Omega\epsilon_n{}^2}\sum\frac{dy}{d\nu}\square_\epsilon(\bar{\epsilon},y)\end{aligned}$$

$$=-\frac{1}{\Omega\epsilon_n{}^2\omega_\nu}\begin{vmatrix}A, & H, & G, & L, & \epsilon_1, & \omega_1\\ H, & B, & F, & M, & \epsilon_2, & \omega_2\\ G, & F, & C, & N, & \epsilon_3, & \omega_3\\ L, & M, & N, & D, & \epsilon_4, & \omega_4\\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0, & 0\\ [\epsilon]_1, & [\epsilon]_2, & [\epsilon]_3, & [\epsilon]_4, & 0, & 0\end{vmatrix}=-\frac{1}{\Omega\epsilon_n{}^2\omega_\nu}\square_\epsilon(\bar{\epsilon},\omega).$$

Similarly, if the inclination of the ω-regional geodesic to the domainal normal of the ϵ-region be denoted by β_n, we find

$$\frac{1}{\sigma_\omega}\cos\beta_n=-\frac{1}{\Omega\omega_\nu{}^2\epsilon_n}\begin{vmatrix}A, & H, & G, & L, & \omega_1, & \epsilon_1\\ H, & B, & F, & M, & \omega_2, & \epsilon_2\\ G, & F, & C, & N, & \omega_3, & \epsilon_3\\ L, & M, & N, & D, & \omega_4, & \epsilon_4\\ \omega_1, & \omega_2, & \omega_3, & \omega_4, & 0, & 0\\ [\omega]_1, & [\omega]_2, & [\omega]_3, & [\omega]_4, & 0, & 0\end{vmatrix}=-\frac{1}{\Omega\omega_\nu{}^2\epsilon_n}\square_\omega(\bar{\omega},\epsilon).$$

Binormal of a superficial geodesic.

347. The typical direction-cosine λ_3 of the binormal of the superficial geodesic, lying in the tangent plane of the surface at right angles to the tangent, is given (§ 106) by

$$V_0\lambda_3 = \bar{u}_1\bar{y}_2 - \bar{u}_2\bar{y}_1.$$

Now

$$\left.\begin{aligned} s_{12}\bar{y}_1 &= s_{12}y_1 - s_{23}y_3 - s_{24}y_4 \\ s_{12}\bar{y}_2 &= s_{12}y_2 - s_{31}y_3 - s_{41}y_4 \end{aligned}\right\}, \qquad \left.\begin{aligned} s_{12}\bar{u}_1 &= s_{12}u_1 - s_{23}u_3 - s_{24}u_4 \\ s_{12}\bar{u}_2 &= s_{12}u_2 - s_{31}u_3 - s_{41}u_4 \end{aligned}\right\};$$

and therefore, using the relations (§ 270) among the orientation-variables s_{ij} as connected with the derivatives of ϵ and ω, we have

$$V_0\lambda_3 = \frac{1}{\epsilon_3\omega_4 - \epsilon_4\omega_3} \begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ y_1, & y_2, & y_3, & y_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix}.$$

But also

$$V_0(\epsilon_3\omega_4 - \epsilon_4\omega_3) = \Omega^{\frac{1}{2}}\epsilon_n\omega_\nu \sin\iota;$$

consequently

$$\lambda_3\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu \sin\iota = \begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ y_1, & y_2, & y_3, & y_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix}.$$

Let ϕ_3 denote the inclination of this superficial geodesic binormal to the binormal of the domainal geodesic, which has a typical direction-cosine l_3 given by the adapted Frenet equation (§ 8)

$$\frac{l_3}{\sigma} - \frac{y'}{\rho} = Y'.$$

Then, as

$$\sum \lambda_3 l_3 = \cos\phi_3, \quad \sum \lambda_3 y' = 0, \quad \sum y_\mu Y' = -v_\mu,$$

the last holding for all values $\mu = 1, 2, 3, 4$, we have

$$\frac{\Omega^{\frac{1}{2}}}{\sigma}\epsilon_n\omega_\nu \sin\iota\cos\phi_3 = -\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix} = \Delta_v,$$

with the notation of p. 485.

Next, let ϕ_4 denote the inclination of the superficial geodesic binormal to the

trinormal of the domainal geodesic, which has its typical direction-cosine l_4 given (§ 8) by

$$\frac{l_4}{\sigma\tau}+l_3\frac{d}{ds}\left(\frac{1}{\sigma}\right)-Y\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)-y'\frac{d}{ds}\left(\frac{1}{\rho}\right)=Y''.$$

Now

$$\sum l_4\lambda_3=\cos\phi_4,\quad \sum l_3\lambda_3=\cos\phi_3,\quad \sum Y\lambda_3=0,\quad \sum \lambda_3 y'=0\,;$$

and therefore, multiplying this equation throughout by λ_3, and adding for the range of the plenary space,

$$\begin{aligned}\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota\left\{\frac{1}{\sigma\tau}\cos\phi_4+\frac{d}{ds}\left(\frac{1}{\sigma}\right)\cos\phi_3\right\}&=\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota\sum\lambda_3 Y''\\
&=\begin{vmatrix} u_1, & \sum y_1Y'', & \epsilon_1, & \omega_1\\ u_2, & \sum y_2Y'', & \epsilon_2, & \omega_2\\ u_3, & \sum y_3Y'', & \epsilon_3, & \omega_3\\ u_4, & \sum y_4Y'', & \epsilon_4, & \omega_4\end{vmatrix}\\
&=\begin{vmatrix} w_1, & w_2, & w_3, & w_4\\ u_1, & u_2, & u_3, & u_4\\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4\\ \omega_1, & \omega_2, & \omega_3, & \omega_4\end{vmatrix}\\
&=\Delta_w,\end{aligned}$$

on using the domainal relations

$$\sum y_\mu Y''=-w_\mu,\quad (\mu=1,2,3,4),$$

established in § 285, *Ex.* 1.

Domainal torsion, regional torsions, of a superficial geodesic.

348. Again, as

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+g_\epsilon\frac{dy}{dn}+g_\omega\frac{dy}{d\nu},$$

we have

$$\frac{1}{\rho_0}\sum Y_0y_{\lambda\mu}=\frac{1}{\rho}\sum Yy_{\lambda\mu}+g_\epsilon\sum y_{\lambda\mu}\frac{dy}{dn}+g_\omega\sum y_{\lambda\mu}\frac{dy}{d\nu}.$$

But

$$\begin{aligned}\sum y_{\lambda\mu}\frac{dy}{dn}&=\sum y_{\lambda\mu}\left(y_1\frac{dp}{dn}+y_2\frac{dq}{dn}+y_3\frac{dr}{dn}+y_4\frac{dt}{dn}\right)\\
&=\sum\left\{\frac{dp}{dn}\sum A\Gamma_{\lambda\mu}+\frac{dq}{dn}\sum H\Gamma_{\lambda\mu}+\frac{dr}{dn}\sum G\Gamma_{\lambda\mu}+\frac{dt}{dn}\sum L\Gamma_{\mu\lambda}\right\}\\
&=\frac{1}{\epsilon_n}(\Gamma_{\lambda\mu}\epsilon_1+\Delta_{\lambda\mu}\epsilon_2+\Theta_{\lambda\mu}\epsilon_3+\Phi_{\lambda\mu}\epsilon_4),\end{aligned}$$

and

$$\sum y_{\lambda\mu}\frac{dy}{d\nu}=\frac{1}{\omega_\nu}(\Gamma_{\lambda\mu}\omega_1+\varDelta_{\lambda\mu}\epsilon_2+\Theta_{\lambda\mu}\epsilon_3+\Phi_{\lambda\mu}\epsilon_4),$$

while $\sum Yy_{\lambda\mu}=\bar{A}_{\lambda\mu}$; consequently

$$\frac{1}{\rho_0}\sum Y_0 y_{\lambda\mu}=\frac{1}{\rho}\bar{A}_{\lambda\mu}+\frac{g_\epsilon}{\epsilon_n}\sum \epsilon_1\Gamma_{\lambda\mu}+\frac{g_\omega}{\omega_\nu}\sum\omega_1\Gamma_{\lambda\mu},$$

and therefore

$$\frac{1}{\rho_0}\sum Y_0 y_\lambda'=\frac{1}{\rho}v_\lambda+\frac{g_\epsilon}{\epsilon_n}\left[\sum_\alpha\sum_\beta\{\lambda\alpha,\beta\}\,x_\alpha'\epsilon_\beta\right]+\frac{g_\omega}{\omega_\nu}\left[\sum_\alpha\sum_\beta\{\lambda\alpha,\beta\}\,x_\alpha'\omega_\beta\right].$$

Further, from the same initial relation, it follows that

$$\sum y_\lambda Y_0=\frac{\rho_0 g_\epsilon}{\epsilon_n}\epsilon_\lambda+\frac{\rho_0 g_\omega}{\omega_\nu}\omega_\lambda;$$

hence

$$\begin{aligned}\sum y_\lambda Y_0'&=-\sum Y_0 y_\lambda'+\frac{\rho_0 g_\epsilon}{\epsilon_n}\frac{d\epsilon_\lambda}{ds}+\frac{\rho_0 g_\omega}{\omega_\nu}\frac{d\omega_\lambda}{ds}+\epsilon_\lambda\frac{d}{ds}\left(\frac{\rho_0 g_\epsilon}{\epsilon_n}\right)+\omega_\lambda\frac{d}{ds}\left(\frac{\rho_0 g_\omega}{\omega_\nu}\right)\\&=-\frac{\rho_0}{\rho}v_\lambda+\frac{\rho_0 g_\epsilon}{\epsilon_n}\bar{\epsilon}_\lambda+\frac{\rho_0 g_\omega}{\omega_\nu}\bar{\omega}_\lambda+\epsilon_\lambda\frac{d}{ds}\left(\frac{\rho_0 g_\epsilon}{\epsilon_n}\right)+\omega_\lambda\frac{d}{ds}\left(\frac{\rho_0 g_\omega}{\omega_\nu}\right).\end{aligned}$$

The spatial torsion of the superficial geodesic is given by the Frenet equation

$$\frac{\lambda_3}{\sigma_0}-\frac{y'}{\rho_0}=Y_0',$$

so that

$$\frac{1}{\sigma_0}=\sum\lambda_3 Y_0'.$$

When the foregoing determinantal form of λ_3 is inserted, we find

$$\frac{\Omega^{\frac{1}{2}}}{\sigma_0}\epsilon_n\omega_\nu\sin\iota=\begin{vmatrix}u_1, & \sum y_1 Y_0', & \epsilon_1, & \omega_1\\ u_2, & \sum y_2 Y_0', & \epsilon_2, & \omega_2\\ u_3, & \sum y_3 Y_0', & \epsilon_3, & \omega_3\\ u_4, & \sum y_4 Y_0', & \epsilon_4, & \omega_4\end{vmatrix}.$$

When the values of the quantities $\sum y_\lambda Y_0'$ are inserted in this determinant, their terms involving ϵ_λ disappear because of the third column, and their terms involving ω_λ disappear because of the fourth column; consequently

$$\frac{\Omega^{\frac{1}{2}}}{\rho_0\sigma_0}\epsilon_n\omega_\nu\cos\iota=\begin{vmatrix} u_1, & -\frac{v_1}{\rho}+\frac{g_\epsilon}{\epsilon_n}\bar{\epsilon}_1+\frac{g_\omega}{\omega_\nu}\bar{\omega}_1, & \epsilon_1, & \omega_1 \\ u_2, & -\frac{v_2}{\rho}+\frac{g_\epsilon}{\epsilon_n}\bar{\epsilon}_2+\frac{g_\omega}{\omega_\nu}\bar{\omega}_2, & \epsilon_2, & \omega_2 \\ u_3, & -\frac{v_3}{\rho}+\frac{g_\epsilon}{\epsilon_n}\bar{\epsilon}_3+\frac{g_\omega}{\omega_\nu}\bar{\omega}_3, & \epsilon_3, & \omega_3 \\ u_4, & -\frac{v_4}{\rho}+\frac{g_\epsilon}{\epsilon_n}\bar{\epsilon}_4+\frac{g_\omega}{\omega_\nu}\bar{\omega}_4, & \epsilon_4, & \omega_4 \end{vmatrix}$$

$$=\frac{1}{\rho}\Delta_v-\left(\frac{g_\epsilon}{\epsilon_n}\Delta_{\bar{\epsilon}}+\frac{g_\omega}{\omega_\nu}\Delta_{\bar{\omega}}\right),$$

in accordance with the result and the notation of p. 485, already used for the spatial torsion of the superficial geodesic.

The characteristic Frenet equation for the domainal torsion $1/\sigma_D$ of the superficial geodesic is

$$\frac{dl}{ds'}=\frac{\lambda_3}{\sigma_D}-\frac{y'}{\gamma},$$

where l, the typical direction-cosine of the radius of domainal flexure of that geodesic, is given by

$$\frac{l}{\gamma}=g_\epsilon\frac{dy}{dn}+g_\omega\frac{dy}{d\nu}.$$

From the last relation, we have

$$\frac{dl}{ds'}\frac{1}{\gamma}+l\frac{d}{ds'}\left(\frac{1}{\gamma}\right)=g_\epsilon\frac{d}{ds'}\left(\frac{dy}{dn}\right)+g_\omega\frac{d}{ds'}\left(\frac{dy}{d\nu}\right)+\frac{dy}{dn}\frac{dg_\epsilon}{ds'}+\frac{dy}{d\nu}\frac{dg_\omega}{ds'}.$$

Now

$$\frac{1}{\sigma_D}=\sum\lambda_3\frac{dl}{ds'};$$

and

$$\sum\lambda_3 l=0,\quad \sum\lambda_3\frac{dy}{dn}=0,\quad \sum\lambda_3\frac{dy}{d\nu}=0;$$

consequently

$$\frac{1}{\gamma\sigma_D}=g_\epsilon\left\{\sum\lambda_3\frac{d}{ds'}\left(\frac{dy}{dn}\right)\right\}+g_\omega\left\{\sum\lambda_3\frac{d}{ds'}\left(\frac{dy}{d\nu}\right)\right\}.$$

As the quantities

$$\frac{d}{ds'}\left(\frac{dy}{dn}\right),\quad \frac{d}{ds'}\left(\frac{dy}{d\nu}\right),$$

are the same for the surface and for each of the two regions as for the domain, because the direction-variables $\frac{dy}{dn}$, $\frac{dy}{d\nu}$, do not involve the direction-variables of

the geodesic, we use the determinantal value of λ_3 of § 347, which is

$$\lambda_3 \Omega^{\frac{1}{2}} \epsilon_n \omega_\nu \sin \iota = \begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ y_1, & y_2, & y_3, & y_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix};$$

and we take the foregoing sums in connection with the row of constituents y_1, y_2, y_3, y_4. By the result of § 324, we have

$$\sum y_\mu \frac{d}{ds}\left(\frac{dy}{dn}\right) = \epsilon_\mu \frac{d}{ds}\left(\frac{1}{\epsilon_n}\right) + \frac{\bar{\epsilon}_\mu}{\epsilon_n},$$

for $\mu = 1, 2, 3, 4$; and similarly

$$\sum y_\mu \frac{d}{ds}\left(\frac{dy}{d\nu}\right) = \omega_\mu \frac{d}{ds}\left(\frac{1}{\omega_\nu}\right) + \frac{\bar{\omega}_\mu}{\omega_\nu}.$$

In the magnitude $\sum \lambda_3 \frac{d}{ds}\left(\frac{dy}{dn}\right)$, when substitution is made for $\sum y_\mu \frac{d}{ds}\left(\frac{dy}{dn}\right)$, the terms involving ϵ_μ disappear because of the third row of constituents in λ_3; and similarly in the magnitude $\sum \lambda_3 \frac{d}{ds}\left(\frac{dy}{d\nu}\right)$, when substitution is made for $\sum y_\mu \frac{d}{ds}\left(\frac{dy}{d\nu}\right)$, the terms involving ω_μ disappear because of the fourth row of constituents in λ_3. Hence

$$\frac{1}{\gamma\sigma_D} \Omega^{\frac{1}{2}} \epsilon_n \omega_\nu \sin \iota = \frac{g_\epsilon}{\epsilon_n} \begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix} + \frac{g_\omega}{\omega_\nu} \begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ \bar{\omega}_1, & \bar{\omega}_2, & \bar{\omega}_3, & \bar{\omega}_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix}$$

$$= -\left(\frac{g_\epsilon}{\epsilon_n} \Delta_\epsilon + \frac{g_\omega}{\omega_\nu} \Delta_\omega\right),$$

thus giving the domainal torsion of the superficial geodesic. In passing, we note the relation

$$\left(\frac{1}{\rho_0\sigma_0} - \frac{1}{\gamma\sigma_D}\right) \Omega^{\frac{1}{2}} \epsilon_n \omega_\nu \sin \iota = \frac{1}{\rho} \Delta_v;$$

and therefore

$$\frac{1}{\rho_0\sigma_0} - \frac{1}{\gamma\sigma_D} = \frac{1}{\rho\sigma} \cos \phi_3.$$

Let θ_ϵ and θ_ω denote the respective angles made by the binormal of the superficial geodesic with the binormals of the geodesics in the ϵ-region and the ω-region, so that

$$\cos \theta_\epsilon = \sum \lambda_3 \bar{\lambda}_\epsilon, \quad \cos \theta_\omega = \sum \lambda_3 \bar{\lambda}_\omega.$$

Now

$$\frac{\bar{\lambda}_\epsilon}{\sigma_\epsilon} - \frac{y'}{\rho_\epsilon} = \frac{dY_\epsilon}{ds_\epsilon},$$

and $\sum \lambda_3 y' = 0$; hence

$$\frac{\cos\theta_\epsilon}{\sigma_\epsilon} = \sum \lambda_3 \frac{dY_\epsilon}{ds_\epsilon}.$$

By the result on p. 448, we have

$$\sum y_\mu \frac{dY_\epsilon}{ds_\epsilon} = -\frac{\rho_\epsilon}{\rho} v_\mu + \frac{\rho_\epsilon}{\gamma_\epsilon \epsilon_n} \bar{\epsilon}_\mu + \epsilon_\mu \frac{d}{ds_\epsilon}\left(\frac{\rho_\epsilon}{\gamma_\epsilon \epsilon_n}\right);$$

and therefore, when the determinantal form of λ_3 is used, so as to combine the second row of constituents with the arc-derivative of Y_ϵ, and when this result is used, we find

$$\frac{1}{\rho_\epsilon \sigma_\epsilon} \Omega^{\frac{1}{2}} \epsilon_n \omega_\nu \sin\iota \cos\theta_\epsilon = \frac{1}{\rho} \Delta_v - \frac{1}{\gamma_\epsilon \epsilon_n} \Delta_{\bar{\epsilon}}.$$

Similarly we find

$$\frac{1}{\rho_\omega \sigma_\omega} \Omega^{\frac{1}{2}} \epsilon_n \omega_\nu \sin\iota \cos\theta_\omega = \frac{1}{\rho} \Delta_v - \frac{1}{\gamma_\omega \omega_\nu} \Delta_{\bar{\omega}}.$$

Also we have

$$\frac{\bar{\lambda}_\epsilon}{\bar{\sigma}_\epsilon} - \frac{y'}{\gamma_\epsilon} = \frac{d}{ds_\epsilon}\left(\frac{dy}{dn}\right);$$

and therefore, in the same way,

$$\frac{\Omega^{\frac{1}{2}}}{\bar{\sigma}_\epsilon} \epsilon_n \omega_\nu \sin\iota \cos\theta_\epsilon = \left[\sum \lambda_3 \frac{d}{ds_\epsilon}\left(\frac{dy}{dn}\right)\right] \Omega^{\frac{1}{2}} \epsilon_n \omega_\nu \sin\iota$$

$$= -\frac{1}{\epsilon_n} \Delta_{\bar{\epsilon}},$$

by using the formula for $\sum y_\mu \dfrac{dY_\epsilon}{ds_\epsilon}$; and likewise

$$\frac{\Omega^{\frac{1}{2}}}{\bar{\sigma}_\omega} \epsilon_n \omega_\nu \sin\iota \cos\theta_\omega = -\frac{1}{\omega_\nu} \Delta_{\bar{\omega}}.$$

From these relations, various combinations can be effected. Thus, by the elimination of $\cos\theta_\epsilon$ between the two equations in which it occurs, we find

$$\frac{1}{\rho \bar{\sigma}_\epsilon} \Delta_v = \frac{1}{\epsilon_n} \Delta_{\bar{\epsilon}} \left(\frac{1}{\gamma_\epsilon \bar{\sigma}_\epsilon} - \frac{1}{\rho_\epsilon \sigma_\epsilon}\right);$$

and, by the corresponding elimination of $\cos\theta_\omega$, we find

$$\frac{1}{\rho \bar{\sigma}_\omega} \Delta_v = \frac{1}{\omega_\nu} \Delta_{\bar{\omega}} \left(\frac{1}{\gamma_\omega \bar{\sigma}_\omega} - \frac{1}{\rho_\omega \sigma_\omega}\right).$$

By the elimination of $\Delta_{\bar{\epsilon}}$, we find

$$\left(\frac{1}{\rho_\epsilon\sigma_\epsilon} - \frac{1}{\gamma_\epsilon\bar{\sigma}_\epsilon}\right)\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu \sin\iota\cos\theta_3 = \frac{1}{\rho}\Delta_\nu,$$

and therefore

$$\left(\frac{1}{\rho_\epsilon\sigma_\epsilon} - \frac{1}{\gamma_\epsilon\bar{\sigma}_\epsilon}\right)\cos\theta_\epsilon = \frac{1}{\rho_0\sigma_0} - \frac{1}{\gamma\sigma_D}.$$

Similarly, by the elimination of $\Delta_{\bar{\omega}}$, we find

$$\left(\frac{1}{\rho_\omega\sigma_\omega} - \frac{1}{\gamma_\omega\bar{\sigma}_\omega}\right)\cos\theta_\omega = \frac{1}{\rho_0\sigma_0} - \frac{1}{\gamma\sigma_D}.$$

Moreover, the relations provide alternative forms for the three concomitants Δ_ν, Δ_ϵ, $\Delta_{\bar{\omega}}$, respectively equivalent in virtue of these relations.

Diagram of some principal lines for a superficial geodesic.

349. We now can construct a diagram for the schematic arrangement of some of the characteristic lines which, lying within the tangent block of the domain, belong to the ϵ-region, to the ω-region, and to the surface arising as the intersection of those regions. That tangent block contains a flat which is orthogonal to the tangent line OT common to the superficial geodesic, the ϵ-regional geodesic, the ω-regional geodesic, and the domainal geodesic; and therefore the three gremial lines of the domainal geodesic with typical direction-cosines l_3, l_4, l_5, can be taken as three leading lines for the flat.

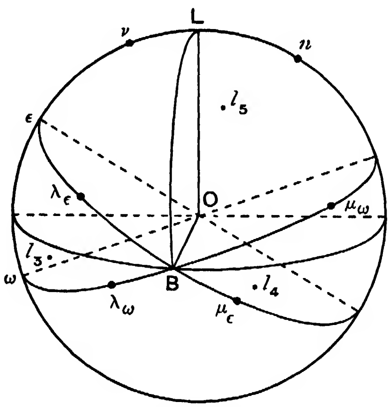

FIG. 34.

Now consider the superficial geodesic; its binormal and its radius of domainal flexure lie within the block and are at right angles to the tangent OT, and therefore they lie in the selected flat. The domainal normal On to the ϵ-region, having $\frac{dy}{dn}$ for its typical cosine, is perpendicular to the tangent OT and lies in the flat; and the domainal normal $O\nu$ to the ω-region, having $\frac{dy}{d\nu}$ for its typical cosine, also is perpendicular to OT and lies in the flat. Both On and $O\nu$ are at right angles to the tangent plane of the surface, and so the plane $nO\nu$ is a plane which, lying in the tangent block of the domain, is orthogonal to the tangent plane of the surface. In that orthogonal plane $nO\nu$, the direction OL of the radius of domainal flexure is drawn, by the construction in § 342.

Accordingly, for the present purpose, we take the directions of the lines OB, OL, On, $O\nu$, as radii of a sphere of representation in the selected flat. The typical direction-cosine l of OL is connected with the typical direction-cosines of On and $O\nu$ by the relation

$$\frac{l}{\gamma}=g_\epsilon \frac{dy}{dn}+g_\omega \frac{dy}{d\nu},$$

where

$$g_\epsilon+g_\omega \cos\iota=\frac{1}{\gamma_\epsilon}, \quad g_\epsilon \cos\iota+g_\omega=\frac{1}{\gamma_\omega},$$

$$\frac{\sin^2\iota}{\gamma^2}=\frac{1}{\gamma_\epsilon{}^2}-\frac{2\cos\iota}{\gamma_\epsilon\gamma_\omega}+\frac{1}{\gamma_\omega{}^2},$$

the angle $nO\nu$ being denoted by ι.

The surface lies in the ϵ-region; and the radius of ϵ-regional flexure of the superficial geodesic lies along $O\epsilon$, the regional normal to the surface. All normals within the domain (and therefore the ϵ-regional normal) to the surface lie in the domainal orthogonal plane $nO\nu$ of the surface. The ϵ-regional normal to the surface lies in the region and its direction is therefore at right angles to On, which is the domainal normal; hence $O\epsilon$ lies in the plane $nO\nu$ and is perpendicular to On. We therefore draw $O\epsilon$ in the diagram, where $n\epsilon=\frac{1}{2}\pi$. Similarly $O\omega$, the direction of the radius of ω-regional flexure of the superficial geodesic lies in the same orthogonal plane $nO\nu$ and is perpendicular to $O\nu$; we therefore draw $O\omega$ in the diagram, where $\nu\omega=\frac{1}{2}\pi$.

We denote by λ_ϵ and μ_ϵ the typical direction-cosines of the binormal and the trinormal of the ϵ-regional geodesic tangent, respectively. The investigation of § 169 shews that these directions lie in the plane of which On (the domainal normal of the region) is the axis in the spherical representation; that is, $O\lambda_\epsilon$ and $O\mu_\epsilon$ (representing the binormal and the trinormal) lie in the plane $BO\epsilon$. Similarly, when λ_ω and μ_ω denote the typical direction-cosines of the binormal and the trinormal of the ω-regional geodesic tangent respectively, $O\lambda_\omega$ and $O\mu_\omega$ (representing that binormal and that trinormal) lie in the plane $BO\omega$. Thus On, $O\lambda_\epsilon$, $O\mu_\epsilon$, are a set of three orthogonal lines in the flat, specially associated with the ϵ-region; and $O\nu$, $O\lambda_\omega$, and $O\mu_\omega$, are another set of orthogonal lines in the flat, these being specially associated with the ω-region.

As the direction $O\epsilon$ lies in the plane $nO\nu$, leading-lines of which are On and $O\nu$ with typical direction-cosines $\frac{dy}{dn}$ and $\frac{dy}{d\nu}$ respectively, the typical direction-cosine (denoted by y_ϵ) of $O\epsilon$ is

$$y_\epsilon=\frac{1}{\sin\iota}\left(-\frac{dy}{d\nu}+\frac{dy}{dn}\cos\iota\right);$$

and similarly the typical direction-cosine (denoted by y_ω) of $O\omega$ is

$$y_\omega=\frac{1}{\sin\iota}\left(-\frac{dy}{dn}+\frac{dy}{d\nu}\cos\iota\right).$$

Ex. Prove that the ϵ-regional flexure $1/G_\epsilon$ and the ω-regional flexure $1/G_\omega$ of the superficial geodesic are given by

$$\frac{1}{G_\epsilon}=\frac{1}{\sin\iota}\left(\frac{1}{\gamma_\omega}-\frac{\cos\iota}{\gamma_\epsilon}\right),\quad \frac{1}{G_\omega}=\frac{1}{\sin\iota}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\iota}{\gamma_\omega}\right).$$

To discuss geometrical relations of the surface, alike in reference to the two regions and to the whole domain, in so far as these are either gremial to the domain or also relate to the curvatures and the flexures, the foregoing diagram, with the lines On, $O\nu$, OB, as central lines of reference for the flat, can be used in conjunction with the diagram on p. 468, in which the lines OF_ω, OF_0, OF_ϵ, are respectively the lines $O\nu$, OL, On, of the preceding diagram. We still have to indicate the positions, in the flat, of the binormal Ol_3, the trinormal Ol_4, and the quartinormal Ol_5, of the domainal geodesic, the common tangent OT being orthogonal to the flat, so that the plane containing OT and OY (the prime normal of the domainal geodesic) is orthogonal to the flat.

Let $\chi_{\epsilon 3}$ and $\chi_{\omega 3}$ denote the inclinations of the binormal of the domainal geodesic, which has the typical direction-cosine l_3, to the respective domainal normals of the ϵ-region and the ω-region, so that (§ 317)

$$-\frac{\Omega\epsilon_n}{\sigma}\cos\chi_{\epsilon 3}=\sum av_1\epsilon_1,\quad -\frac{\Omega\omega_\nu}{\sigma}\cos\chi_{\omega 3}=\sum av_1\omega_1,$$

where $1/\sigma$ is the torsion of the domainal geodesic. Also, the inclination ϕ_3 of the binormal of the superficial geodesic to the binormal of the domainal geodesic tangent is given (§ 347) by

$$\frac{\Omega^{\frac{1}{2}}}{\sigma}\epsilon_n\omega_\nu\sin\iota\cos\phi_3=\Delta_v,$$

so that, if Ol_3 in the diagram represents the direction of the binormal of the domainal geodesic,

$$Bl_3=\phi_3,\quad nl_3=\chi_{\epsilon 3},\quad \nu l_3=\chi_{\omega 3}.$$

Thus the position of l_3 in the diagram is settled: the two angles $\chi_{\epsilon 3}$ and $\chi_{\omega 3}$ are sufficient for the purpose, owing to the relation

$$\cos^2\phi_3+\frac{1}{\sin^2\iota}(\cos^2\chi_{\epsilon 3}-2\cos\iota\cos\chi_{\epsilon 3}\cos\chi_{\omega 3}+\cos^2\chi_{\omega 3})=1.$$

Similarly for the position of l_4 in the diagram. We denote by $\chi_{\epsilon 4}$ and $\chi_{\omega 4}$ respectively the inclinations of the trinormal of the domainal geodesic to the domainal normals of the ϵ-region and the ω-region; and then by the results of § 339, we have

$$\frac{1}{\sigma\tau}\cos\chi_{\epsilon 4}+\frac{d}{ds}\left(\frac{1}{\sigma}\right)\cos\chi_{\epsilon 3}=-\frac{1}{\Omega\epsilon_n}\sum aw_1\epsilon_1,$$

$$\frac{1}{\sigma\tau}\cos\chi_{\omega 4}+\frac{d}{ds}\left(\frac{1}{\sigma}\right)\cos\chi_{\omega 3}=-\frac{1}{\Omega\omega_\nu}\sum aw_1\omega_1.$$

The two angles $\chi_{\epsilon 4}$ and $\chi_{\omega 4}$ settle the position of l_4. The inclination ϕ_4 of the trinormal of the domainal geodesic to the superficial binormal is (§ 347) given by

$$\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu \sin\iota\left\{\frac{1}{\sigma\tau}\cos\phi_4+\frac{d}{ds}\left(\frac{1}{\sigma}\right)\cos\phi_3\right\}=\Delta_w;$$

and again there is the same type of relation connecting the three angles ϕ_4, $\chi_{\epsilon 4}$, $\chi_{\omega 4}$, in the form

$$\cos^2\phi_4+\frac{1}{\sin^2\iota}(\cos^2\chi_{\epsilon 4}-2\cos\iota\cos\chi_{\epsilon 4}\cos\chi_{\omega 4}+\cos^2\chi_{\omega 4})=1.$$

Finally, for the position of l_5, which is the pole of the great circle l_3l_4 in the diagram, the value of l_5 is (§ 288) given by

$$\frac{\Omega^{\frac{1}{2}}}{\sigma^2\tau}l_5=\begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

Then

$$\cos nOl_5=\sum l_5\frac{dy}{dn},\quad \cos\nu Ol_5=\sum l_5\frac{dy}{d\nu},\quad \cos BOl_5=\sum l_5\lambda_3.$$

Hence

$$\frac{\Omega^{\frac{1}{2}}\epsilon_n}{\sigma^2\tau}\cos nOl_5=\begin{vmatrix} \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix},\quad \frac{\Omega^{\frac{1}{2}}\omega_\nu}{\sigma^2\tau}\cos\nu Ol_5=\begin{vmatrix} \omega_1, & \omega_2, & \omega_3, & \omega_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}.$$

Also, with the value of λ_3 in § 347, we have

$$\begin{aligned}\frac{\Omega\epsilon_n\omega_\nu\sin\iota}{\sigma^2\tau}\cos BOl_5&=-\left(\sum\eta_1u_2v_3w_4\right)\left(\sum y_1u_2\epsilon_3\omega_4\right)\\ &=-\sum A\,|\,u_2v_3w_4\,|\,|\,u_2\epsilon_3\omega_4\,|\\ &=-\frac{1}{\Omega^2}\begin{vmatrix} \sum au_1^2, & \sum au_1v_1, & \sum au_1w_1 \\ \sum a\epsilon_1u_1, & \sum a\epsilon_1v_1, & \sum a\epsilon_1w_1 \\ \sum a\omega_1u_1, & \sum a\omega_1v_1, & \sum a\omega_1w_1 \end{vmatrix}.\end{aligned}$$

Now

$$\sum au_1^2=\Omega,\quad \sum au_1\epsilon_1=0,\quad \sum au_1\omega_1=0;$$

the last two relations are $\sum\epsilon_1p'=0$, $\sum\omega_1p'=0$, which are satisfied for the arc on the surface. Also, the values of

$$\sum a\epsilon_1v_1,\quad \sum a\epsilon_1w_1,\quad \sum a\omega_1v_1,\quad \sum a\omega_1w_1,$$

are known in terms of the angles $\chi_{\epsilon 3}$, $\chi_{\epsilon 4}$, $\chi_{\omega 3}$, $\chi_{\omega 4}$; when they are substituted, we have a relation, in the covariantive form

$$-\frac{\Omega^2 \epsilon_n \omega_\nu \sin\iota}{\sigma^2 \tau}\cos BOl_5 = \begin{vmatrix} \sum a\epsilon_1 v_1, & \sum a\epsilon_1 w_1 \\ \sum a\omega_1 v_1, & \sum a\omega_1 w_1 \end{vmatrix}$$

and a relation in the trigonometrical form

$$\sin\iota\cos BOl_5 = \begin{vmatrix} \cos\chi_{\epsilon 3}, & \cos\chi_{\epsilon 4} \\ \cos\chi_{\omega 3}, & \cos\chi_{\omega 4} \end{vmatrix}.$$

Also there is the relation

$$\cos^2 BOl_5 + \frac{1}{\sin^2\iota}(\cos^2 nOl_5 - 2\cos\iota\cos nOl_5\cos\nu Ol_5 + \cos^2\nu Ol_5) = 1;$$

and there are relations, similar to the foregoing,

$$\sin\iota\cos\phi_3 = \begin{vmatrix} \cos\chi_{\epsilon 4}, & \cos\chi_{\omega 4} \\ \cos nOl_5, & \cos\nu Ol_5 \end{vmatrix},$$

$$\sin\iota\cos\phi_4 = \begin{vmatrix} \cos nOl_5, & \cos\nu Ol_5 \\ \cos\chi_{\epsilon 3}, & \cos\chi_{\omega 3} \end{vmatrix}.$$

The trigonometrical relations of the arcs in the complete diagram are satisfied in virtue of the covariantive expressions which are themselves the representation of the inclinations of the various characteristic lines.

Curves of domainal circular curvature on a surface.

350. The circular curvature of the domainal geodesic tangent of a superficial geodesic is a function of the direction-variables of the tangent common to both geodesics; and therefore there are directions on the surface which give a maximum or a minimum value to this curvature. Such curves on the surface may be called its curves of domainal circular curvature.

To determine the directions of these curves at any point on the surface, we assign the critical equations under which the magnitude $1/\rho$ can obtain a maximum or a minimum for the variables p', q', r', t', subject to the conditions

$$\sum Ap'^2 = 1, \quad \sum \epsilon_1 p' = 0, \quad \sum \omega_1 p' = 0.$$

These critical equations are

$$v_a = Pu_a + Q\epsilon_a + R\omega_a,$$

for $a = 1, 2, 3, 4$, the quantities P, Q, R, being left undetermined in the formation of the equations.

The value of P can be obtained at once. Multiply the equations by p', q', r', t', for the successive values of a, and add the results; then, under the limiting conditions, we find

$$\frac{1}{\rho} = P,$$

so that the equations now become

$$v_a - \frac{1}{\rho} u_a = Q\epsilon_a + R\omega_a.$$

Before determining Q and R, one geometrical property can be inferred. The typical direction-cosine l_3 of the binormal of the domainal geodesic is given (§ 284) by

$$l_3 = y_1\lambda + y_2\mu + y_3\nu + y_4\varpi,$$

where the quantities λ, μ, ν, ϖ, are determined by equations of the type

$$\frac{1}{\sigma}(A_{a1}\lambda + A_{a2}\mu + A_{a3}\nu + A_{a4}\varpi) = \frac{1}{\rho} u_a - v_a = -(Q\epsilon_a + R\omega_a).$$

But

$$\frac{\epsilon_a}{\epsilon_n} = A_{a1}\frac{dp}{dn} + A_{a2}\frac{dq}{dn} + A_{a3}\frac{dr}{dn} + A_{a4}\frac{dt}{dn},$$

$$\frac{\omega_a}{\omega_\nu} = A_{a1}\frac{dp}{d\nu} + A_{a2}\frac{dq}{d\nu} + A_{a3}\frac{dr}{d\nu} + A_{a4}\frac{dt}{d\nu};$$

and therefore, as the equations hold for $a=1, 2, 3, 4$, we have

$$\frac{\lambda}{\sigma} = -\left(Q\epsilon_n\frac{dp}{dn} + R\omega_\nu\frac{dp}{d\nu}\right)$$

$$\frac{\mu}{\sigma} = -\left(Q\epsilon_n\frac{dq}{dn} + R\omega_\nu\frac{dq}{d\nu}\right),$$

and so for ν, ϖ. Let these four equations, equivalent to the former four, be multiplied by y_1, y_2, y_3, y_4, and the results be added; then

$$-\frac{1}{\sigma} l_3 = Q\epsilon_n\frac{dy}{dn} + R\omega_\nu\frac{dy}{d\nu}.$$

It follows that the binormal of the domainal geodesic, touching a curve of domainal circular curvature on the surface, lies in the plane $nO\nu$, being the domainal orthogonal plane of the surface (Fig. 34, p. 496).

To determine the values of Q and R, we have

$$-\frac{1}{\sigma}\cos\chi_{3\epsilon} = -\frac{1}{\sigma}\sum l_3\frac{dy}{dn} = Q\epsilon_n + R\omega_\nu\cos\iota,$$

$$-\frac{1}{\sigma}\cos\chi_{3\omega} = -\frac{1}{\sigma}\sum l_3\frac{dy}{d\nu} = Q\epsilon_n\cos\iota + R\omega_\nu,$$

and therefore, assuming

$$\chi_{3\epsilon} + \chi_{3\omega} = \iota$$

as the standard case out of the possibilities $\iota \pm \chi_{3\epsilon} \pm \chi_{3\omega} = 0$, we have

$$Q\epsilon_n\sin\iota = -\frac{1}{\sigma}\sin\chi_{3\omega}, \quad R\omega_\nu\sin\iota = -\frac{1}{\sigma}\sin\chi_{3\epsilon}.$$

The foregoing equation for l_3 now assumes the form, obvious after the establishment of the property that the domainal binormal lies in the plane nOv,

$$l_3 \sin\iota = \frac{dy}{dn}\sin\chi_{3\omega} + \frac{dy}{dv}\sin\chi_{3\epsilon};$$

and the equations for the directions of the curves of domainal circular curvature now become

$$\frac{1}{\rho}u_a - v_a = \frac{1}{\sigma\sin\iota}\left(\frac{\epsilon_a}{\epsilon_n}\sin\chi_{3\omega} + \frac{\omega_a}{\omega_v}\sin\chi_{3\epsilon}\right),$$

for $a=1, 2, 3, 4$.

Further, as the direction Ol_3 lies in the plane nOv, it follows that Ol_3 and OB are at right angles; and therefore

$$\phi_3 = \tfrac{1}{2}\pi.$$

Also, eliminating $1/\rho$, $\sin\chi_{3\omega}$, and $\sin\chi_{3\epsilon}$, from the four equations of the directions of the curves of domainal circular curvature, we have

$$\begin{vmatrix} v_1, & v_2, & v_3, & v_4 \\ u_1, & u_2, & u_3, & u_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix} = 0,$$

along these directions: that is, with the notation of § 347,

$$\Delta_v = 0,$$

a result in accordance with the relation

$$\frac{\Omega^{\frac{1}{2}}}{\sigma}\epsilon_n\omega_v\sin\iota\cos\phi_3 = \Delta_v.$$

Consequently, for the curvatures of the various geodesics which touch a curve of domainal curvature, we have

$$\frac{1}{\rho_0\sigma_0} - \frac{1}{\gamma\sigma_D} = 0,$$

$$\frac{1}{\rho_\epsilon\sigma_\epsilon} - \frac{1}{\gamma_\epsilon\bar{\sigma}_\epsilon} = 0,$$

$$\frac{1}{\rho_\omega\sigma_\omega} - \frac{1}{\gamma_\omega\bar{\sigma}_\omega} = 0,$$

which are results that follow from § 348.

Ex. Shew that, if the inclination of the radius of domainal flexure of the surface to the binormal of the domainal geodesic be denoted by θ, then for a superficial geodesic along a curve of domainal circular curvature

$$\frac{\sin\iota\cos\theta}{\gamma} = \frac{\sin\chi_{3\omega}}{\gamma_\epsilon} + \frac{\sin\chi_{3\epsilon}}{\gamma_\omega}.$$

Spatial tilt of a superficial geodesic.

351. An expression for the spatial tilt of a geodesic, on a surface in a domain, can be obtained as follows.

For the quantities $\bar{\eta}_{ij}$ connected with the circular curvature of the geodesic, we have the relations (§ 345)

$$\bar{\eta}_{ij}=[\eta_{ij}]+\xi_{ij},$$

for the three combinations $ij=11, 12, 22$. We write

$$\bar{\eta}_1=\bar{\eta}_{11}p'+\bar{\eta}_{12}q', \quad [\eta_1]=[\eta_{11}]p'+[\eta_{12}]q',$$
$$\bar{\eta}_2=\bar{\eta}_{12}p'+\bar{\eta}_{22}q', \quad [\eta_2]=[\eta_{12}]p'+[\eta_{22}]q',$$

so that

$$\bar{\eta}_1=[\eta_1]+\xi_{11}p'+\xi_{12}q',$$
$$\bar{\eta}_2=[\eta_2]+\xi_{12}p'+\xi_{22}q'.$$

Let $\bar{l}_4$ denote the typical direction-cosine of the trinormal of the superficial geodesic, and $1/\tau_0$ denote its spatial torsion; then (§ 132), with the preceding notation, we have

$$V_0\left(\frac{\bar{l}_4}{\tau_0}-\frac{Y_0}{\sigma_0}\right)=\bar{u}_1\bar{\eta}_2-\bar{u}_2\bar{\eta}_1.$$

On squaring both sides of the equation and adding for the range of the plenary homaloidal space, it follows that

$$V_0^2\left(\frac{1}{\tau_0^2}+\frac{1}{\sigma_0^2}\right)=\bar{u}_1^2(\sum\bar{\eta}_2^2)-2\bar{u}_1\bar{u}_2(\sum\bar{\eta}_1\bar{\eta}_2)+\bar{u}_2^2(\sum\bar{\eta}_1^2).$$

Now

$$\bar{u}_1^2=A_0-V_0^2q'^2, \quad \bar{u}_1\bar{u}_2=H_0+V_0^2q'p', \quad \bar{u}_2^2=B_0-V_0^2p'^2.$$

When these values are substituted on the right-hand side, the coefficient of V_0^2

$$=-\sum(\bar{\eta}_2q'+\bar{\eta}_1p')^2=-\sum\left(\frac{Y_0}{\rho_0}\right)^2=-\frac{1}{\rho_0^2};$$

and therefore the equation becomes

$$V_0^2\left(\frac{1}{\tau_0^2}+\frac{1}{\sigma_0^2}+\frac{1}{\rho_0^2}\right)=A_0(\sum\bar{\eta}_2^2)-2H_0(\sum\bar{\eta}_1\bar{\eta}_2)+B_0(\sum\bar{\eta}_1^2).$$

The right-hand side has to be evaluated, no longer having the simple form of the corresponding equation for a free surface.

For all values of integers $i, j, \lambda, \mu, =1, 2$, in all combinations, we have

$$\sum[\eta_{ij}]\xi_{\lambda\mu}=0,$$

the summation extending over the plenary range, because the directional quantities $\xi_{\lambda\mu}$ are linear in y_1, y_2, y_3, y_4, and

$$\sum\eta_{ij}y_k=0,$$

for all values $k=1, 2, 3, 4$. Consequently, also

$$\sum [\eta_\iota]\xi_{\lambda\mu}=0\,;$$

and therefore

$$\begin{aligned}\sum \bar{\eta}_1^2&=\sum [\eta_1]^2+\sum (\xi_{11}p'+\xi_{12}q')^2,\\ \sum \bar{\eta}_1\bar{\eta}_2&=\sum [\eta_1][\eta_2]+\sum (\xi_{11}p'+\xi_{12}q')(\xi_{12}p'+\xi_{22}q'),\\ \sum \bar{\eta}_2^2&=\sum [\bar{\eta}_2]^2+\sum (\xi_{12}p'+\xi_{22}q')^2.\end{aligned}$$

Thus the right-hand side of the equation consists of two aggregates of terms Φ and Φ_D, where

$$\begin{aligned}\Phi&=A_0\sum [\bar{\eta}_2]^2-2H_0\sum [\eta_1][\eta_2]+B_0\sum [\bar{\eta}_1]^2,\\ \Phi_D&=A_0\sum (\xi_{12}p'+\xi_{22}q')^2+B_0(\xi_{11}p'+\xi_{12}q')^2\\ &\quad-2H_0\sum (\xi_{11}p'+\xi_{12}q')(\xi_{12}p'+\xi_{22}q').\end{aligned}$$

First, for the aggregate Φ. In § 297, there is the relation

$$\eta_k=Yv_k+l_6\xi_k,$$

for $k=1, 2, 3, 4$; and therefore, as

$$[\eta_1]=\eta_1-\frac{s_{23}}{s_{12}}\eta_3-\frac{s_{24}}{s_{12}}\eta_4,\quad [\eta_2]=\eta_2-\frac{s_{31}}{s_{12}}\eta_3-\frac{s_{41}}{s_{12}}\eta_4,$$

we have

$$[\eta_1]=Y\left(v_1-\frac{s_{23}}{s_{12}}v_3-\frac{s_{24}}{s_{12}}v_4\right)+l_6\left(\xi_1-\frac{s_{23}}{s_{12}}\xi_3-\frac{s_{24}}{s_{12}}\xi_4\right).$$

By the values of the quantities $\xi_1, \xi_2, \xi_3, \xi_4$, as obtained in § 300, we have

$$\begin{aligned}\frac{1}{\rho}\left(\xi_1-\frac{s_{23}}{s_{12}}\xi_3-\frac{s_{24}}{s_{12}}\xi_4\right)&=-m_{12}q'-m_{13}r'-m_{14}t'\\ &\quad-\frac{s_{23}}{s_{12}}(m_{13}p'+m_{23}q'+m_{34}t')\\ &\quad-\frac{s_{24}}{s_{12}}(m_{14}p'+m_{24}q'+m_{34}r')\\ &=-\frac{1}{s_{12}}q'S,\end{aligned}$$

where S is defined by the relation

$$S=s_{12}m_{12}+s_{13}m_{13}+s_{14}m_{14}+s_{23}m_{23}+s_{24}m_{24}+s_{34}m_{34}\,;$$

and similarly we have

$$\frac{1}{\rho}\left(\xi_2-\frac{s_{31}}{s_{12}}\xi_3-\frac{s_{41}}{s_{12}}\xi_4\right)=\frac{1}{s_{12}}p'S.$$

Also (§ 345)

$$v_1-\frac{s_{23}}{s_{12}}v_3-\frac{s_{24}}{s_{12}}v_4=\bar{v}_1,\quad v_2-\frac{s_{31}}{s_{12}}v_3-\frac{s_{41}}{s_{12}}v_4=\bar{v}_2.$$

Consequently,

$$[\eta_1]=X\bar{v}_1-\frac{1}{s_{12}}q'\rho Sl_6,\quad [\eta_2]=X\bar{v}_2+\frac{1}{s_{12}}p'\rho Sl_6\,;$$

and therefore also

$$\sum[\eta_1]^2=\ \bar{v}_1{}^2+\frac{1}{s_{12}{}^2}\rho^2S^2q'^2,$$

$$\sum[\eta_1][\eta_2]=\bar{v}_1\bar{v}_2-\frac{1}{s_{12}{}^2}\rho^2S^2q'p',$$

$$\sum[\eta_2]^2=\ \bar{v}_2{}^2+\frac{1}{s_{12}{}^2}\rho^2S^2p'^2.$$

But

$$A_0\bar{v}_2{}^2-2H_0\bar{v}_2\bar{v}_1+B_0\bar{v}_1{}^2=\frac{1}{(\epsilon_3\omega_4-\omega_3\epsilon_4)^2}\Theta(v,v),$$

where

$$\Theta(\theta,\phi)=\begin{vmatrix} A, & H, & G, & L, & \theta_1, & \epsilon_1, & \omega_1 \\ H, & B, & F, & M, & \theta_2, & \epsilon_2, & \omega_2 \\ G, & F, & C, & N, & \theta_3, & \epsilon_3, & \omega_3 \\ L, & M, & N, & D, & \theta_4, & \epsilon_4, & \omega_4 \\ \phi_1, & \phi_2, & \phi_3, & \phi_4, & 0, & 0, & 0 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0, & 0, & 0 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4, & 0, & 0, & 0 \end{vmatrix},$$

for all quantities θ, ϕ; and (§ 270)

$$\epsilon_3\omega_4-\omega_3\epsilon_4=s_{12}\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu\sin\iota,$$

while (§ 345)

$$V_0{}^2(\epsilon_3\omega_4-\epsilon_4\omega_3)^2=\Omega\epsilon_n{}^2\omega_\nu{}^2\sin^2\iota,\quad V_0s_{12}=1.$$

Hence we have

$$\Phi=V_0{}^2\left\{\frac{\Theta(v,v)}{\Omega\epsilon_n{}^2\omega_\nu{}^2\sin^2\iota}+\rho^2S^2\right\}.$$

Next, for the aggregate Φ_D. By the results in § 345, we have

$$\xi_{11}p'+\xi_{12}q'=\frac{1}{\epsilon_n}\left(\bar{\epsilon}_1-\frac{s_{23}}{s_{12}}\bar{\epsilon}_3-\frac{s_{24}}{s_{12}}\bar{\epsilon}_4\right)y_\omega+\frac{1}{\omega_\nu}\left(\bar{\omega}_1-\frac{s_{23}}{s_{12}}\bar{\omega}_3-\frac{s_{24}}{s_{12}}\bar{\omega}_4\right)y_\epsilon$$

$$=\frac{1}{\epsilon_n}\bar{E}_1y_\omega+\frac{1}{\omega_\nu}\bar{\Omega}_1y_\epsilon,$$

and

$$\xi_{12}p'+\xi_{22}q'=\frac{1}{\epsilon_n}\left(\bar{\epsilon}_2-\frac{s_{31}}{s_{12}}\bar{\epsilon}_3-\frac{s_{41}}{s_{12}}\bar{\epsilon}_4\right)y_\omega+\frac{1}{\omega_\nu}\left(\bar{\omega}_2-\frac{s_{31}}{s_{12}}\bar{\omega}_3-\frac{s_{41}}{s_{12}}\bar{\epsilon}_4\right)y_\epsilon$$

$$=\frac{1}{\epsilon_n}\bar{E}_2y_\omega+\frac{1}{\omega_\nu}\bar{\Omega}_2y_\epsilon\,;$$

and therefore

$$\Phi_D = A_0\left(\frac{1}{\epsilon_n^2}\bar{E}_2^2 - \frac{2\cos\iota}{\epsilon_n\omega_\nu}\bar{E}_2\bar{\Omega}_2 + \frac{1}{\omega_\nu^2}\bar{\Omega}_2^2\right)$$
$$- 2H_0\left\{\frac{1}{\epsilon_n^2}\bar{E}_1\bar{E}_2 - \frac{\cos\iota}{\epsilon_n\omega_\nu}(\bar{E}_1\bar{\Omega}_2 + \bar{E}_2\bar{\Omega}_1) + \frac{1}{\omega_\nu^2}\bar{\Omega}_1\bar{\Omega}_2\right\}$$
$$+ B_0\left(\frac{1}{\epsilon_n^2}\bar{E}_1^2 - \frac{2\cos\iota}{\epsilon_n\omega_\nu}\bar{E}_1\bar{\Omega}_1 + \frac{1}{\omega_\nu^2}\bar{\Omega}_1^2\right).$$

But

$$A_0\bar{E}_2^2 - 2H_0\bar{E}_1\bar{E}_2 + B_0\bar{E}_1^2 = \frac{V_0^2}{\Omega\epsilon_n^2\omega_\nu^2\sin^2\iota}\Theta(\bar{\epsilon}, \bar{\epsilon}),$$

$$A_0\bar{E}_2\bar{\Omega}_2 - H_0(\bar{E}_1\bar{\Omega}_2 + \bar{E}_2\bar{\Omega}_1) + B_0\bar{E}_1\bar{\Omega}_1 = \frac{V_0^2}{\Omega\epsilon_n^2\omega_\nu^2\sin^2\iota}\Theta(\bar{\epsilon}, \bar{\omega}),$$

$$A_0\bar{\Omega}_2^2 - 2H_0\bar{\Omega}_1\bar{\Omega}_2 + B_0\bar{\Omega}_1^2 = \frac{V_0^2}{\Omega\epsilon_n^2\omega_\nu^2\sin^2\iota}\Theta(\bar{\omega}, \bar{\omega});$$

and therefore

$$\Phi_D = \frac{V_0^2}{\Omega\epsilon_n^2\omega_\nu^2\sin^2\iota}\left\{\frac{1}{\epsilon_n^2}\Theta(\bar{\epsilon}, \bar{\epsilon}) - \frac{2\cos\iota}{\epsilon_n\omega_\nu}\Theta(\bar{\epsilon}, \bar{\omega}) + \frac{1}{\omega_\nu^2}\Theta(\bar{\omega}, \bar{\omega})\right\}.$$

Accordingly, the equation for the spatial tilt of the superficial geodesic has the form

$$\left(\frac{1}{\tau_0^2} + \frac{1}{\sigma_0^2} + \frac{1}{\rho_0^2} + \rho^2 S^2\right)\Omega\epsilon_n^2\omega_\nu^2\sin^2\iota$$
$$= \Theta(v, v) + \frac{1}{\epsilon_n^2}\Theta(\bar{\epsilon}, \bar{\epsilon}) - \frac{2\cos\iota}{\epsilon_n\omega_\nu}\Theta(\bar{\epsilon}, \bar{\omega}) + \frac{1}{\omega_\nu^2}\Theta(\bar{\omega}, \bar{\omega}).$$

Circular curvature of a superficial geodesic in the bi-parametric representation of a surface.

352. The fundamental relation

$$\frac{Y_0}{\rho_0} = \frac{Y}{\rho} + \frac{l}{\gamma},$$

connecting the circular curvature and the domainal flexure of a superficial geodesic with the circular curvature of the domainal geodesic tangent, can be established in connection with the bi-parametric representation of the surface, instead of the bi-regional representation used in the preceding investigation. The course of the analysis leads to a modified expression for the domainal flexure of the superficial geodesic.

In the bi-parametric representation, the four domainal parameters are postulated as functions of two parameters x, z, according to the definitions

$$p = p(x, z), \quad q = q(x, z), \quad r = r(x, z), \quad t = t(x, z).$$

We write

$$\sigma_{23}=q_x r_z - r_x q_z, \qquad \sigma_{14}=p_x t_z - t_x p_z,$$
$$\sigma_{31}=r_x p_z - p_x r_z, \qquad \sigma_{24}=q_x t_z - t_x q_z,$$
$$\sigma_{12}=p_x q_z - q_x p_z, \qquad \sigma_{34}=r_x t_z - t_x r_z\,;$$

and these variables σ_{ij}, differing each from the corresponding variable s_{ij} only by a factor which is the same for all, are orientation-variables of the surface.

Any line in the tangent plane of the surface is given by the typical equation

$$\frac{\bar{y}-y}{y'}=\theta,$$

that is,

$$\bar{y}-y=\theta y'=\kappa y_1+\lambda y_2+\mu y_3+\varpi y_4,$$

where θ is the current variable of length along the line; and κ, λ, μ, ϖ, denote parametric magnitudes, the values of which are given by

$$\kappa=\theta p'=\theta(p_x x'+p_z z')=\alpha p_x+\beta p_z,$$
$$\lambda=\theta q'=\theta(q_x x'+q_z z')=\alpha q_x+\beta q_z,$$
$$\mu=\theta r'=\theta(r_x x'+r_z z')=\alpha r_x+\beta r_z,$$
$$\varpi=\theta t'=\theta(t_x x'+t_z z')=\alpha t_x+\beta t_z.$$

Thus

$$\left\Vert \begin{matrix} \kappa, & \lambda, & \mu, & \varpi \\ p_x, & q_x, & r_x, & t_x \\ p_z, & q_z, & r_z, & t_z \end{matrix} \right\Vert=0,$$

and (what is the equivalent)

$$\begin{aligned} &\lambda\sigma_{34}+\mu\sigma_{24}+\varpi\sigma_{23}=0,\\ -\kappa\sigma_{34}\qquad &+\mu\sigma_{14}+\varpi\sigma_{31}=0,\\ \kappa\sigma_{24}-\lambda\sigma_{14}\qquad &+\varpi\sigma_{12}=0,\\ \kappa\sigma_{23}+\lambda\sigma_{31}+\mu\sigma_{12}\qquad &=0\,; \end{aligned}$$

as usual, these four relations are equivalent to only a couple of independent relations. Hence the equations of the tangent plane of the surface are

$$\bar{y}-y=\kappa y_1+\lambda y_2+\mu y_3+\varpi y_4,$$

where the parameters κ, λ, μ, ϖ, of the plane are subject to the preceding set of relations.

Expressions for the direction-cosines and for the magnitude of the circular curvature of a superficial geodesic can be obtained as before*, by drawing,

* The analysis which follows is an alternative to the analysis in § 343, in connection with which reference may be made also to the general investigation in § 69.

upon the tangent plane at O, a perpendicular from a contiguous point O' of the surface at a distance (ultimately to be a geodesic distance δ) from O. The critical equations of § 343 for this perpendicular now take the form

$$\begin{aligned}\sum y_1(\eta-y_0)&=P\sigma_{23}+Q\sigma_{24},\\ \sum y_2(\eta-y_0)&=P\sigma_{31}+Q\sigma_{41},\\ \sum y_3(\eta-y_0)&=P\sigma_{12}\qquad\quad,\\ \sum y_4(\eta-y_0)&=\qquad\qquad Q\sigma_{12},\end{aligned}$$

where η denotes the typical coordinate of the neighbouring point O', where y_0 is the typical coordinate of the foot of the perpendicular given by

$$y_0=y+\kappa y_1+\lambda y_2+\mu y_3+\varpi y_4,$$

and where P, Q, are two multipliers left undetermined in the construction of the critical equations. When the value of y_0 is inserted, the first of the foregoing equations becomes

$$A\kappa+H\lambda+G\mu+L\varpi+P\sigma_{23}+Q\sigma_{24}=\sum y_1(\eta-y),$$

and the others acquire corresponding forms.

Expanding up to the second order of small quantities inclusive, we have

$$\eta=y+(y_1p'+y_2q'+y_3r'+y_4t')\delta+\tfrac{1}{2}y_0''\delta^2,$$

where y_0'' implies second derivation along the surface, so that, with the like significance for p_0'', q_0'', r_0'', t_0'',

$$\begin{aligned}y_0''&=y_1p_0''+y_2q_0''+y_3r_0''+y_4t_0''+\sum y_{11}p'^2\\ &=y_1(p_0''+\sum\Gamma_{11}p'^2)+y_2(q_0''+\sum\Delta_{11}p'^2)\\ &\qquad+y_3(r_0''+\sum\Theta_{11}p'^2)+y_4(t_0''+\sum\Phi_{11}p'^2)+\frac{Y}{\rho}.\end{aligned}$$

As before, there are quantities $\bar\kappa$, $\bar\lambda$, $\bar\mu$, $\bar\varpi$, defined by the relations

$$\begin{aligned}\bar\kappa&=\kappa-p'\delta-\tfrac{1}{2}(p_0''+\sum\Gamma_{11}p'^2)\delta^2=\kappa-p'\delta-\tfrac{1}{2}P_0\delta^2,\\ \bar\lambda&=\lambda-q'\delta-\tfrac{1}{2}(q_0''+\sum\Delta_{11}p'^2)\delta^2=\lambda-q'\delta-\tfrac{1}{2}Q_0\delta^2,\\ \bar\mu&=\mu-r'\delta-\tfrac{1}{2}(r_0''+\sum\Theta_{11}p'^2)\delta^2=\mu-r'\delta-\tfrac{1}{2}R_0\delta^2,\\ \bar\varpi&=\varpi-t'\delta-\tfrac{1}{2}(t_0''+\sum\Phi_{11}p'^2)\delta^2=\varpi-t'\delta-\tfrac{1}{2}T_0\delta^2;\end{aligned}$$

and the four critical equations acquire the form

$$\left.\begin{aligned}A\bar\kappa+H\bar\lambda+G\bar\mu+L\bar\varpi&=-P\sigma_{23}-Q\sigma_{24}\\ H\bar\kappa+B\bar\lambda+F\bar\mu+M\bar\varpi&=-P\sigma_{31}-Q\sigma_{41}\\ G\bar\kappa+F\bar\lambda+C\bar\mu+N\bar\varpi&=-P\sigma_{12}\\ L\bar\kappa+M\bar\lambda+N\bar\mu+D\bar\varpi&=\qquad\qquad-Q\sigma_{12}\end{aligned}\right\}.$$

With these values, we find (as in § 343)

$$l_0\Pi=\tfrac{1}{2}\frac{Y}{\rho}\delta^2-(y_1\bar\kappa+y_2\bar\lambda+y_3\bar\mu+y_4\bar\varpi),$$

and therefore, by a comparison with preceding results,

$$-\tfrac{1}{2}\frac{l}{\gamma}\delta^2 = y_1\bar{\kappa} + y_2\bar{\lambda} + y_3\bar{\mu} + y_4\bar{\varpi},$$

where γ denotes the magnitude and l denotes the typical direction-cosine of the radius of domainal flexure of the superficial geodesic.

In order to evaluate the right-hand side of this equation, it is necessary to determine the values of P and Q. Let the four critical equations, in their latest form, be resolved for $\bar{\kappa}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\varpi}$, so that

$$\begin{aligned}
\Omega\bar{\kappa} &= -P(a\sigma_{23} + h\sigma_{31} + g\sigma_{12}) - Q(a\sigma_{24} + h\sigma_{41} + l\sigma_{12}),\\
\Omega\bar{\lambda} &= -P(h\sigma_{23} + b\sigma_{31} + f\sigma_{12}) - Q(h\sigma_{24} + b\sigma_{41} + m\sigma_{12}),\\
\Omega\bar{\mu} &= -P(g\sigma_{23} + f\sigma_{31} + c\sigma_{12}) - Q(g\sigma_{24} + f\sigma_{41} + n\sigma_{12}),\\
\Omega\bar{\varpi} &= -P(l\sigma_{23} + m\sigma_{31} + n\sigma_{12}) - Q(l\sigma_{24} + m\sigma_{41} + d\sigma_{12}).
\end{aligned}$$

From the relations connecting κ, λ, μ, ϖ, we select two conditions

$$\kappa\sigma_{23} + \lambda\sigma_{31} + \mu\sigma_{12} = 0, \quad \kappa\sigma_{34} + \lambda\sigma_{41} + \varpi\sigma_{12} = 0,$$

and from the fact that the variables p', q', r', t', determine a direction in the tangent plane, we have the two equations

$$p'\sigma_{23} + q'\sigma_{31} + r'\sigma_{12} = 0, \quad p'\sigma_{34} + q'\sigma_{41} + t'\sigma_{12} = 0\,;$$

and therefore, when the relations defining $\bar{\kappa}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\varpi}$, are used, we find

$$\begin{aligned}
\bar{\kappa}\sigma_{23} + \bar{\lambda}\sigma_{31} + \bar{\mu}\sigma_{12} &= -\tfrac{1}{2}(\sigma_{23}P_0 + \sigma_{31}Q_0 + \sigma_{12}R_0)\delta^2,\\
\bar{\kappa}\sigma_{24} + \bar{\lambda}\sigma_{41} + \bar{\varpi}\sigma_{12} &= -\tfrac{1}{2}(\sigma_{24}P_0 + \sigma_{41}Q_0 + \sigma_{12}T_0)\delta^2.
\end{aligned}$$

Let the preceding resolved values of $\bar{\kappa}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\varpi}$, be introduced into these equations: then

$$\begin{aligned}
\tfrac{1}{2}\Omega\Theta_1\delta^2 &= \tfrac{1}{2}\Omega(\sigma_{23}P_0 + \sigma_{31}Q_0 + \sigma_{12}R_0)\delta^2 = P\bar{\alpha} + Q\bar{\beta},\\
\tfrac{1}{2}\Omega\Theta_2\delta^2 &= \tfrac{1}{2}\Omega(\sigma_{24}P_0 + \sigma_{41}Q_0 + \sigma_{12}T_0)\delta^2 = P\bar{\beta} + Q\bar{\gamma},
\end{aligned}$$

where, if we use umbral symbols k_1, k_2, k_3, k_4, such that

$$(a, b, c, f, g, h, l, m, n\between\xi_1, \xi_2, \xi_3, \xi_4)^2 = (k_1\xi_1 + k_2\xi_2 + k_3\xi_3 + k_4\xi_4)^2,$$

the values of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, can be expressed in the umbral forms

$$\begin{aligned}
\bar{\alpha} &= (k_1\sigma_{23} + k_2\sigma_{31} + k_3\sigma_{12})^2,\\
\bar{\beta} &= (k_1\sigma_{23} + k_2\sigma_{31} + k_3\sigma_{12})(k_1\sigma_{24} + k_1\sigma_{41} + k_4\sigma_{12}),\\
\bar{\gamma} &= (k_1\sigma_{24} + k_2\sigma_{41} + k_4\sigma_{12})^2.
\end{aligned}$$

The value of $\bar{\alpha}\bar{\gamma} - \bar{\beta}^2$ is given by

$$\bar{\alpha}\bar{\gamma} - \bar{\beta}^2 = \sigma_{12}{}^2\{\textstyle\sum(ab - h^2)\sigma_{34}{}^2\} = \sigma_{12}{}^2\Omega\Box,$$

where

$$\square = \begin{vmatrix} A, & H, & G, & L, & p_x, & p_y \\ H, & B, & F, & M, & q_x, & q_y \\ G, & F, & C, & N, & r_x, & r_y \\ L, & M, & N, & D, & t_x, & t_y \\ p_x, & q_x, & r_x, & t_x, & 0, & 0 \\ p_y, & q_y, & r_y, & t_y, & 0, & 0 \end{vmatrix}.$$

It is convenient to introduce the umbral combinations

$$\left.\begin{aligned} e_1 &= \phantom{k_1\sigma_{34}+{}} k_2\sigma_{34}+k_3\sigma_{42}+k_4\sigma_{23} \\ -e_2 &= k_1\sigma_{34} \phantom{{}+k_2\sigma_{41}} +k_3\sigma_{41}+k_4\sigma_{13} \\ e_3 &= k_1\sigma_{24}+k_2\sigma_{41} \phantom{{}+k_3\sigma_{41}} +k_4\sigma_{12} \\ -e_4 &= k_1\sigma_{23}+k_2\sigma_{31}-k_3\sigma_{12} \end{aligned}\right\}.$$

Then we have

$$P(\bar{a}\bar{\gamma}-\bar{\beta}^2)=\tfrac{1}{2}\Omega(\ \ \Theta_1\bar{\gamma}-\Theta_2\bar{\beta})\,\delta^2=\tfrac{1}{2}\Omega\Phi e_3\sigma_{12}\delta^2,$$
$$Q(\bar{a}\bar{\gamma}-\bar{\beta}^2)=\tfrac{1}{2}\Omega(-\Theta_1\bar{\beta}+\Theta_2\bar{a})\,\delta^2=\tfrac{1}{2}\Omega\Phi e_4\sigma_{12}\delta^2,$$

on reduction, where

$$\Phi=P_0e_1+Q_0e_2+R_0e_3+T_0e_4.$$

Now that the values of P and Q are known, they can be substituted in the resolved equations for $\bar{\kappa}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\varpi}$. Owing to the relations

$$e_1\sigma_{12}=e_3\sigma_{23}+e_4\sigma_{24},$$
$$e_2\sigma_{12}=e_3\sigma_{31}+e_4\sigma_{41},$$

we have

$$-\Omega\bar{\kappa}(\bar{a}\bar{\gamma}-\bar{\beta}^2)=\tfrac{1}{2}\Omega\Phi(ae_1+\ he_2+ge_3+\ le_4)\sigma_{12}{}^2\delta^2;$$

and similarly

$$-\Omega\bar{\lambda}(\bar{a}\bar{\gamma}-\bar{\beta}^2)=\tfrac{1}{2}\Omega\Phi(he_1+\ be_2+fe_3+me_4)\sigma_{12}{}^2\delta^2,$$
$$-\Omega\bar{\mu}(\bar{a}\bar{\gamma}-\bar{\beta}^2)=\tfrac{1}{2}\Omega\Phi(ge_1+\ fe_2+ce_3+\ ne_4)\sigma_{12}{}^2\delta^2,$$
$$-\Omega\bar{\varpi}(\bar{a}\bar{\gamma}-\bar{\beta}^2)=\tfrac{1}{2}\Omega\Phi(le_1+me_2+ne_3+\ de_4)\sigma_{12}{}^2\delta^2.$$

When we use the value of $\bar{a}\bar{\gamma}-\bar{\beta}^2$, and insert these values of $\bar{\kappa}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\varpi}$, in the value of l/γ, we find

$$\frac{\Omega\square}{\gamma}\,l=\Phi\Psi,$$

where

$$\Psi=\sum y_1(ae_1+he_2+ge_3+le_4)=\sum ae_1y_1.$$

Now

$$P_0=p_0''+\sum\Gamma_{11}p'^2,$$
$$p_0''=x''p_x+z''p_z+x'^2p_{xx}+2x'z'p_{xz}+z'^2p_{zz};$$

and therefore, the value of P_0 is

$$P_0 = x''p_x + z''p_z + \bar{P},$$

where

$$\bar{P} = x'^2(p_{xx} + \sum \Gamma_{11}p_x^2) + 2x'z'(p_{xz} + \sum \Gamma_{11}p_xp_z) + z'^2(p_{zz} + \sum \Gamma_{11}p_z^2).$$

Similarly

$$\begin{aligned} Q_0 &= x''q_x + z''q_z + \bar{Q}, \\ R_0 &= x''r_x + z''r_z + \bar{R}, \\ T_0 &= x''t_x + z''t_z + \bar{T}, \end{aligned}$$

where

$$\begin{aligned} \bar{Q} &= x'^2(q_{xx} + \sum \Delta_{11}p_x^2) + 2x'z'(q_{xz} + \sum \Delta_{11}p_xp_z) + z'^2(q_{zz} + \sum \Delta_{11}p_z^2), \\ \bar{R} &= x'^2(r_{xx} + \sum \Theta_{11}p_x^2) + 2x'z'(r_{xz} + \sum \Theta_{11}p_xp_z) + z'^2(r_{zz} + \sum \Theta_{11}p_z^2), \\ \bar{T} &= x'^2(t_{xx} + \sum \Phi_{11}p_x^2) + 2x'z'(t_{xz} + \sum \Phi_{11}p_xp_z) + z'^2(t_{zz} + \sum \Phi_{11}p_z^2). \end{aligned}$$

Thus, for the value of Φ, we find

$$\begin{aligned} \Phi &= P_0e_1 + Q_0e_2 + R_0e_3 + T_0e_4 \\ &= \begin{vmatrix} x''p_x + z''p_z + \bar{P}, & k_1, & p_x, & p_z \\ x''q_x + z''q_z + \bar{Q}, & k_2, & q_x, & q_z \\ x''r_x + z''r_z + \bar{R}, & k_3, & r_x, & r_z \\ x''t_x + z''t_z + \bar{T}, & k_4, & t_x, & t_z \end{vmatrix} \\ &= -\begin{vmatrix} k_1, & \bar{P}, & p_x, & p_z \\ k_2, & \bar{Q}, & q_x, & q_z \\ k_3, & \bar{R}, & r_x, & r_z \\ k_4, & \bar{T}, & t_x, & t_z \end{vmatrix}. \end{aligned}$$

Also

$$\begin{aligned} \Psi &= \sum ae_1y_1 \\ &= -\begin{vmatrix} k_1, & y_1a + y_2h + y_3g + y_4l, & p_x, & p_z \\ k_2, & y_1h + y_2b + y_3f + y_4m, & q_x, & q_z \\ k_3, & y_1g + y_2f + y_3c + y_4n, & r_x, & r_z \\ k_4, & y_1l + y_2m + y_3n + y_4d, & t_x, & t_z \end{vmatrix}. \end{aligned}$$

Thus, finally, the value of l/γ is given by

$$\frac{l}{\gamma} = \frac{1}{\Omega\square}\Phi\Psi,$$

where Φ and Ψ are the two foregoing umbral determinants.

The direction-variables x' and z' of the surface occur only in the magnitudes $\bar{P}$, $\bar{Q}$, $\bar{R}$, $\bar{T}$: when regard is paid to their form, the expression for l/γ becomes

$$\frac{l}{\gamma}=\xi_{11}x'^2+2\xi_{12}x'z'+\xi_{22}z'^2,$$

analogous to the (less generalised) form in § 345, where the parameters p, q, of the domain are retained for the surface.

Domainal flexure in the bi-parametric representation of a surface.

353. Finally, we can construct an expression for the value of the domainal flexure of the superficial geodesic. Let

$$\bar{\kappa}=\tfrac{1}{2}\kappa_0\delta^2,\quad \bar{\lambda}=\tfrac{1}{2}\lambda_0\delta^2,\quad \bar{\mu}=\tfrac{1}{2}\mu_0\delta^2,\quad \bar{\varpi}=\tfrac{1}{2}\varpi_0\delta^2\,;$$

then we have, by the formula on p. 509,

$$-\frac{l}{\gamma}=y_1\kappa_0+y_2\lambda_0+y_3\mu_0+y_4\varpi_0,$$

so that

$$\frac{1}{\gamma^2}=\sum A\kappa_0^2\,;$$

and the quantity on the right-hand side must be evaluated.

In § 352, it was proved that

$$A\bar{\kappa}+H\bar{\lambda}+G\bar{\mu}+L\bar{\varpi}=-P\sigma_{23}-Q\sigma_{24},$$

and therefore, inserting the values of P and Q obtained on p. 510, we have

$$\begin{aligned}-(\bar{a}\bar{\gamma}-\bar{\beta}^2)(A\bar{\kappa}+H\bar{\lambda}+G\bar{\mu}+L\bar{\varpi})&=\tfrac{1}{2}\Omega\Phi(e_3\sigma_{23}+e_4\sigma_{24})\sigma_{12}\delta^2\\&=\tfrac{1}{2}\Omega\Phi e_1\sigma_{12}^2\delta^2,\end{aligned}$$

that is,

$$-\Box(A\kappa_0+H\lambda_0+G\mu_0+L\omega_0)=\Phi e_1.$$

Now introduce new variables S_1, S_2, S_3, S_4, defined by the equations

$$S_1,\ S_2,\ S_3,\ S_4=\left\|\begin{matrix}\bar{P}, & \bar{Q}, & \bar{R}, & \bar{T}\\ p_x, & q_x, & r_x, & t_x\\ p_z, & q_z, & r_z, & t_z\end{matrix}\right\|\,;$$

then

$$\Phi=-(k_1S_1+k_2S_2+k_3S_3+k_4S_4),$$

and therefore

$$\Box(A\kappa_0+H\lambda_0+G\mu_0+L\varpi_0)=e_1(k_1S_1+k_2S_2+k_3S_3+k_4S_4)=E_1.$$

Similarly

$$\begin{aligned}\Box(H\kappa_0+B\lambda_0+F\mu_0+M\varpi_0)&=e_2(k_1S_1+k_2S_2+k_3S_3+k_4S_4)=E_2,\\ \Box(G\kappa_0+F\lambda_0+C\mu_0+N\varpi_0)&=e_3(k_1S_1+k_2S_2+k_3S_3+k_4S_4)=E_3,\\ \Box(L\kappa_0+M\lambda_0+N\mu_0+D\varpi_0)&=e_4(k_1S_1+k_2S_2+k_3S_3+k_4S_4)=E_4,\end{aligned}$$

the symbols E_1, E_2, E_3, E_4, being only for temporary use.

We have

$$\Omega \sum A\kappa_0{}^2 = \sum a(A\kappa_0 + H\lambda_0 + G\mu_0 + L\varpi_0)^2,$$

and therefore

$$\begin{aligned}\Omega \Box^2 \sum A\kappa_0{}^2 &= \sum a E_1{}^2 \\ &= (k_1'E_1 + k_2'E_2 + k_3'E_3 + k_4'E_4)^2,\end{aligned}$$

where k_1', k_2', k_3', k_4', denote umbral symbols congruent with k_1, k_2, k_3, k_4. Now

$$\begin{aligned}E_1 &= e_1(k_1S_1 + k_2S_2 + k_3S_3 + k_4S_4) \\ &= \sigma_{34}(hS_1 + bS_2 + fS_3 + mS_4) \\ &\quad + \sigma_{42}(gS_1 + fS_2 + cS_3 + nS_4) \\ &\quad + \sigma_{23}(lS_1 + mS_2 + nS_3 + dS_4) \\ &= \begin{vmatrix} hS_1 + bS_2 + fS_3 + mS_4, & q_x, & q_z \\ gS_1 + fS_2 + cS_3 + nS_4, & r_x, & r_z \\ lS_1 + mS_2 + nS_3 + dS_4, & t_x, & t_z \end{vmatrix},\end{aligned}$$

with similar expressions for E_2, E_3, E_4; and therefore

$$\Omega \Box^2 \frac{1}{\gamma^2} = \begin{vmatrix} k_1', & aS_1 + hS_2 + gS_3 + lS_4, & p_x, & p_z \\ k_2', & hS_1 + bS_2 + fS_3 + mS_4, & q_x, & q_z \\ k_3', & gS_1 + fS_2 + cS_3 + nS_4, & r_x, & r_z \\ k_4', & lS_1 + mS_2 + nS_3 + dS_4, & t_x, & t_z \end{vmatrix}^2,$$

where the umbral symbols k_1', k_2', k_3', k_4', are such as to give $k_1'^2 = a$, $k_1'k_2' = h$, ... , $k_4'^2 = d$. Accordingly we have an expression for the domainal flexure of the superficial geodesic, in terms of the magnitudes of the domain and of the quantities in the bi-parametric equations defining the surface.

CHAPTER XXX

DOMAINAL CURVATURES OF SURFACES

Sphericity of a domainal surface.

354. To obtain the sphericity of the domainal surface which is the intersection of the ϵ-region and the ω-region, we recall the magnitudes connected with the domainal flexures of the ϵ-regional geodesics alone and of the ω-regional geodesics alone. It has appeared that, in addition to the principal linear measures of domainal flexure of the regional geodesics (§ 319), there are also two superficial measures of domainal flexure of regional geodesics for geodesics originating in a superficial orientation (§ 321). For the immediate purpose, we select the superficial orientation to be the same for the two regions, being that of the surface which is the intersection of the regions.

With the notation of § 345, we have

$$U_0 = \quad 1 = A_0 p'^2 + 2H_0 p'q' + G_0 q'^2,$$

$$U_\epsilon = -\frac{\epsilon_n}{\gamma_\epsilon} = \bar{E}_{11} p'^2 + 2\bar{E}_{12} p'q' + \bar{E}_{22} q'^2,$$

$$U_\omega = -\frac{\omega_\nu}{\gamma_\omega} = \bar{\Omega}_{11} p'^2 + 2\bar{\Omega}_{12} p'q' + \bar{\Omega}_{22} q'^2.$$

Now as in § 319, the principal values of the domainal flexure of the ϵ-regional geodesics are obtained by making U_ϵ a maximum or a minimum for values of p' and q' lying in the surface and therefore satisfying the single condition $U_0 = 1$. These principal values of γ_ϵ are the roots of the equation

$$\begin{vmatrix} \bar{E}_{11} + \dfrac{\epsilon_n}{\gamma_\epsilon} A_0, & \bar{E}_{12} + \dfrac{\epsilon_n}{\gamma_\epsilon} H_0 \\ \bar{E}_{12} + \dfrac{\epsilon_n}{\gamma_\epsilon} H_0, & \bar{E}_{22} + \dfrac{\epsilon_n}{\gamma_\epsilon} B_0 \end{vmatrix} = 0;$$

when they are denoted by $\gamma_{1\epsilon}$ and $\gamma_{2\epsilon}$, and when K_ϵ and H_ϵ denote the two superficial measures of domainal flexure, we have

$$K_\epsilon = \frac{1}{\gamma_{1\epsilon}\gamma_{2\epsilon}} = \frac{1}{\epsilon_n{}^2 V_0{}^2} (\bar{E}_{11}\bar{E}_{22} - \bar{E}_{12}{}^2),$$

$$H_\epsilon = \frac{1}{\gamma_{1\epsilon}} + \frac{1}{\gamma_{2\epsilon}} = -\frac{1}{\epsilon_n V_0{}^2} (B_0\bar{E}_{11} - 2H_0\bar{E}_{12} + A_0\bar{E}_{22}).$$

Similarly, there are two principal values of γ_ω, denoted by $\gamma_{1\omega}$ and $\gamma_{2\omega}$; and

there are the two superficial measures K_ω and H_ω of domainal flexure of the ω-regional geodesics; the quantities $\gamma_{1\omega}$ and $\gamma_{2\omega}$ are the roots of the equation

$$\begin{vmatrix} \bar{\Omega}_{11}+\frac{\omega_\nu}{\gamma_\omega}A_0, & \bar{\Omega}_{12}+\frac{\omega_\nu}{\gamma_\omega}H_0 \\ \bar{\Omega}_{12}+\frac{\omega_\nu}{\gamma_\omega}H_0, & \bar{\Omega}_{22}+\frac{\omega_\nu}{\gamma_\omega}B_0 \end{vmatrix}=0,$$

while K_ω and H_ω are given by

$$K_\omega=\frac{1}{\gamma_{1\omega}\gamma_{2\omega}}=\frac{1}{\omega_\nu^2 V_0^2}(\bar{\Omega}_{11}\bar{\Omega}_{22}-\bar{\Omega}_{12}^2),$$

$$H_\omega=\frac{1}{\gamma_{1\omega}}+\frac{1}{\gamma_{2\omega}}=-\frac{1}{\omega_\nu V_0^2}(B_0\bar{\Omega}_{11}-2H_0\bar{\Omega}_{12}+A_0\bar{\Omega}_{22}).$$

The quantities V_0^2, $\bar{E}_{11}\bar{E}_{22}-\bar{E}_{12}^2$, $\bar{\Omega}_{11}\bar{\Omega}_{22}-\bar{\Omega}_{12}^2$, are the discriminants of the three quadratic forms U_0, U_ϵ, U_ω, respectively; and the quantities

$$B_0\bar{E}_{11}-2H_0\bar{E}_{12}+A_0\bar{E}_{22},\quad B_0\bar{\Omega}_{11}-2H_0\bar{\Omega}_{12}+A_0\bar{\Omega}_{22},$$

are intermediate invariants of U_0 and U_ϵ, and of U_0 and U_ω, respectively. There is also the intermediate invariant $I_{\epsilon\omega}$ of U_ϵ and U_ω, with the value

$$I_{\epsilon\omega}=\frac{1}{\epsilon_n\omega_\nu}(\bar{\Omega}_{11}\bar{E}_{22}-2\bar{\Omega}_{12}\bar{E}_{12}+\bar{\Omega}_{22}\bar{E}_{11});$$

a more geometrical form for $I_{\epsilon\omega}$ can be obtained as follows. Let $4J_{\lambda\mu}$ denote the Jacobian of two quadratics of the type U_λ and U_μ, so that

$$4J_{\lambda\mu}=\begin{vmatrix} \frac{\partial U_\lambda}{\partial p'}, & \frac{\partial U_\lambda}{\partial q'} \\ \frac{\partial U_\mu}{\partial p'}, & \frac{\partial U_\mu}{\partial q'} \end{vmatrix};$$

then as $p'\frac{\partial U_\lambda}{\partial p'}+q'\frac{\partial U_\lambda}{\partial q'}=2U_\lambda$ for each of the forms, there is an identity

$$U_0J_{\epsilon\omega}+U_\epsilon J_{\omega 0}+U_\omega J_{0\epsilon}=0.$$

The binariants of the system of U_0 and U_ϵ are connected by a relation

$$\begin{aligned} J_{0\epsilon}^2 &= U_0U_\epsilon I_{0\epsilon}-U_0^2(\bar{E}_{11}\bar{E}_{22}-\bar{E}_{12}^2)-U_\epsilon^2(A_0B_0-H_0^2) \\ &=\epsilon_n^2V_0^2\left(\frac{1}{\gamma_\epsilon}H_\epsilon-K_\epsilon-\frac{1}{\gamma_\epsilon^2}\right) \\ &=\epsilon_n^2V_0^2\left(\frac{1}{\gamma_\epsilon}-\frac{1}{\gamma_{1\epsilon}}\right)\left(\frac{1}{\gamma_{2\epsilon}}-\frac{1}{\gamma_\epsilon}\right); \end{aligned}$$

so that, if ψ_ϵ denote the inclination of the direction p', q', in the surface to the direction of a principal value $\gamma_{1\epsilon}$, where

$$\frac{1}{\gamma_\epsilon}=\frac{\cos^2\psi_\epsilon}{\gamma_{1\epsilon}}+\frac{\sin^2\psi_\epsilon}{\gamma_{2\epsilon}},$$

we have

$$J_{0\epsilon}=\epsilon_n V_0\left(\frac{1}{\gamma_{2\epsilon}}-\frac{1}{\gamma_{1\epsilon}}\right)\sin\psi_\epsilon\cos\psi_\epsilon.$$

Similarly, with a corresponding significance for ψ_ω, we have

$$J_{0\omega}=\omega_\nu V_0\left(\frac{1}{\gamma_{2\omega}}-\frac{1}{\gamma_{1\omega}}\right)\sin\psi_\omega\cos\psi_\omega.$$

Consequently, as $U_0=1$, a value for $J_{\epsilon\omega}$ is given by

$$\begin{aligned}J_{\epsilon\omega}&=\epsilon_n\omega_\nu V_0\left\{\frac{1}{\gamma_\omega}\left(\frac{1}{\gamma_{2\epsilon}}-\frac{1}{\gamma_{1\epsilon}}\right)\sin\psi_\epsilon\cos\psi_\epsilon-\frac{1}{\gamma_\epsilon}\left(\frac{1}{\gamma_{2\omega}}-\frac{1}{\gamma_{1\omega}}\right)\sin\psi_\omega\cos\psi_\omega\right\}\\&=\epsilon_n\omega_\nu V_0\Phi,\end{aligned}$$

for brevity.

But there is also the relation

$$J_{\epsilon\omega}^2=U_\epsilon U_\omega I_{\epsilon\omega}-U_\omega^2(\bar{E}_{11}\bar{E}_{22}-\bar{E}_{12}^2)-U_\epsilon^2(\bar{\Omega}_{11}\bar{\Omega}_{22}-\bar{\Omega}_{12}^2);$$

and therefore

$$\frac{\epsilon_n\omega_\nu}{\gamma_\epsilon\gamma_\omega}I_{\epsilon\omega}=\epsilon_n^2\omega_\nu^2V_0^2\left\{\Phi^2+\frac{1}{\gamma_\omega^2\gamma_{1\epsilon}\gamma_{2\epsilon}}+\frac{1}{\gamma_\epsilon^2\gamma_{1\omega}\gamma_{2\omega}}\right\},$$

giving a value for $I_{\epsilon\omega}$; and the value admits of further reduction.

The reduction to a standard form can be attained by another method. Let P_ϵ', Q_ϵ', be direction-variables, in the surface, of one of the principal directions in the superficial orientation for giving the domainal flexure $1/\gamma_{1\epsilon}$ of ϵ-regional geodesics, so that

$$\left(\bar{E}_{11}+\frac{\epsilon_n}{\gamma_{1\epsilon}}A_0\right)P_\epsilon'+\left(\bar{E}_{12}+\frac{\epsilon_n}{\gamma_{1\epsilon}}H_0\right)Q_\epsilon'=0,$$

$$\left(\bar{E}_{12}+\frac{\epsilon_n}{\gamma_{1\epsilon}}H_0\right)P_\epsilon'+\left(\bar{E}_{22}+\frac{\epsilon_n}{\gamma_{1\epsilon}}B_0\right)Q_\epsilon'=0.$$

Consequently we have

$$\begin{aligned}\frac{\bar{E}_{11}+\dfrac{\epsilon_n}{\gamma_{1\epsilon}}A_0}{Q_\epsilon'^2}=\frac{\bar{E}_{12}+\dfrac{\epsilon_n}{\gamma_{1\epsilon}}H_0}{-P_\epsilon'Q_\epsilon'}=\frac{\bar{E}_{22}+\dfrac{\epsilon_n}{\gamma_{1\epsilon}}B_0}{P_\epsilon'^2}\\=B_0\left(\bar{E}_{11}+\frac{\epsilon_n}{\gamma_{1\epsilon}}A_0\right)-2H_0\left(E_{12}+\frac{\epsilon_n}{\gamma_{1\epsilon}}H_0\right)+A_0\left(\bar{E}_{22}+\frac{\epsilon_n}{\gamma_{1\epsilon}}B_0\right)\\=\epsilon_nV_0^2\left(\frac{1}{\gamma_{1\epsilon}}-\frac{1}{\gamma_{2\epsilon}}\right).\end{aligned}$$

Similarly, if P_ω', Q_ω', be direction-variables, in the surface, of the principal direction in the superficial orientation giving the domainal flexure $1/\gamma_{1\omega}$ of ω-regional geodesics, we have

$$\frac{\bar{\Omega}_{11}+\frac{\omega_\nu}{\gamma_{1\omega}}A_0}{Q_\omega'^2}=\frac{\bar{\Omega}_{12}+\frac{\omega_\nu}{\gamma_{1\omega}}H_0}{-P_\omega'Q_\omega'}=\frac{\bar{\Omega}_{22}+\frac{\omega_\nu}{\gamma_{1\omega}}B_0}{P_\omega'^2}=\omega_\nu V_0^2\left(\frac{1}{\gamma_{1\omega}}-\frac{1}{\gamma_{2\omega}}\right).$$

Let μ denote the inclination, to one another, of these selected principal directions of the respective domainal flexures of geodesics originating in the surface; then

$$\sin^2\mu = V_0^2(P_\epsilon'Q_\omega'-Q_\epsilon'P_\omega')^2.$$

Hence

$$\begin{aligned}\epsilon_n\omega_\nu V_0^2\left(\frac{1}{\gamma_{1\epsilon}}-\frac{1}{\gamma_{2\epsilon}}\right)&\left(\frac{1}{\gamma_{1\omega}}-\frac{1}{\gamma_{2\omega}}\right)\sin^2\mu\\ &=\left(\bar{E}_{22}+\frac{\epsilon_n}{\gamma_{1\epsilon}}B_0\right)\left(\bar{\Omega}_{11}+\frac{\omega_\nu}{\gamma_{1\omega}}A_0\right)\\ &\quad-2\left(\bar{E}_{12}+\frac{\epsilon_n}{\gamma_{1\epsilon}}H_0\right)\left(\bar{\Omega}_{12}+\frac{\omega_\nu}{\gamma_{1\omega}}H_0\right)\\ &\quad+\left(\bar{E}_{11}+\frac{\epsilon_n}{\gamma_{1\epsilon}}A_0\right)\left(\bar{\Omega}_{22}+\frac{\omega_\nu}{\gamma_{1\omega}}B_0\right)\\ &=\epsilon_n\omega_\nu I_{\epsilon\omega}-\epsilon_n\omega_\nu V_0^2\left(\frac{1}{\gamma_{1\epsilon}\gamma_{2\omega}}+\frac{1}{\gamma_{1\omega}\gamma_{2\epsilon}}\right)\end{aligned}$$

so that

$$\frac{I_{\epsilon\omega}}{V_0^2}=\left(\frac{1}{\gamma_{1\epsilon}\gamma_{1\omega}}+\frac{1}{\gamma_{2\epsilon}\gamma_{2\omega}}\right)\sin^2\mu+\left(\frac{1}{\gamma_{1\epsilon}\gamma_{2\omega}}+\frac{1}{\gamma_{1\omega}\gamma_{2\epsilon}}\right)\cos^2\mu.$$

Thus the algebraical invariants and covariants of the system of three quadratic forms are known. The geometrical magnitudes implicitly involved are the two measures of domainal superficial flexure of the ϵ-region and the like two measures of the ω-region, together with the inclination of principal directions of the domainal linear flexure, all estimated in the orientation of the surface which is the intersection of the two regions.

To the general covariantive expressions, not specially referred to the surface, a return will be made later.

Superficial measure of domainal flexure.

355. As regards the sphericity of the surface, we have (with the notation of § 345)

$$\frac{Y_0}{\rho_0}=\bar{\eta}_{11}p'^2+2\bar{\eta}_{12}p'q'+\bar{\eta}_{22}q'^2,$$

so that, if K_S denote that sphericity,

$$V_0^2K_S=\sum(\bar{\eta}_{11}\bar{\eta}_{22}-\bar{\eta}_{12}^2).$$

But

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma};$$

also, we have

$$\frac{Y}{\rho}=[\eta_{11}]p'^2+2[\eta_{12}]p'q'+[\eta_{22}]q'^2,$$

and, if

$$V_0^2K_D=\sum\{[\eta_{11}][\eta_{22}]-[\eta_{12}]^2\},$$

K_D is the sphericity of the domain in the orientation of the surface. Further,

$$\begin{aligned}\frac{l}{\gamma}&=g_\epsilon\frac{dy}{dn}+g_\omega\frac{dy}{d\nu}\\&=\frac{1}{\sin^2\iota}\left(\frac{1}{\gamma_\epsilon}-\frac{\cos\iota}{\gamma_\omega}\right)\frac{dy}{dn}+\frac{1}{\sin^2\iota}\left(\frac{1}{\gamma_\omega}-\frac{\cos\iota}{\gamma_\epsilon}\right)\frac{dy}{d\nu}\\&=\frac{1}{\gamma_\epsilon\sin^2\iota}\left(\frac{dy}{dn}-\frac{dy}{d\nu}\cos\iota\right)+\frac{1}{\gamma_\omega\sin^2\iota}\left(\frac{dy}{d\nu}-\frac{dy}{dn}\cos\iota\right)\\&=-\frac{1}{\gamma_\epsilon\sin\iota}y_\omega-\frac{1}{\gamma_\omega\sin\iota}y_\epsilon,\end{aligned}$$

where y_ϵ and y_ω denote (§ 349) the typical direction-cosines of the ϵ-regional normal and the ω-regional normal respectively to the surface. We therefore take

$$\frac{l}{\gamma}=\xi_{11}p'^2+2\xi_{12}p'q'+\xi_{22}q'^2,$$

where

$$\begin{aligned}\xi_{11}\sin\iota&=\frac{\overline{E}_{11}}{\epsilon_n}y_\omega+\frac{\overline{\Omega}_{11}}{\omega_\nu}y_\epsilon,\\\xi_{12}\sin\iota&=\frac{\overline{E}_{12}}{\epsilon_n}y_\omega+\frac{\overline{\Omega}_{12}}{\omega_\nu}y_\epsilon,\\\xi_{22}\sin\iota&=\frac{\overline{E}_{22}}{\epsilon_n}y_\omega+\frac{\overline{\Omega}_{22}}{\omega_\nu}y_\epsilon,\end{aligned}$$

with the preceding notation; and thus we have

$$\bar{\eta}_{11}=[\eta_{11}]+\xi_{11},\quad\bar{\eta}_{12}=[\eta_{12}]+\xi_{12},\quad\bar{\eta}_{22}=[\eta_{22}]+\xi_{22}^2,$$

the equivalent of the relations on p. 483.

To develop the expression for K_S, we note that (§ 345) each of the quantities $[\eta_{ij}]$ is a linear combination of the quantities $\eta_{\alpha\beta}$ and that the quantities g_{ij} are linear combinations of y_ϵ and y_ω, that is, of y_1, y_2, y_3, y_4; also, for all values of α, β, λ, we have

$$\sum\eta_{\alpha\beta}y_\lambda=0.$$

Hence all the combinations

$$\sum\{[\eta_{11}]\xi_{22}\},\quad\sum\{[\eta_{12}]\xi_{12}\},\quad\sum\{[\eta_{22}]\xi_{11}\},$$

vanish ; and therefore

$$\begin{aligned} V_0^2 K_S &= \sum (\bar\eta_{11}\bar\eta_{22} - \bar\eta_{12}^2) \\ &= \sum \{[\eta_{11}][\eta_{22}] - [\eta_{12}]^2\} + \sum (\xi_{11}\xi_{22} - \xi_{12}^2) \\ &= V_0^2 (K_D + K_F), \end{aligned}$$

where K_D is the sphericity of the domain in the orientation of the surface, and K_F is a new magnitude defined by the equation

$$V_0^2 K_F = \sum (\xi_{11}\xi_{22} - \xi_{12}^2).$$

Having regard to the similarity of the forms

$$\frac{X}{\rho} = \bar\eta_{11} p'^2 + 2\bar\eta_{12} p'q' + \bar\eta_{22} q'^2, \quad \frac{l}{\gamma} = \xi_{11} p'^2 + 2\xi_{12} p'q' + \xi_{22} q'^2,$$

the one belonging to the prime normal and the circular curvature, and the other to the direction and the magnitude of the radius of domainal flexure, all the quantities $\bar\eta_{ij}$ and ξ_{ij} being independent of the direction-variables of the superficial geodesic, and to the fact that the quantity

$$\frac{1}{V_0^2} \sum (\bar\eta_{11}\bar\eta_{22} - \bar\eta_{12}^2)$$

is the Riemann measure of sphericity of the surface as a configuration in the plenary homaloidal space, we may regard K_F as a measure of domainal flexure of the surface, corresponding (though in a different range) to K_S. Thus the sphericity of the surface is the sum of (i), the sphericity of the domain in the orientation of the surface and (ii), the measure of domainal flexure of the surface.

The value of the measure of domainal flexure K_F, developed from

$$V_0^2 K_F = \sum (\xi_{11}\xi_{22} - \xi_{12}^2),$$

arises by substituting the foregoing values of ξ_{11}, ξ_{12}, ξ_{22}. Because

$$\sum y_\epsilon y_\omega = -\cos\iota,$$

we have

$$\begin{aligned} &\sum (\xi_{11}\xi_{22} - \xi_{12}^2) \sin^2\iota \\ &= \frac{1}{\epsilon_n^2} (\bar E_{11}\bar E_{22} - \bar E_{12}^2) - \frac{\cos\iota}{\epsilon_n \omega_\nu} (\bar E_{11}\bar\Omega_{22} - 2\bar E_{12}\bar\Omega_{12} + \bar E_{22}\bar\Omega_{11}) + \frac{1}{\omega_\nu^2} (\bar\Omega_{11}\bar\Omega_{22} - \bar\Omega_{12}^2) \\ &= V_0^2 (K_\epsilon - I_{\epsilon\omega}\cos\iota + K_\omega). \end{aligned}$$

Therefore

$$K_F = \frac{1}{\sin^2\iota} (K_\epsilon - I_{\epsilon\omega}\cos\iota + K_\omega),$$

where K_ϵ is the domainal flexure of the ϵ-region, K_ω is the domainal flexure of the ω-region, and $I_{\epsilon\omega}$ is a composite measure: all the measures being estimated in the superficial orientation of the surface that is the intersection of the regions.

When the values of the principal radii of flexure in the ϵ-region and the ω-region respectively are inserted, and account is taken of the inclinations of the superficial directions in the orientation, the expression giving K_F becomes

$$K_F \sin^2 \iota = \frac{1}{\gamma_{1\epsilon}\gamma_{2\epsilon}} + \frac{1}{\gamma_{1\omega}\gamma_{2\omega}} - \left\{\left(\frac{1}{\gamma_{1\epsilon}\gamma_{1\omega}} + \frac{1}{\gamma_{2\epsilon}\gamma_{2\omega}}\right)\sin^2\mu + \left(\frac{1}{\gamma_{1\epsilon}\gamma_{2\omega}} + \frac{1}{\gamma_{1\omega}\gamma_{2\epsilon}}\right)\cos^2\mu\right\}\cos\iota.$$

Other forms can be obtained. Thus, for instance, we find

$$K_\epsilon = \frac{1}{\epsilon_n{}^2 V_0{}^2}(\bar{E}_{11}\bar{E}_{22} - \bar{E}_{12}{}^2) = \frac{1}{\Omega\epsilon_n{}^2\omega_\nu{}^2\sin^2\iota}\begin{vmatrix} \bar{\epsilon}_{11}, & \bar{\epsilon}_{12}, & \bar{\epsilon}_{13}, & \bar{\epsilon}_{14}, & \epsilon_1, & \omega_1 \\ \bar{\epsilon}_{12}, & \bar{\epsilon}_{22}, & \bar{\epsilon}_{23}, & \bar{\epsilon}_{24}, & \epsilon_2, & \omega_2 \\ \bar{\epsilon}_{13}, & \bar{\epsilon}_{23}, & \bar{\epsilon}_{33}, & \bar{\epsilon}_{34}, & \epsilon_3, & \omega_3 \\ \bar{\epsilon}_{14}, & \bar{\epsilon}_{24}, & \bar{\epsilon}_{34}, & \bar{\epsilon}_{44}, & \epsilon_4, & \omega_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0, & 0 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4, & 0, & 0 \end{vmatrix};$$

and, for all values of α and β,

$$\alpha^2 K_\epsilon + \alpha\beta I_{\epsilon\omega} + \beta^2 K_\omega$$
$$= \frac{1}{\Omega\epsilon_n{}^2\omega_\nu{}^2\sin^2\iota}\begin{vmatrix} \alpha\frac{\bar{\epsilon}_{11}}{\epsilon_n} + \beta\frac{\bar{\omega}_{11}}{\omega_\nu}, & \dots, & \alpha\frac{\bar{\epsilon}_{14}}{\epsilon_n} + \beta\frac{\bar{\omega}_{14}}{\omega_\nu}, & \epsilon_1, & \omega_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha\frac{\bar{\epsilon}_{14}}{\epsilon_n} + \beta\frac{\bar{\omega}_{14}}{\omega_\nu}, & \dots, & \alpha\frac{\bar{\epsilon}_{44}}{\epsilon_n} + \beta\frac{\bar{\omega}_{44}}{\omega_\nu}, & \epsilon_4, & \omega_4 \\ \epsilon_1, & \dots, & \epsilon_4, & 0, & 0 \\ \omega_1, & \dots, & \omega_4, & 0, & 0 \end{vmatrix}.$$

But the form that will be used, in what follows immediately, is

$$V_0{}^2 K_F = \sum (\xi_{11}\xi_{22} - \xi_{12}{}^2),$$

taken in connection with (i), the arc-relation of the domain, when specialised for the surface, that is,

$$A_0 p'^2 + 2H_0 p'q' + B_0 q'^2 = 1,$$

and (ii), with the typical equation

$$\frac{l}{\gamma} = \xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2,$$

for the direction-cosines and the magnitude of the radius of domainal flexure of the superficial geodesic.

Alternative bi-parametric representation of a domainal surface.

356. Thus far, the source of the domainal surface has been bi-regional within the domain; the equations of the surface have been those of the two regions, taken in combination; and some of the properties of the surface have been

dominated by its relations to the regions. An alternative representation is provided by postulating the domainal parameters p, q, r, t, as functions of two independent parameters x and z; or by taking r and t as two independent functions of p and q. Such a representation is equivalent to the two equations $\epsilon=0$, $\omega=0$, the equivalence being obvious when these two equations are combined, first so as to eliminate t, and next so as to eliminate r; and, indeed, the bi-regional representation is not unique, for any two equations

$$\alpha\epsilon+\beta\omega=0, \quad \alpha'\epsilon+\beta'\omega=0,$$

where $\alpha\beta'-\alpha'\beta$ is not zero, would provide the same surface, while the detailed expressions for magnitudes and relations of the surface would be concerned with the modified regions. But whatever transformations of this character be adopted, geometrical magnitudes represented by expressions such as

$$\frac{1}{\sin^2 \iota}\left(\frac{1}{\gamma_\epsilon^2}-\frac{2\cos\iota}{\gamma_\epsilon\gamma_\omega}+\frac{1}{\gamma_\omega^2}\right),$$

$$\frac{1}{\sin^2 \iota}(K_\epsilon - I_{\epsilon\omega}\cos\iota + K_\omega),$$

are absolutely invariantive in analytical expression.

In particular, the transition to the second method can be made by taking

$$\epsilon(p, q, r, t)=\phi(p, q)-r=0,$$
$$\omega(p, q, r, t)=\psi(p, q)-t=0,$$

where ϕ and ψ are independent functions of p and q; and then the superficial parameters p and q can be changed to any other independent parameters x and z.

When this bi-parametric representation of the domainal surface is adopted, we take the element of arc in the form

$$ds^2=A_0\,dp^2+2H_0\,dp\,dq+B_0dq^2;$$

and we use the equations

$$\frac{l}{\gamma}=\xi_{11}p'^2+2\xi_{12}p'q'+\xi_{22}q'^2$$

for the domainal flexure. Some inferences from these equations will be made; where desirable or useful, reference will be made to the relations

$$\xi_{m\mu}\sin\iota=\frac{\overline{E}_{m\mu}}{\epsilon_n}y_\omega+\frac{\overline{\Omega}_{m\mu}}{\omega_\nu}y_\epsilon,$$

for the combinations $m\mu=11$, 12, 22.

We proceed as for the (spatial) circular curvature of geodesics on a surface existing freely in its plenary homaloidal space. The parametric derivatives of the

typical spatial variable y will be denoted by $\bar{y}_1$ and $\bar{y}_2$, so that (as compared with the domainal parametric derivatives of y) we have (§ 347)

$$\bar{y}_1 = y_1 - \frac{s_{23}}{s_{12}} y_3 - \frac{s_{24}}{s_{12}} y_4, \quad \bar{y}_2 = y_2 - \frac{s_{31}}{s_{12}} y_3 - \frac{s_{41}}{s_{12}} y_4,$$

the variables $s_{\alpha\beta}$ being the orientation-variables of the surface in the domain. It is easy to verify that

$$\sum \bar{y}_1 y_\omega = 0, \quad \sum \bar{y}_1 y_\epsilon = 0, \quad \sum \bar{y}_2 y_\omega = 0, \quad \sum \bar{y}_2 y_\epsilon = 0,$$

and therefore

$$\sum \bar{y}_1 \xi_{m\mu} = 0, \quad \sum \bar{y}_2 \xi_{m\mu} = 0,$$

for all values of $m\mu$; while

$$\sum l\bar{y}_1 = 0, \quad \sum l\bar{y}_2 = 0,$$

relations to be expected, because the radius of domainal flexure (with the typical direction-cosine l) is perpendicular to the tangent plane of the surface.

Let two sets of quantities be introduced, under the definitions

$$\sum l\xi_{11} = \bar{e}, \qquad \sum l\xi_{12} = \bar{f}, \qquad \sum l\xi_{22} = \bar{g},$$

for the one set; and

$$\sum \xi_{11}{}^2 = \mathfrak{a}, \qquad \sum \xi_{12}{}^2 = \mathfrak{b}, \qquad \sum \xi_{22}{}^2 = \mathfrak{c},$$
$$\sum \xi_{12}\xi_{22} = \mathfrak{f}, \quad \sum \xi_{22}\xi_{11} = \mathfrak{g}, \quad \sum \xi_{11}\xi_{12} = \mathfrak{h},$$

for the other set. It is convenient to introduce a symbol $\mathfrak{k}$ such that

$$\mathfrak{g} + 2\mathfrak{b} = 3\mathfrak{k};$$

and we evidently have

$$\mathfrak{g} - \mathfrak{b} = V_0{}^2 K_F.$$

Then there are the relations

$$\frac{1}{\gamma} = \bar{e}p'^2 + 2\bar{f}p'q' + \bar{g}q'^2;$$

$$\left.\begin{aligned}\frac{\bar{e}}{\gamma} &= \mathfrak{a}p'^2 + 2\mathfrak{h}p'q' + \mathfrak{g}q'^2\\ \frac{\bar{f}}{\gamma} &= \mathfrak{h}p'^2 + 2\mathfrak{b}p'q' + \mathfrak{f}q'^2\\ \frac{\bar{g}}{\gamma} &= \mathfrak{g}p'^2 + 2\mathfrak{f}p'q' + \mathfrak{c}q'^2\end{aligned}\right\}:$$

and

$$\frac{1}{\gamma^2} = \mathfrak{a}p'^4 + 4\mathfrak{h}p'^3q' + 6\mathfrak{k}p'^2q'^2 + 4\mathfrak{f}p'q'^3 + \mathfrak{c}q'^4.$$

Moreover, in connection with the bi-regional representation of the surface, we have equations

$$\mathfrak{a}\sin^2\iota = \frac{\bar{E}_{11}^2}{\epsilon_n^2} - 2\frac{\bar{E}_{11}\bar{\Omega}_{11}}{\epsilon_n\omega_\nu}\cos\iota + \frac{\bar{\Omega}_{11}^2}{\omega_\nu^2},$$

$$\mathfrak{b}\sin^2\iota = \frac{\bar{E}_{12}^2}{\epsilon_n^2} - 2\frac{\bar{E}_{12}\bar{\Omega}_{12}}{\epsilon_n\omega_\nu}\cos\iota + \frac{\bar{\Omega}_{12}^2}{\omega_\nu^2},$$

$$\mathfrak{c}\sin^2\iota = \frac{\bar{E}_{22}^2}{\epsilon_n^2} - 2\frac{\bar{E}_{22}\bar{\Omega}_{22}}{\epsilon_n\omega_\nu}\cos\iota + \frac{\bar{\Omega}_{22}^2}{\omega_\nu^2},$$

$$\mathfrak{f}\sin^2\iota = \frac{\bar{E}_{12}\bar{E}_{22}}{\epsilon_n^2} - \frac{\cos\iota}{\epsilon_n\omega_\nu}(\bar{E}_{12}\bar{\Omega}_{22} + \bar{E}_{22}\bar{\Omega}_{12}) + \frac{\bar{\Omega}_{12}\bar{\Omega}_{22}}{\omega_\nu^2},$$

$$\mathfrak{g}\sin^2\iota = \frac{\bar{E}_{22}\bar{E}_{11}}{\epsilon_n^2} - \frac{\cos\iota}{\epsilon_n\omega_\nu}(\bar{E}_{22}\bar{\Omega}_{11} + \bar{E}_{11}\bar{\Omega}_{22}) + \frac{\bar{\Omega}_{22}\bar{\Omega}_{11}}{\omega_\nu^2},$$

$$\mathfrak{h}\sin^2\iota = \frac{\bar{E}_{11}\bar{E}_{12}}{\epsilon_n^2} - \frac{\cos\iota}{\epsilon_n\omega_\nu}(\bar{E}_{11}\bar{\Omega}_{12} + \bar{E}_{12}\bar{\Omega}_{11}) + \frac{\bar{\Omega}_{11}\bar{\Omega}_{12}}{\omega_\nu^2}.$$

357. On account of the relations

$$\sum \bar{y}_1\xi_{m\mu} = 0, \quad \sum \bar{y}_2\xi_{m\mu} = 0,$$

for $m\mu = 11,\ 12,\ 22$, it follows that the three directions typified by direction-cosines

$$\mathfrak{a}^{-\frac{1}{2}}\xi_{11}, \quad \mathfrak{b}^{-\frac{1}{2}}\xi_{12}, \quad \mathfrak{c}^{-\frac{1}{2}}\xi_{22},$$

lie in the domainal plane orthogonal to the surface; and therefore there must exist a homogeneous linear relation connecting ξ_{11}, ξ_{12}, ξ_{22}, with coefficients the same throughout the whole set of which the linear relation is typical. It might, in fact, be taken in the form

$$\begin{vmatrix} \xi_{11}, & \bar{E}_{11}, & \bar{\Omega}_{11} \\ \xi_{12}, & \bar{E}_{12}, & \bar{\Omega}_{12} \\ \xi_{22}, & \bar{E}_{22}, & \bar{\Omega}_{22} \end{vmatrix} = 0;$$

but there is an analytical advantage in assuming it to be of the form

$$\xi_{12} = P\xi_{11} + Q\xi_{22},$$

with coefficients P and Q to be determined.

To determine P and Q, we multiply the relation by ξ_{11} and add: also by ξ_{12} and add: also by ξ_{22} and add: and we obtain equations

$$\mathfrak{h} = P\mathfrak{a} + Q\mathfrak{g},$$

$$\mathfrak{b} = P\mathfrak{h} + Q\mathfrak{f},$$

$$\mathfrak{f} = P\mathfrak{g} + Q\mathfrak{c}.$$

Of these equations, two uses are to be made.

In the first place, the elimination of P and Q leads to the condition

$$Y=\begin{vmatrix} \mathfrak{a}, & \mathfrak{h}, & \mathfrak{g} \\ \mathfrak{h}, & \mathfrak{b}, & \mathfrak{f} \\ \mathfrak{g}, & \mathfrak{f}, & \mathfrak{c} \end{vmatrix}=0,$$

a condition which is satisfied identically when the foregoing values of the constituents in terms of the quantities $\overline{E}_{ij}$, $\overline{\Omega}_{ij}$, are substituted. As the determinant vanishes, there are relations connecting its first minors which can be constructed as follows.

Owing to the identity

$$\sin^2(\phi-\iota)-2\sin(\phi-\iota)\sin\phi\cos\iota+\sin^2\phi=\sin^2\iota,$$

where ϕ is any quantity, we can modify the expressions for $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, by taking

$$\frac{\overline{E}_{11}}{\epsilon_n}=\mathfrak{a}^{\frac{1}{2}}\sin(\theta_1-\iota),\quad \frac{\overline{\Omega}_{11}}{\omega_\nu}=\mathfrak{a}^{\frac{1}{2}}\sin\theta_1,$$

$$\frac{\overline{E}_{12}}{\epsilon_n}=\mathfrak{b}^{\frac{1}{2}}\sin(\theta_2-\iota),\quad \frac{\overline{\Omega}_{12}}{\omega_\nu}=\mathfrak{b}^{\frac{1}{2}}\sin\theta_2,$$

$$\frac{\overline{E}_{22}}{\epsilon_n}=\mathfrak{c}^{\frac{1}{2}}\sin(\theta_3-\iota),\quad \frac{\overline{\Omega}_{22}}{\omega_\nu}=\mathfrak{c}^{\frac{1}{2}}\sin\theta_3.$$

Let these values of $\overline{E}_{ij}$ and $\overline{\Omega}_{ij}$ be substituted in the expressions for $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$; then, after simple reductions, we find

$$\mathfrak{f}=(\mathfrak{b}\mathfrak{c})^{\frac{1}{2}}\cos(\theta_2-\theta_3),\quad \mathfrak{g}=(\mathfrak{a}\mathfrak{c})^{\frac{1}{2}}\cos(\theta_3-\theta_1),\quad \mathfrak{h}=(\mathfrak{a}\mathfrak{b})^{\frac{1}{2}}\cos(\theta_1-\theta_2).$$

It will appear immediately that no two of the angles θ_1, θ_2, θ_3, can be equal; and as square roots of the positive minors $\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2$, $\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2$, $\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2$, will be required, we associate a positive sign with each of those real square roots. As the standard of reference, we assume

$$\theta_1>\theta_2>\theta_3;$$

and thus we have

$$(\mathfrak{b}\mathfrak{c})^{\frac{1}{2}}\sin(\theta_2-\theta_3)=(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}},$$

$$(\mathfrak{a}\mathfrak{c})^{\frac{1}{2}}\sin(\theta_1-\theta_3)=(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}},$$

$$(\mathfrak{a}\mathfrak{b})^{\frac{1}{2}}\sin(\theta_1-\theta_2)=(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}.$$

With the foregoing values obtained for $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, we find

$$\begin{aligned}\mathfrak{g}\mathfrak{h}-\mathfrak{a}\mathfrak{f}&=\ \ \mathfrak{a}(\mathfrak{b}\mathfrak{c})^{\frac{1}{2}}\{\cos(\theta_1-\theta_3)\cos(\theta_1-\theta_2)-\cos(\theta_2-\theta_3)\}\\ &=-\mathfrak{a}(\mathfrak{b}\mathfrak{c})^{\frac{1}{2}}\sin(\theta_3-\theta_1)\sin(\theta_2-\theta_1)\\ &=-(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}};\end{aligned}$$

and similarly

$$\mathfrak{h}\mathfrak{f}-\mathfrak{b}\mathfrak{g}=\quad(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}},$$
$$\mathfrak{f}\mathfrak{g}-\mathfrak{c}\mathfrak{h}=-(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}},$$

which are the standard relations * among the minors, all being valid in virtue of the relation $Y=0$.

In the second place, the values of P and Q are given by the two equations

$$\mathfrak{h}=P\mathfrak{a}+Q\mathfrak{g},\quad \mathfrak{f}=P\mathfrak{g}+Q\mathfrak{c},$$

so that

$$P=\frac{\mathfrak{c}\mathfrak{h}-\mathfrak{f}\mathfrak{g}}{\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2}=\frac{(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}}{(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}},$$
$$Q=\frac{\mathfrak{a}\mathfrak{f}-\mathfrak{g}\mathfrak{h}}{\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2}=\frac{(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}}{(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}};$$

and therefore the linear relation connecting the three typical magnitudes ξ_{11}, ξ_{12}, ξ_{13}, becomes

$$(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}\xi_{11}-(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}\xi_{12}+(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}\xi_{22}=0.$$

Ex. 1. Let the last equation be multiplied by l, and the result be added for all equations of which it is typical; then

$$(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}\bar{e}-(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}\bar{f}+(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}\bar{g}=0.$$

Ex. 2. When the equation is multiplied by ξ_{11}, and the results are added for all the equations, we have

$$\mathfrak{a}(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}-\mathfrak{h}(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}+\mathfrak{g}(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}=0;$$

and similarly

$$\mathfrak{h}(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}-\mathfrak{b}(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}+\mathfrak{f}(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}=0,$$
$$\mathfrak{g}(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}-\mathfrak{f}(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}+\mathfrak{c}(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}=0,$$

all valid in virtue of the single equation $Y=0$.

Ex. 3. By direct substitution, we have

$$\frac{1}{\epsilon_n\omega_\nu}(\bar{E}_{12}\bar{\Omega}_{22}-\bar{E}_{22}\bar{\Omega}_{12})=(\mathfrak{b}\mathfrak{c})^{\frac{1}{2}}\{\sin(\theta_2-\iota)\sin\theta_3-\sin(\theta_3-\iota)\sin\theta_2\}$$
$$=(\mathfrak{b}\mathfrak{c})^{\frac{1}{2}}\sin\iota\sin(\theta_2-\theta_3)=(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}\sin\iota;$$

and similarly

$$\frac{1}{\epsilon_n\omega_\nu}(\bar{E}_{22}\bar{\Omega}_{11}-\bar{E}_{11}\bar{\Omega}_{22})=-(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}\sin\iota,\quad \frac{1}{\epsilon_n\omega_\nu}(\bar{E}_{11}\bar{\Omega}_{12}-\bar{E}_{12}\bar{\Omega}_{11})=(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}\sin\iota.$$

* The results are similar to those in the corresponding investigation connected with the circular curvature of a geodesic on a surface existing freely in a quadruple homaloidal plenary space: see *G.F.D.*, vol i, p. 401.

Thus the relation (p. 523)

$$\begin{vmatrix} \xi_{11}, & \bar{E}_{11}, & \bar{\Omega}_{11} \\ \xi_{12}, & \bar{E}_{12}, & \bar{\Omega}_{12} \\ \xi_{22}, & \bar{E}_{22}, & \bar{\Omega}_{22} \end{vmatrix} = 0$$

becomes the preceding relation

$$(\mathfrak{bc} - \mathfrak{f}^2)^{\frac{1}{2}} \xi_{11} - (\mathfrak{ac} - \mathfrak{g}^2)^{\frac{1}{2}} \xi_{12} + (\mathfrak{ab} - \mathfrak{h}^2)^{\frac{1}{2}} \xi_{22} = 0.$$

Ex. 4. We had relations (p. 521)

$$\xi_{11} \sin \iota = \frac{\bar{E}_{11}}{\epsilon_n} y_\omega + \frac{\bar{\Omega}_{11}}{\omega_\nu} y_\epsilon$$

where

$$y_\epsilon \sin \iota = \frac{dy}{dn} \cos \iota - \frac{dy}{d\nu}, \quad y_\omega \sin \iota = \frac{dy}{d\nu} \cos \iota - \frac{dy}{dn}.$$

Let the relation be multiplied by l, using the equation

$$\frac{l}{\gamma} = g_\epsilon \frac{dy}{dn} + g_\omega \frac{dy}{d\nu}$$

for the evaluation of the right-hand side; then, as

$$\frac{1}{\gamma} \sum l y_\epsilon = -g_\omega \sin \iota, \quad \frac{1}{\gamma} \sum l y_\omega = -g_\epsilon \sin \iota,$$

we have

$$\frac{\bar{e}}{\gamma} = -\frac{\bar{E}_{11}}{\epsilon_n} g_\epsilon - \frac{\bar{\Omega}_{11}}{\omega_\nu} g_\omega;$$

and similarly

$$\frac{\bar{f}}{\gamma} = -\frac{\bar{E}_{12}}{\epsilon_n} g_\epsilon - \frac{\bar{\Omega}_{12}}{\omega_\nu} g_\omega,$$

$$\frac{\bar{g}}{\gamma} = -\frac{\bar{E}_{22}}{\epsilon_n} g_\epsilon - \frac{\bar{\Omega}_{22}}{\omega_\nu} g_\omega.$$

When regard is paid to the relations

$$g_\epsilon + g_\omega \cos \iota = \frac{1}{\gamma_\epsilon}, \quad g_\omega + g_\epsilon \cos \iota = \frac{1}{\gamma_\omega},$$

these results can be transformed to the expressions on p. 482.

Also the relations on p. 484 can be expressed in the form

$$\frac{L_0}{\rho_0} = \frac{\bar{A}_0}{\rho} + \frac{\bar{e}}{\gamma}, \quad \frac{M_0}{\rho_0} = \frac{\bar{H}_0}{\rho} + \frac{\bar{f}}{\gamma}, \quad \frac{N_0}{\rho_0} = \frac{\bar{B}_0}{\rho} + \frac{\bar{g}}{\gamma}.$$

Ex. 5. When the surface is referred (as in the text) to p and q as its parameters, these being original parameters of the domain, the reference can be made by taking r and t as two independent functions of p and q, so that we take

$$\epsilon = r(p, q) - r = 0, \quad \omega = t(p, q) - t = 0.$$

Denoting derivatives of $r(p, q)$ and $t(p, q)$ with respect to p and to q by a suffix 1 and a suffix 2, we have

$$\begin{array}{llll} \epsilon_1 = r_1, & \epsilon_2 = r_2, & \epsilon_3 = -1, & \epsilon_4 = 0, \\ \omega_1 = t_1, & \omega_2 = t_2, & \omega_3 = 0, & \omega_4 = -1, \end{array}$$

so that the superficial variables s are

$$\frac{s_{34}}{s_{12}} = r_1 t_2 - r_2 t_1, \quad \frac{s_{24}}{s_{12}} = -t_1, \quad \frac{s_{41}}{s_{12}} = -t_2, \quad \frac{s_{31}}{s_{12}} = -r_2, \quad \frac{s_{23}}{s_{12}} = -r_1.$$

Also

$$\bar{\epsilon}_{ij} = r_{ij} - r_1 \Gamma_{ij} - r_2 \Delta_{ij} + \Theta_{ij}, \quad \text{or} \quad = -r_1 \Gamma_{ij} - r_2 \Delta_{ij} + \Theta_{ij},$$

for $ij = 11$, 12, 22, in the first instance, and for $ij =$ any combination of the integers 1, 2, 3, 4, involving 3 or 4 or both 3 and 4; and similarly, for like combinations,

$$\bar{\omega}_{ij} = \omega_{ij} - t_1 \Gamma_{ij} - t_2 \Delta_{ij} + \Phi_{ij}, \quad \text{or} \quad = -t_1 \Gamma_{ij} - t_2 \Delta_{ij} + \Phi_{ij}.$$

Then we have

$$\begin{aligned} \bar{E}_{11} &= \bar{\epsilon}_{11} + 2r_1 \bar{\epsilon}_{13} + 2t_1 \bar{\epsilon}_{14} + r_1^2 \bar{\epsilon}_{33} + 2r_1 t_1 \bar{\epsilon}_{34} + t_1^2 \bar{\epsilon}_{44}, \\ \bar{E}_{12} &= \bar{\epsilon}_{12} + r_2 \bar{\epsilon}_{13} + r_1 \bar{\epsilon}_{23} + t_2 \bar{\epsilon}_{14} + t_1 \bar{\epsilon}_{24} + r_1 r_2 \bar{\epsilon}_{33} + (r_1 t_2 + r_2 t_1) \bar{\epsilon}_{34} + t_1 t_2 \bar{\epsilon}_{44}, \\ \bar{E}_{22} &= \bar{\epsilon}_{22} + 2r_2 \bar{\epsilon}_{23} + 2t_2 \epsilon_{24} + r_2^2 \epsilon_{33} + 2r_2 t_2 \epsilon_{34} + t_2^2 \epsilon_{44}; \end{aligned}$$

while $\bar{\Omega}_{11}$, $\bar{\Omega}_{12}$, $\bar{\Omega}_{22}$, are the same combinations of the foregoing magnitudes ω_{ij} as $\bar{E}_{11}$, $\bar{E}_{12}$, $\bar{E}_{22}$, are of the magnitudes $\bar{\epsilon}_{ij}$.

358. Substituting the values (p. 524) of $\bar{E}_{11}$, $\bar{E}_{12}$, $\bar{E}_{22}$, in the equation

$$-\frac{\epsilon_n}{\gamma_\epsilon} = \bar{E}_{11} p'^2 + 2\bar{E}_{12} p'q' + \bar{E}_{22} q'^2,$$

we have

$$\begin{aligned} -\frac{1}{\gamma_\epsilon} &= \mathfrak{a}^{\frac{1}{2}} p'^2 \sin(\theta_1 - \iota) + 2\mathfrak{b}^{\frac{1}{2}} p'q' \sin(\theta_2 - \iota) + \mathfrak{c}^{\frac{1}{2}} q'^2 \sin(\theta_3 - \iota) \\ &= \{\mathfrak{a}^{\frac{1}{2}} p'^2 + 2\mathfrak{b}^{\frac{1}{2}} p'q' \cos(\theta_1 - \theta_2) + \mathfrak{c}^{\frac{1}{2}} q'^2 \cos(\theta_1 - \theta_3)\} \sin(\theta_1 - \iota) \\ &\qquad - \{2\mathfrak{b}^{\frac{1}{2}} p'q' \sin(\theta_1 - \theta_2) + \mathfrak{c}^{\frac{1}{2}} q'^2 \sin(\theta_1 - \theta_3)\} \cos(\theta_1 - \iota). \end{aligned}$$

On the right-hand side, the coefficient of $\sin(\theta_1 - \iota)$

$$= \frac{1}{\mathfrak{a}^{\frac{1}{2}}} (\mathfrak{a} p'^2 + 2\mathfrak{h} p'q' + \mathfrak{g} q'^2) = \frac{1}{\mathfrak{a}^{\frac{1}{2}}} \frac{\bar{e}}{\gamma};$$

the coefficient of $\cos(\theta_1 - \iota)$ is $T_1/\mathfrak{a}^{\frac{1}{2}}$, where

$$T_1 = -2(\mathfrak{a}\mathfrak{b} - \mathfrak{h}^2)^{\frac{1}{2}} p'q' - (\mathfrak{a}\mathfrak{c} - \mathfrak{g}^2)^{\frac{1}{2}} q'^2;$$

and therefore

$$-\frac{\mathfrak{a}^{\frac{1}{2}}}{\gamma_\epsilon} = \frac{\bar{e}}{\gamma} \sin(\theta_1 - \iota) + T_1 \cos(\theta_1 - \iota).$$

Proceeding similarly from the value of γ_ω, we find

$$-\frac{\mathfrak{a}^{\frac{1}{2}}}{\gamma_\omega}=\frac{\bar{e}}{\gamma}\sin\theta_1+T_1\cos\theta_1.$$

When these two relations are resolved for $\bar{e}/\gamma$ and T_1, we have

$$\frac{\bar{e}}{\gamma}\sin\iota=-\mathfrak{a}^{\frac{1}{2}}\left\{-\frac{\cos\theta_1}{\gamma_\epsilon}+\frac{\cos(\theta_1-\iota)}{\gamma_\omega}\right\},$$

$$T_1\sin\iota=-\mathfrak{a}^{\frac{1}{2}}\left\{-\frac{\sin\theta_1}{\gamma_\epsilon}+\frac{\sin(\theta_1-\iota)}{\gamma_\omega}\right\}.$$

It is easy to verify that

$$\left(\frac{\bar{e}}{\gamma}\right)^2+T_1^2=\frac{\mathfrak{a}}{\gamma^2},$$

the three results being in accord because of the relation

$$\frac{\sin^2\iota}{\gamma^2}=\frac{1}{\gamma_\epsilon^2}-\frac{2\cos\iota}{\gamma_\epsilon\gamma_\omega}+\frac{1}{\gamma_\omega^2}.$$

Proceeding from the same initial equation, and retaining (instead of θ_1) the quantities θ_2 and θ_3 in turn, we find similar expressions for $\bar{f}$ and $\bar{g}$, with modified two-term expressions T_2 and T_3: the values are

$$-\frac{\mathfrak{b}^{\frac{1}{2}}}{\gamma_\epsilon}=\frac{\bar{f}}{\gamma}\sin(\theta_2-\iota)+T_2\cos(\theta_2-\iota),$$

$$-\frac{\mathfrak{c}^{\frac{1}{2}}}{\gamma_\epsilon}=\frac{\bar{g}}{\gamma}\sin(\theta_3-\iota)+T_3\cos(\theta_3-\iota),$$

$$-\frac{\mathfrak{b}^{\frac{1}{2}}}{\gamma_\omega}=\frac{\bar{f}}{\gamma}\sin\theta_2+T_2\cos\theta_2,$$

$$-\frac{\mathfrak{c}^{\frac{1}{2}}}{\gamma_\omega}=\frac{\bar{g}}{\gamma}\sin\theta_3+T_3\cos\theta_3,$$

where

$$T_2=(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}p'^2-\ (\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}q'^2,$$

$$T_3=(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}p'^2+2(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}p'q'.$$

Moreover, the values of T_1, T_2, T_3, satisfy the relation

$$T_1p'^2+2T_2p'q'+T_3q'^2=0,$$

while

$$\left(\frac{\bar{f}}{\gamma}\right)^2+T_2^2=\frac{\mathfrak{b}}{\gamma^2},\quad \left(\frac{\bar{g}}{\gamma}\right)^2+T_3^2=\frac{\mathfrak{c}}{\gamma^2};$$

and we have

$$T_1p'+T_2q'=-q'W,\quad T_2p'+T_3q'=p'W,$$

where

$$W=(\mathfrak{a}\mathfrak{b}-\mathfrak{h}^2)^{\frac{1}{2}}p'^2+(\mathfrak{a}\mathfrak{c}-\mathfrak{g}^2)^{\frac{1}{2}}p'q'+(\mathfrak{b}\mathfrak{c}-\mathfrak{f}^2)^{\frac{1}{2}}q'^2,$$

a quantity that will occur in the expression for the domainal tilt of the superficial geodesic. And, by using the results of *Ex.* 3, p. 525, we have

$$\frac{1}{\epsilon_n \omega_\nu} \begin{vmatrix} \bar{E}_{11}p' + \bar{E}_{12}q', & \bar{E}_{12}p' + \bar{E}_{22}q' \\ \bar{\Omega}_{11}p' + \bar{\Omega}_{12}q', & \bar{\Omega}_{12}p' + \bar{\Omega}_{22}q' \end{vmatrix} = W \sin \iota.$$

Domainal torsion and tilt of a geodesic on a domainal surface.

359. To estimate the domainal flexural curvatures, we require a frame of four lines (as the domain is four-dimensional) which, orthogonal to one another, shall constitute an organic frame for a superficial geodesic. Of these, two must be gremial to the surface; they are the tangent of the geodesic with a typical direction-cosine y', and the binormal of that geodesic with a typical direction-cosine λ_3. The other two lines, within the tangent block of the domain, must lie in the orthogonal plane of the geodesic represented by the equations

$$\left\| \bar{y} - y, \quad \frac{dy}{dn}, \quad \frac{dy}{d\nu} \right\| = 0.$$

One of these two is the direction of the radius of domainal flexure, with a typical direction-cosine l such that

$$\frac{l}{\gamma} = g_\epsilon \frac{dy}{dn} + g_\omega \frac{dy}{d\nu},$$

where

$$g_\epsilon + g_\omega \cos \iota = \frac{1}{\gamma_\epsilon}, \quad g_\omega + g_\epsilon \cos \iota = \frac{1}{\gamma_\omega}.$$

Let m denote the typical direction-cosine of the remaining line, lying in this orthogonal plane and at right angles to the radius of domainal flexure. We easily find

$$\frac{m}{\gamma} \sin \iota = \frac{1}{\gamma_\omega} \frac{dy}{dn} - \frac{1}{\gamma_\epsilon} \frac{dy}{d\nu},$$

$$\sum m\bar{y}_1 = 0, \quad \sum m\bar{y}_2 = 0;$$

and this line can be regarded as a trinormal within the domainal range. Let $1/\sigma_D$ and $1/\tau_D$ denote the domainal torsion and the domainal tilt of a superficial geodesic: then Frenet domainal equations for a superficial geodesic are

$$\frac{dy'}{ds'} = \frac{l}{\gamma},$$

$$\frac{dl}{ds'} = \frac{\lambda_3}{\sigma_D} - \frac{y'}{\gamma},$$

$$\frac{d\lambda_3}{ds'} = \frac{m}{\tau_D} - \frac{l}{\sigma_D},$$

where d/ds' denotes differentiation along the superficial geodesic and is the same as d/ds only when applied to magnitudes of position. We proceed to find expressions for the domainal torsion and the domainal tilt which, in the first instance, are given by

$$\frac{1}{\sigma_D} = \sum \lambda_3 \frac{dl}{ds'},$$

$$\frac{1}{\tau_D} = \sum m \frac{d\lambda_3}{ds'} = -\sum \lambda_3 \frac{dm}{ds'},$$

because of the relation $\sum m\lambda_3 = 0$.

From the equation

$$\frac{l}{\gamma} = g_\epsilon \frac{dy}{dn} + g_\omega \frac{dy}{d\nu},$$

we have

$$\frac{1}{\gamma}\frac{dl}{ds'} + l\frac{d}{ds'}\left(\frac{1}{\gamma}\right) = g_\epsilon \frac{d}{ds'}\left(\frac{dy}{dn}\right) + g_\omega \frac{d}{ds'}\left(\frac{dy}{d\nu}\right) + \frac{dy}{dn}\frac{dg_\epsilon}{ds'} + \frac{dy}{d\nu}\frac{dg_\omega}{ds'}.$$

Now

$$\sum \lambda_3 l = 0, \quad \sum \lambda_3 \frac{dy}{dn} = 0, \quad \sum \lambda_3 \frac{dy}{d\nu},$$

because the binormal of the superficial geodesic is at right angles to the orthogonal plane; and the first arc-derivatives of $\frac{dy}{dn}$ and $\frac{dy}{d\nu}$ along the superficial geodesic are the same as those taken along the domainal geodesic. Hence, multiplying the equation throughout by λ_3, and adding for the range of the plenary homaloidal space, we have

$$\frac{1}{\gamma\sigma_D} = g_\epsilon \sum \left\{\lambda_3 \frac{d}{ds}\left(\frac{dy}{dn}\right)\right\} + g_\omega \sum \left\{\lambda_3 \frac{d}{ds}\left(\frac{dy}{d\nu}\right)\right\}.$$

Similarly, from the equation

$$\frac{m}{\gamma} = \frac{1}{\gamma_\omega}\frac{dy}{dn} - \frac{1}{\gamma_\epsilon}\frac{dy}{d\nu},$$

we have

$$\left\{\frac{1}{\gamma}\frac{dm}{ds'} + m\frac{d}{ds'}\left(\frac{1}{\gamma}\right)\right\}\sin\iota = \frac{1}{\gamma_\omega}\frac{d}{ds'}\left(\frac{dy}{dn}\right) - \frac{1}{\gamma_\epsilon}\frac{d}{ds'}\left(\frac{dy}{d\nu}\right) + \frac{dy}{dn}\frac{d}{ds'}\left(\frac{1}{\gamma_\omega}\right) - \frac{dy}{d\nu}\frac{d}{ds'}\left(\frac{1}{\gamma_\epsilon}\right).$$

We have $\sum \lambda_3 m = 0$; and therefore, multiplying by λ_3 and proceeding as before, we have

$$-\frac{\sin\iota}{\gamma\tau_D} = \frac{1}{\gamma_\omega}\sum\left\{\lambda_3 \frac{d}{ds}\left(\frac{dy}{dn}\right)\right\} - \frac{1}{\gamma_\epsilon}\sum\left\{\lambda_3 \frac{d}{ds}\left(\frac{dy}{d\nu}\right)\right\}.$$

Now, as in § 345, we have

$$-\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu \sin\iota \sum\left\{\lambda_3 \frac{d}{ds}\left(\frac{dy}{dn}\right)\right\} = \frac{1}{\epsilon_n}\Delta_{\bar{\epsilon}},$$

$$-\Omega^{\frac{1}{2}}\epsilon_n\omega_\nu \sin\iota \sum\left\{\lambda_3 \frac{d}{ds}\left(\frac{dy}{d\nu}\right)\right\} = \frac{1}{\omega_\nu}\Delta_{\bar{\omega}},$$

where

$$\Delta_{\bar{\epsilon}} = \begin{vmatrix} \bar{\epsilon}_1, & \bar{\epsilon}_2, & \bar{\epsilon}_3, & \bar{\epsilon}_4 \\ u_1, & u_2, & u_3, & u_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4 \\ \omega_1, & \omega_2, & \omega_3, & \omega_4 \end{vmatrix};$$

with a like expression for $\Delta_{\bar{\omega}}$. But

$$\left.\begin{aligned} \bar{E}_{11}p' + \bar{E}_{12}q' &= \bar{\epsilon}_1 - \frac{s_{23}}{s_{12}}\bar{\epsilon}_3 - \frac{s_{24}}{s_{12}}\bar{\epsilon}_4 \\ \bar{E}_{12}p' + \bar{E}_{22}q' &= \bar{\epsilon}_2 - \frac{s_{31}}{s_{12}}\bar{\epsilon}_3 - \frac{s_{41}}{s_{12}}\bar{\epsilon}_4 \end{aligned}\right\},$$

$$\left.\begin{aligned} A_0p' + H_0q' &= u_1 - \frac{s_{23}}{s_{12}}u_3 - \frac{s_{24}}{s_{12}}u_4 \\ H_0p' + B_0q' &= u_2 - \frac{s_{31}}{s_{12}}u_3 - \frac{s_{41}}{s_{12}}u_4 \end{aligned}\right\},$$

and therefore, in the notation of p. 485,

$$\Delta_\epsilon = (\epsilon_3\omega_4 - \epsilon_4\omega_3)\begin{vmatrix} \bar{E}_{11}p' + \bar{E}_{12}q', & \bar{E}_{12}p' + \bar{E}_{22}q' \\ A_0p' + H_0q', & H_0p' + B_0q' \end{vmatrix} = (\epsilon_3\omega_4 - \epsilon_4\omega_3)R_\epsilon.$$

Similarly

$$\Delta_{\bar{\omega}} = (\epsilon_3\omega_4 - \epsilon_4\omega_3)\begin{vmatrix} \bar{\Omega}_{11}p' + \bar{\Omega}_{12}q', & \bar{\Omega}_{12}p' + \bar{\Omega}_{22}q' \\ A_0p' + H_0q', & H_0p' + B_0q' \end{vmatrix} = (\epsilon_3\omega_4 - \epsilon_4\omega_3)R_\omega.$$

Also

$$V_0(\epsilon_3\omega_4 - \epsilon_4\omega_3) = \Omega^{\frac{1}{2}}\epsilon_n\omega_\nu \sin\iota;$$

and therefore

$$-\frac{V_0}{\gamma\sigma_D} = \frac{1}{\epsilon_n}g_\epsilon R_\epsilon + \frac{1}{\omega_\nu}g_\omega R_\omega,$$

$$-\frac{V_0\sin\iota}{\gamma\tau_D} = \frac{1}{\omega_\nu}\frac{1}{\gamma_\epsilon}R_\omega - \frac{1}{\epsilon_n}\frac{1}{\gamma_\omega}R_\epsilon.$$

In the equation for σ_D, the coefficient of $H_0p' + B_0q'$ on the right-hand side

$$= \frac{1}{\epsilon_n}g_\epsilon(\bar{E}_{11}p' + \bar{E}_{12}q') + \frac{1}{\omega_\nu}g_\omega(\bar{E}_{12}p' + \bar{E}_{22}q')$$

$$= -\frac{1}{\gamma}(\bar{e}p' + \bar{f}q'),$$

by the use of *Ex.* 4, p. 526; and, similarly, the coefficient of $A_0 p' + H_0 q'$

$$= +\frac{1}{\gamma}(\bar{f}p' + \bar{g}q');$$

and thus the equation for the domainal torsion of a superficial geodesic becomes

$$\frac{V_0}{\sigma_D} = \begin{vmatrix} \bar{e}p' + \bar{f}q', & \bar{f}p' + \bar{g}q' \\ A_0 p' + H_0 q', & H_0 p' + B_0 q' \end{vmatrix}.$$

Consequently, also,

$$\frac{V_0}{\gamma\sigma_D} = \begin{vmatrix} \mathfrak{a}p'^3 + 3\mathfrak{h}p'^2 q' + 3\mathfrak{k}p'q'^2 + \mathfrak{f}q'^3, & A_0 p' + H_0 q' \\ \mathfrak{h}p'^3 + 3\mathfrak{k}p'^2 q' + 3\mathfrak{f}p'q'^2 + \mathfrak{c}q'^3, & H_0 p' + B_0 q' \end{vmatrix}.$$

Again, in the equation for τ_D, the coefficient of $H_0 p' + B_0 q'$ on the right-hand side

$$= \frac{1}{\omega_\nu \gamma_\epsilon}(\bar{\Omega}_{11} p' + \bar{\Omega}_{12} q') - \frac{1}{\epsilon_n \gamma_\omega}(\bar{E}_{11} p' + \bar{E}_{12} q')$$

$$= \frac{q'}{\epsilon_n \omega_\nu} \begin{vmatrix} \bar{E}_{11} p' + \bar{E}_{12} q', & \bar{E}_{12} p' + \bar{E}_{22} q' \\ \bar{\Omega}_{11} p' + \bar{\Omega}_{12} q', & \bar{\Omega}_{12} p' + \bar{\Omega}_{22} q' \end{vmatrix}$$

$$= Wq' \sin\iota,$$

by the result on p. 529; and similarly the coefficient of $-(A_0 p' + H_0 q')$ on the same right-hand side is

$$= -Wp' \sin\iota.$$

We therefore have

$$-\frac{V_0}{\gamma\tau_D} = W\{(H_0 p' + B_0 q')q' + (A_0 p' + H_0 q')p'\}$$

$$= W$$

$$= (\mathfrak{a}\mathfrak{b} - \mathfrak{h}^2)^{\frac{1}{2}} p'^2 + (\mathfrak{a}\mathfrak{c} - \mathfrak{g}^2)^{\frac{1}{2}} p'q' + (\mathfrak{b}\mathfrak{c} - \mathfrak{f}^2)^{\frac{1}{2}} q'^2.$$

All these expressions, for the domainal flexure, the domainal torsion, and the domainal tilt, of any geodesic on a surface in the domain, are formally similar to the corresponding expressions for the circular curvature, the torsion, and the tilt, of a geodesic on a surface existing freely in four-dimensional homaloidal space.*

360. One other property affecting the relation between the spatial torsion and the domainal torsion of a superficial geodesic may be established here.

In connection with the domainal surface, we have obtained (pp. 484, 526) the relations

$$\frac{L_0}{\rho_0} = \frac{\bar{A}_0}{\rho} + \frac{\bar{e}}{\gamma}, \quad \frac{M_0}{\rho_0} = \frac{\bar{H}_0}{\rho} + \frac{\bar{f}}{\gamma}, \quad \frac{N_0}{\rho_0} = \frac{\bar{B}_0}{\rho} + \frac{\bar{g}}{\gamma},$$

* *G.F.D.*, vol. i, §§ 215, 234, 235.

where L_0, M_0, N_0, are secondary magnitudes for the circular curvature of the superficial geodesic; $\bar{e}$, $\bar{f}$, $\bar{g}$, have the same relation to the domainal flexure of that geodesic as L_0, M_0, N_0, to its circular curvature; and $\bar{A}_0$, $\bar{H}_0$, $\bar{B}_0$, are corresponding magnitudes for the circular curvature of the domainal geodesic in the same direction.

Now consider the surface, which is geodesic to the region at O; these quantities $\bar{A}_0$, $\bar{H}_0$, $\bar{B}_0$, are the secondary magnitudes for the circular curvature of its geodesic (being the domainal geodesic tangent), so that

$$\frac{1}{\rho}=\bar{A}_0 p'^2+2\bar{H}_0 p'q'+\bar{B}_0 q'^2.$$

Let the torsion of the geodesic on this geodesic surface, which is the torsion of the domainal geodesic tangent, be denoted by $1/\sigma_{\epsilon 1}$, so that

$$\frac{V_0}{\sigma_{\epsilon 1}}=\begin{vmatrix} \bar{A}_0 p'+\bar{H}_0 q', & \bar{H}_0 p'+\bar{B}_0 q' \\ A_0 p'+H_0 q', & H_0 p'+B_0 q' \end{vmatrix}.$$

Then, because

$$\frac{V_0}{\sigma_0}=\begin{vmatrix} L_0 p'+M_0 q', & M_0 p'+N_0 q' \\ A_0 p'+H_0 q', & H_0 p'+B_0 q' \end{vmatrix}, \quad \frac{V_0}{\sigma_D}=\begin{vmatrix} \bar{e}p' +\bar{f}q', & \bar{f}p' +\bar{g}q' \\ A_0 p'+H_0 q', & H_0 p'+B_0 q' \end{vmatrix},$$

the foregoing set of relations leads to the result

$$\frac{1}{\rho_0\sigma_0}=\frac{1}{\rho\sigma_{\epsilon 1}}+\frac{1}{\gamma\sigma_D},$$

which is the property in question: and it can be expressed in the form

$$\frac{1}{\sigma_0}=\frac{\cos\psi_0}{\sigma_{\epsilon 1}}+\frac{\sin\psi_0}{\sigma_D},$$

where ψ_0 has the same significance as on p. 469.

Curves of domainal flexure on a surface.

361. We define the curves of domainal flexure on the surface to be those of which the superficial geodesic tangents have a maximum or a minimum value for the domainal flexure, among the values for all superficial directions through the point. To obtain their directions and also an equation for those principal values of the domainal flexure, we assign the critical equations for a maximum or a minimum value of $1/\gamma^2$, where

$$\frac{1}{\gamma^2}=\mathfrak{a}p'^4+4\mathfrak{h}p'^3q'+6\mathfrak{k}p'^2q'^2+4\mathfrak{f}p'q'^3+\mathfrak{c}q'^4,$$

the variables p' and q' being subject to the condition

$$A_0 p'^2+2H_0 p'q'+B_0 q'^2=1.$$

We have

$$\frac{\partial}{\partial p'}\left(\frac{1}{\gamma^2}\right)=4\,(\mathfrak{a}p'^3+3\mathfrak{h}p'^2q'+3\mathfrak{f}p'q'^2+\mathfrak{f}q'^3)$$

$$=4\left(\frac{\bar{e}}{\gamma}p'+\frac{\bar{f}}{\gamma}q'\right),$$

and therefore

$$\frac{\partial}{\partial p'}\left(\frac{1}{\gamma}\right)=2\,(\bar{e}p'+\bar{f}q')\,;$$

and similarly

$$\frac{\partial}{\partial q'}\left(\frac{1}{\gamma}\right)=2\,(\bar{f}p'+\bar{g}q').$$

Thus the critical equations can be taken in the form

$$\bar{e}p'+\bar{f}q'=\lambda(A_0p'+H_0q'),\quad \bar{f}p'+\bar{g}q'=\lambda(H_0p'+B_0q'),$$

the quantity λ being undetermined in the formation of the critical equations. Multiplying the two equations by p' and q' respectively, and adding, we obtain $1/\gamma$ as the value of λ; and therefore

$$\bar{e}p'+\bar{f}q'=\frac{1}{\gamma}(A_0p'+H_0q'),$$

$$\bar{f}p'+\bar{g}q'=\frac{1}{\gamma}(H_0p'+B_0q'),$$

are the two equations which determine the directions and the magnitudes of the principal values.

In the first place, there is a descriptive property of the curves; the elimination of γ leads to the result

$$\begin{vmatrix} \bar{e}p'+\bar{f}q', & A_0p'+H_0q' \\ \bar{f}p'+\bar{g}q', & H_0p'+B_0q' \end{vmatrix}=0.$$

Consequently, we have (§ 359)

$$\frac{1}{\sigma_D}=0:$$

that is, the domainal torsion of superficial geodesic tangents to a curve of domainal flexure vanishes.

In the next place, if we denote by U_0 the quartic expression in p', q', which is the value of $1/\gamma^2$, and by U the quantity

$$U_0-\frac{1}{\gamma^2}(A_0p'^2+2H_0p'q'+B_0q'^2)^2,$$

the two critical equations are

$$\frac{\partial U}{\partial p'}=0,\quad \frac{\partial U}{\partial q'}=0:$$

that is, the discriminant of U_0, regarded as a binary quartic in p', q', is to vanish. We write

$$A_0^2=\mathfrak{a}_0,\ A_0H_0=\mathfrak{h}_0,\ A_0B_0+2H_0^2=3\mathfrak{k}_0,\ B_0H_0=\mathfrak{f}_0,\ B_0^2=\mathfrak{c}_0,\ A_0B_0=\mathfrak{g}_0,\ H_0^2=\mathfrak{b}_0\,;$$

and also

$$\mathfrak{a}-\frac{\mathfrak{a}_0}{\gamma^2}=\alpha_0,\quad \mathfrak{h}-\frac{\mathfrak{h}_0}{\gamma^2}=\alpha_1,\quad \mathfrak{k}-\frac{\mathfrak{k}_0}{\gamma^2}=\alpha_2,\quad \mathfrak{f}-\frac{\mathfrak{f}_0}{\gamma^2}=\alpha_3,\quad \mathfrak{c}-\frac{\mathfrak{c}_0}{\gamma^2}=\alpha_4\,;$$

then

$$U=(\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \between p',\ q')^4.$$

Let I and J denote the quadrinvariant and the cubinvariant of U, so that

$$I=\alpha_0\alpha_4-4\alpha_1\alpha_3+3\alpha_2^2,$$

$$J=\begin{vmatrix} \alpha_0, & \alpha_1, & \alpha_2 \\ \alpha_1, & \alpha_2, & \alpha_3 \\ \alpha_2, & \alpha_3, & \alpha_4 \end{vmatrix}=\alpha_0\alpha_2\alpha_4+2\alpha_1\alpha_2\alpha_3-\alpha_0\alpha_3^2-\alpha_4\alpha_1^2-\alpha_2^3.$$

The discriminant is I^3-27J^2; and thus the equation, giving the principal values of γ, is

$$I^3-27J^2=0.$$

To develop this equation, we denote by I_0 and J_0 the quadrinvariant and the cubinvariant of U_0 alone, so that

$$I_0=\mathfrak{a}\mathfrak{c}-4\mathfrak{h}\mathfrak{f}+3\mathfrak{k}^2,$$

$$J_0=\mathfrak{a}\mathfrak{k}\mathfrak{c}+2\mathfrak{h}\mathfrak{k}\mathfrak{f}-\mathfrak{a}\mathfrak{f}^2-\mathfrak{c}\mathfrak{h}^2-\mathfrak{k}^3.$$

Now the quantities $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, are subject to the condition

$$0=\begin{vmatrix} \mathfrak{a}, & \mathfrak{h}, & \mathfrak{g} \\ \mathfrak{h}, & \mathfrak{b}, & \mathfrak{f} \\ \mathfrak{g}, & \mathfrak{f}, & \mathfrak{c} \end{vmatrix}=\mathfrak{a}\mathfrak{b}\mathfrak{c}+2\mathfrak{f}\mathfrak{g}\mathfrak{h}-\mathfrak{a}\mathfrak{f}^2-\mathfrak{b}\mathfrak{g}^2-\mathfrak{c}\mathfrak{h}^2\,;$$

hence taking a quantity θ, defined by

$$3\theta=\mathfrak{g}-\mathfrak{b},$$

so that $V_0^2K_F=3\theta$, where K_F is the superficial measure of domainal flexure considered in § 355, we find

$$J_0=\theta I_0-4\theta^3.$$

Also we take intermediate invariants I_1 and J_1, according to the definitions

$$\begin{aligned} I_1&=\mathfrak{c}_0\mathfrak{a}-4\mathfrak{f}_0\mathfrak{h}+6\mathfrak{k}_0\mathfrak{k}-4\mathfrak{h}_0\mathfrak{f}+\mathfrak{a}_0\mathfrak{c} \\ &=B_0^2\mathfrak{a}-4H_0B_0\mathfrak{h}+2(A_0B_0+2H_0^2)\mathfrak{k}-4A_0B_0\mathfrak{f}+A_0^2\mathfrak{c}, \end{aligned}$$

being

$$\tfrac{1}{24}\left(B_0\frac{\partial^2}{\partial p'^2}-2H_0\frac{\partial^2}{\partial p'\partial q'}+A_0\frac{\partial^2}{\partial q'^2}\right)^2 U_0\,;$$

and

$$J_1 = \left(\mathfrak{a}_0 \frac{\partial}{\partial \mathfrak{a}} + \mathfrak{h}_0 \frac{\partial}{\partial \mathfrak{h}} + \mathfrak{k}_0 \frac{\partial}{\partial \mathfrak{k}} + \mathfrak{f}_0 \frac{\partial}{\partial \mathfrak{f}} + \mathfrak{c}_0 \frac{\partial}{\partial \mathfrak{c}}\right) J_0,$$

an equivalent value of which is

$$J_1 = \theta I_1 + \tfrac{1}{3} V_0^2 (I_0 - 12\theta^2) + \frac{1}{V_0^2} \{A_0(\mathfrak{bc} - \mathfrak{f}^2)^{\frac{1}{2}} - H_0(\mathfrak{ac} - \mathfrak{g}^2)^{\frac{1}{2}} + B_0(\mathfrak{ab} - \mathfrak{h}^2)^{\frac{1}{2}}\}^2.$$

Then, after substitution, we find

$$I = I_0 - I_1 \frac{1}{\gamma^2} + \tfrac{4}{3} V_0^4 \frac{1}{\gamma^4},$$

$$J = J_0 - J_1 \frac{1}{\gamma^2} + \tfrac{1}{3} V_0^2 I_1 \frac{1}{\gamma^4} - \tfrac{8}{27} V_0^6 \frac{1}{\gamma^6};$$

and after further substitution, the vanishing discriminant $I^3 - 27J^2$ yields the equation

$$D_0 \frac{1}{\gamma^8} - D_1 \frac{1}{\gamma^6} + D_2 \frac{1}{\gamma^4} - D_3 \frac{1}{\gamma^2} + D_4 = 0,$$

where

$$\begin{aligned}
D_0 &= V_0^4 (I_1^2 - 16 V^2 J_1 + \tfrac{16}{3} V_0^4 I_0),\\
D_1 &= I_1^3 - 18 V_0^2 I_1 J_1 + 8 V_0^4 I_0 I_1 - 16 V_0^6 J_0,\\
D_2 &= 3 I_0 I_1^2 - 18 V_0^2 I_1 J_0 - 27 J_1^2 + 4 V_0^4 I_0^2,\\
D_3 &= 3 I_0^2 I_1 - 54 J_0 J_1,\\
D_4 &= I_0^3 - 27 J_0^2,
\end{aligned}$$

the quantity D_4 being the discriminant of U_0 alone.

The four quantities D_μ / D_0, for $\mu = 1, 2, 3, 4$, give measures of the squares of the domainal flexure of superficial geodesics, this domainal flexure being a linear measure. Thus the product of the four principal domainal flexures is $(D_4/D_0)^{\frac{1}{2}}$; and similarly for other measures.

It is again to be noted that, when the domain becomes a (homaloidal) block, the domainal flexure of a superficial geodesic becomes its circular curvature; and the foregoing results attach themselves to the principal values of that circular curvature *.

Ex. Verify the transformations

(i) $D_4 = (I_0 - 3\theta^2)(I_0 - 12\theta^2)^2$;

(ii) $D_0 V_0^{-4} = (I_1 - 8 V_0^2 \theta)^2 - 16 V_0^2 \{A_0(\mathfrak{bc} - \mathfrak{f}^2)^{\frac{1}{2}} - H_0(\mathfrak{ac} - \mathfrak{g}^2)^{\frac{1}{2}} + B_0(\mathfrak{ab} - \mathfrak{h}^2)^{\frac{1}{2}}\}^2$.

Asymptotic curves of flexure.

362. We can define, for a domainal surface, asymptotic curves of domainal flexure (being the domainal counterpart of asymptotic curves on a surface existing freely in a four-dimensional homaloidal plenary space) as the curves, the super-

* *G.F.D.*, vol. i, §§ 238-240.

ficial geodesic tangents along which have zero domainal flexure. The direction-variables of these domainal asymptotic curves are obtained by assigning a zero value to $1/\gamma^2$, and therefore satisfy the equation

$$\mathfrak{a}p'^4+4\mathfrak{h}p'^3q'+6\mathfrak{k}p'^2q'^2+4\mathfrak{f}p'q'^3+\mathfrak{c}q'^3=0.$$

The roots of this quartic may all be real, and then there are four real asymptotic curves through a point on the surface; or two may be real, and the other two complex (and conjugate to one another), so that there are two real asymptotic curves; or all four roots may be complex (in conjugate pairs), and then there are no real asymptotic curves.

To resolve the equation, let $p'=xq'$, so that x is a root of

$$\mathfrak{a}x^4+4\mathfrak{h}x^3+6\mathfrak{k}x^2+4\mathfrak{f}x+\mathfrak{c}=0.$$

Let x_1, x_2, x_3, x_4, denote the roots, and let t be taken so that

$$t=x_1x_2+x_3x_4,$$

so that t is a three-valued function. Then, if

$$\mathfrak{a}t-2\mathfrak{k}=4v,$$

the customary calculations give

$$\mathfrak{a}(x_1+x_2)(x_3+x_4)=4(\mathfrak{k}-v),$$

while v satisfies the equation

$$J_0=vI_0-4v^3,$$

where I_0 and J_0 are the quadrinvariant and the cubinvariant of the quartic in x, being the magnitudes of p. 535. There is the relation

$$J_0=\theta I_0-4\theta^3;$$

and therefore the three values of v are

$$v=\theta,\quad v=-\tfrac{1}{2}(\theta-T),\quad v=-\tfrac{1}{2}(\theta+T),$$

where

$$\theta=\tfrac{1}{3}(\mathfrak{g}-\mathfrak{b}),\quad T=(I_0-3\theta^2)^{\frac{1}{2}}.$$

Then, corresponding to the value $v=\theta$, we find

$$\mathfrak{a}(x_1x_2+x_3x_4)=2\mathfrak{g},\quad \mathfrak{a}(x_1+x_2)(x_3+x_4)=4\mathfrak{b};$$

to the value $v=-\tfrac{1}{2}(\theta-T)$,

$$\mathfrak{a}(x_1x_3+x_2x_4)=2\mathfrak{b}+2T,\quad \mathfrak{a}(x_1+x_3)(x_2+x_4)=2\mathfrak{b}+2\mathfrak{g}-2T;$$

and to the value

$$v=-\tfrac{1}{2}(\theta+T),$$

$$\mathfrak{a}(x_2x_3+x_1x_4)=2\mathfrak{b}-2T,\quad \mathfrak{a}(x_2+x_3)(x_1+x_4)=2\mathfrak{b}+2\mathfrak{g}+2T.$$

The combination of the first set of values with the relations

$$\mathfrak{a}x_1x_2x_3x_4=\mathfrak{c}, \quad \mathfrak{a}(x_1+x_2+x_3+x_4)=-4\mathfrak{h},$$

gives

$$\left.\begin{array}{ll} \mathfrak{a}x_1x_2=\mathfrak{g}+i(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}, & \mathfrak{a}(x_1+x_2)=-2\mathfrak{h}-2i(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}} \\ \mathfrak{a}x_3x_4=\mathfrak{g}-i(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}, & \mathfrak{a}(x_3+x_4)=-2\mathfrak{h}+2i(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}} \end{array}\right\},$$

where the initial attachments of signs in the expressions for x_1x_2 and x_3x_4 are obviously at our choice, and then the attachments of signs in the expressions for x_1+x_2 and x_3+x_4 are a consequence of the standard reference for the signs in § 357 in relation to the minors of the vanishing determinant Y.

Corresponding results, which involve also the magnitude T, can be deduced from the other two sets of values in connection with the same set of relations for the completely symmetrical combinations $x_1x_2x_3x_4$ and $x_1+x_2+x_3+x_4$; but, for each set of results, it is necessary to select the attachments of signs so as to conform to the same standard of reference. The resulting equivalences in the expressions for x_1, x_2, x_3, x_4, are satisfied, either in virtue of $Y=0$ or else identically.

Orthogonal centre of a geodesic on a domainal surface.

363. Connected with any point on a domainal surface, there are two loci which arise through the aggregate of superficial geodesics passing through the point. One of these is the locus of the orthogonal centre connected with a geodesic; the other is the locus of the flexural centre.

The orthogonal centre of any superficial geodesic is defined to be the limiting position of the point of intersection of two domainal orthogonal planes of the surface drawn at consecutive points along the geodesic. As guiding lines of the orthogonal plane, we selected the domainal normals of the two regions in the domain which, by their intersection, provide the surface; then, later (§ 359), there were two organic lines of the superficial geodesic lying in that plane, one being the radius of domainal flexure with a typical direction-cosine l, the other being the domainal trinormal (p. 529) of that geodesic, with a typical direction-cosine denoted by m. Hence the typical equation of the orthogonal plane has the form

$$\bar{y}-y=l\alpha+m\beta,$$

where α and β are parameters of the plane.

Consider the ultimate intersection of two such planes at consecutive points along the superficial geodesic in the direction p', q'. Its coordinates will be given by a combination of these equations of the plane with those which belong to the consecutive plane and are typified by

$$-y'=l\alpha'+m\beta'+l'\alpha+m'\beta,$$

the quantity $\bar{y}$ now denoting the typical space-coordinate of the orthogonal centre, and symbols such as a' implying arc-derivation along the superficial geodesic.

The quantities a, β, a', β', (the two latter being associated solely with the intersection of the consecutive orthogonal planes), have to be determined. For the domainal flexure and other domainal curvatures of the superficial geodesic, the equations of the Frenet type are

$$\frac{dy'}{ds}=\frac{l}{\gamma},\quad \frac{dl}{ds}=\frac{\lambda_3}{\sigma_D}-\frac{y'}{\gamma},\quad \frac{d\lambda_3}{ds}=\frac{m}{\tau_D}-\frac{l}{\sigma_D}.$$

From these equations, we have (on taking summations for the range of the plenary space)

$$\sum m\frac{dl}{ds}=\frac{1}{\sigma_D}(\sum m\lambda_3)-\frac{1}{\gamma}(\sum my')=0,$$

because the trinormal, lying in the orthogonal plane, is at right angles to the binormal and the tangent which lie in the tangent plane; and therefore, as $\sum lm=0$, we also have

$$\sum l\frac{dm}{ds}=0.$$

Obviously, from the Frenet equations,

$$\sum y'\frac{dl}{ds}=-\frac{1}{\gamma};$$

and as $\sum y'm=0$, we have

$$\sum y'\frac{dm}{ds}=-\sum m\frac{dy'}{ds}=-\frac{1}{\gamma}\sum ml=0.$$

Again, we have $\sum m\lambda_3=0$, and therefore

$$\begin{aligned}\sum \lambda_3\frac{dm}{ds}&=-\sum m\frac{d\lambda_3}{ds}\\&=-\frac{1}{\tau_D},\end{aligned}$$

from the Frenet equations; and therefore

$$\begin{aligned}\sum \frac{dl}{ds}\frac{dm}{ds}&=\frac{1}{\sigma_D}\left(\sum \lambda_3\frac{dm}{ds}\right)-\frac{1}{\gamma}\left(\sum y'\frac{dm}{ds}\right)\\&=-\frac{1}{\sigma_D\tau_D}.\end{aligned}$$

These results are used in connection with the typical equation

$$-y'=la'+m\beta'+l'a+m'\beta.$$

Multiply throughout by l, and add; then, as

$$\sum ly'=0, \quad \sum lm=0, \quad \sum ll'=0, \quad \sum lm'=0,$$

it follows that

$$\alpha'=0.$$

Similarly, on multiplying by m and adding, we find

$$\beta'=0.$$

(These results are only related to the point of intersection of the planes and are not continuous along the superficial geodesic.) Next, multiply by y' and add; then, as

$$\sum y'l'=-\frac{1}{\gamma}, \quad \sum y'm'=0,$$

we find

$$\alpha=\gamma.$$

Finally, multiply by l' and add; then, as

$$\sum l'^2=\frac{1}{\sigma_D{}^2}+\frac{1}{\gamma^2}, \quad \sum l'm'=-\frac{1}{\sigma_D\tau_D},$$

we have

$$\frac{1}{\gamma}=\alpha\left(\frac{1}{\sigma_D{}^2}+\frac{1}{\gamma^2}\right)-\beta\frac{1}{\sigma_D\tau_D},$$

and therefore

$$\beta=\frac{\gamma\tau_D}{\sigma_D}.$$

Hence the typical space-coordinate $\bar{y}$ of the orthogonal centre is

$$\bar{y}=y+l\gamma+m\frac{\gamma\tau_D}{\sigma_D};$$

and the length $\bar{\rho}$ of the orthogonal radius of the surface is given by the equation

$$\frac{\bar{\rho}^2}{\gamma^2}=1+\frac{\tau_D{}^2}{\sigma_D{}^2},$$

this orthogonal radius being equal to the radius of domainal flexure for all geodesics touching a curve of domainal flexure, because then (§ 361) the domainal torsion of the superficial geodesic tangent vanishes.

The result may be obtained also as follows. The homaloidal configuration, which is orthogonal to the tangent plane of the surface (a plane having y' and λ_3 as typical direction-cosines of its leading lines), can be represented by the equations

$$\sum\{(\bar{y}-y)y'\}=0, \quad \sum\{(\bar{y}-y)\lambda_3\}=0.$$

The ultimate intersection of this homaloid with the like homaloid at a consecutive

point along the superficial geodesic is represented analytically by combining these two equations with

$$\sum\left\{(\bar{y}-y)\frac{dy'}{ds}\right\}-\sum y'^2=0,$$

$$\sum\left\{(\bar{y}-y)\frac{d\lambda_3}{ds}\right\}-\sum y'\lambda_3=0.$$

But, for the domainal curvatures,

$$\frac{dy'}{ds}=\frac{l}{\gamma},\quad \sum y'^2=1,$$

$$\frac{d\lambda_3}{ds}=\frac{m}{\tau_D}-\frac{l}{\sigma_D},\quad \sum y'\lambda_3=0;$$

and therefore the new equations are equivalent to

$$\sum\{(\bar{y}-y)l\}=\gamma,$$

$$\sum\{(\bar{y}-y)m\}=\gamma\frac{\tau_D}{\sigma_D}.$$

The orthogonal centre lies in the domainal orthogonal plane represented by the typical equation

$$\bar{y}-y=l\alpha+m\beta;$$

and therefore when this set of equations is combined with the foregoing set, we have

$$\alpha=\gamma,\quad \beta=\gamma\frac{\tau_D}{\sigma_D}:$$

that is, the typical spatial coordinate of the orthogonal centre is given by

$$\bar{y}-y=l\gamma+m\gamma\frac{\tau_D}{\sigma_D}.$$

Locus of orthogonal centres of concurrent geodesics on a domainal surface.

364. This orthogonal centre of a superficial geodesic lies in the orthogonal plane of the surface; and therefore its locus, for all the geodesics, lies in that plane.

We had

$$\frac{l}{\gamma}=\xi_{11}p'^2+2\xi_{12}p'q'+\xi_{22}q'^2,$$

where

$$\xi_{ij}\sin\iota=\frac{\bar{E}_{ij}}{\epsilon_n}y_\omega+\frac{\bar{\Omega}_{ij}}{\omega_\nu}y_\epsilon.$$

Also

$$\frac{m}{\gamma}\sin\iota=\frac{1}{\gamma_\omega}\frac{dy}{dn}-\frac{1}{\gamma_\epsilon}\frac{dy}{d\nu},$$

and therefore we take

$$\frac{m}{\gamma}=\zeta_{11}p'^2+2\zeta_{12}p'q'+\zeta_{22}q'^2,$$

where

$$\zeta_{ij}\sin\iota=\frac{\bar{E}_{ij}}{\epsilon_n}\frac{dy}{d\nu}-\frac{\bar{\Omega}_{ij}}{\omega_\nu}\frac{dy}{dn}.$$

Now

$$y_\epsilon\sin\iota=\frac{dy}{dn}\cos\iota-\frac{dy}{d\nu},\quad y_\omega\sin\iota=\frac{dy}{d\nu}\cos\iota-\frac{dy}{dn},$$

so that

$$\sum y_\epsilon\frac{dy}{dn}=0,\quad \sum y_\epsilon\frac{dy}{d\nu}=-\sin\iota,\quad \sum y_\omega\frac{dy}{dn}=-\sin\iota,\quad \sum y_\omega\frac{dy}{d\nu}=0.$$

Hence, by the results in § 357, *Ex.* 4, we have

$$\left.\begin{aligned}\sum\xi_{11}\zeta_{11}&=0\\ \sum\xi_{11}\zeta_{12}&=(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}\\ \sum\xi_{11}\zeta_{22}&=(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}\end{aligned}\right\},\quad \left.\begin{aligned}\sum\xi_{12}\zeta_{11}&=-(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}\\ \sum\xi_{12}\zeta_{12}&=0\\ \sum\xi_{12}\zeta_{22}&=(\mathfrak{bc}-\mathfrak{f}^2)^{\frac{1}{2}}\end{aligned}\right\},\quad \left.\begin{aligned}\sum\xi_{22}\zeta_{11}&=-(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}\\ \sum\xi_{22}\zeta_{12}&=-(\mathfrak{bc}-\mathfrak{f}^2)^{\frac{1}{2}}\\ \sum\xi_{22}\zeta_{22}&=0\end{aligned}\right\};$$

and therefore

$$\begin{aligned}\frac{1}{\gamma}\sum m\xi_{11}&=2(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}p'q'+(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}q'^2=-T_1,\\ \frac{1}{\gamma}\sum m\xi_{12}&=-(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}p'^2+(\mathfrak{bc}-\mathfrak{f}^2)^{\frac{1}{2}}q'^2=-T_2,\\ \frac{1}{\gamma}\sum m\xi_{22}&=-(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}p'^2-2(\mathfrak{bc}-\mathfrak{f}^2)^{\frac{1}{2}}p'q'=-T_3,\end{aligned}$$

with the significance of T_1, T_2, T_3, as defined in § 358, so that

$$T_1p'+T_2q'=-q'W=q'\frac{V_0}{\gamma\tau_D},\quad T_2p'+T_3q'=p'W=-p'\frac{V_0}{\gamma\tau_D}.$$

And we defined (p. 522)

$$\sum l\xi_{11}=\bar{e},\quad \sum l\xi_{12}=\bar{f},\quad \sum l\xi_{22}=\bar{g}.$$

The magnitudes ξ_{11}, ξ_{12}, ξ_{22}, independent of p' and q', satisfy the relation (§ 357)

$$(\mathfrak{bc}-\mathfrak{f}^2)^{\frac{1}{2}}\xi_{11}-(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}\xi_{12}+(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}\xi_{22}=0\,;$$

and the quantities $\mathfrak{a}^{-\frac{1}{2}}\xi_{11}$, $\mathfrak{b}^{-\frac{1}{2}}\xi_{12}$, $\mathfrak{c}^{-\frac{1}{2}}\xi_{22}$, are typical direction-cosines of three directions in the orthogonal plane, independent of directions in the tangent plane. We therefore take the first and the third of the three directions as coordinate axes of reference, with coordinates $\bar{x}$ and $\bar{z}$, denoting by μ the inclination of these axes so that

$$(\mathfrak{ac})^{\frac{1}{2}}\cos\mu=\mathfrak{g},$$

and we retain the second of the three directions as a subsidiary line of reference. Then for the orthogonal centre, we take

$$\mathfrak{a}^{-\frac{1}{2}}\{\sum (\bar{y}-y)\xi_{11}\} = \bar{X} = \bar{x} + \bar{z}\cos\mu,$$
$$\mathfrak{b}^{-\frac{1}{2}}\{\sum (\bar{y}-y)\xi_{12}\} = \bar{Y},$$
$$\mathfrak{c}^{-\frac{1}{2}}\{\sum (\bar{y}-y)\xi_{22}\} = \bar{Z} = \bar{x}\cos\mu + \bar{z},$$

while there is the homogeneous linear relation between $\bar{X}$, $\bar{Y}$, $\bar{Z}$,

$$\{\mathfrak{a}(\mathfrak{bc}-\mathfrak{f}^2)\}^{\frac{1}{2}}\bar{X} - \{\mathfrak{b}(\mathfrak{ac}-\mathfrak{g}^2)\}^{\frac{1}{2}}\bar{Y} + \{\mathfrak{c}(\mathfrak{ab}-\mathfrak{h}^2)\}^{\frac{1}{2}}\bar{Z} = 0.$$

The typical space-coordinate of the orthogonal centre is given by

$$\bar{y} - y = l\gamma + m\gamma\frac{\tau_D}{\sigma_D};$$

and therefore

$$\bar{X}\mathfrak{a}^{\frac{1}{2}} = \gamma(\sum l\xi_{11}) + \gamma\frac{\tau_D}{\sigma_D}(\sum m\xi_{11}) = \gamma\bar{e} - \gamma^2\frac{\tau_D}{\sigma_D}T_1,$$

$$\bar{Y}\mathfrak{b}^{\frac{1}{2}} = \gamma(\sum l\xi_{12}) + \gamma\frac{\tau_D}{\sigma_D}(\sum m\xi_{12}) = \gamma\bar{f} - \gamma^2\frac{\tau_D}{\sigma_D}T_2,$$

$$\bar{Z}\mathfrak{c}^{\frac{1}{2}} = \gamma(\sum l\xi_{22}) + \gamma\frac{\tau_D}{\sigma_D}(\sum m\xi_{22}) = \gamma\bar{g} - \gamma^2\frac{\tau_D}{\sigma_D}T_3.$$

Hence

$$p'\bar{X}\mathfrak{a}^{\frac{1}{2}} + q'\bar{Y}\mathfrak{b}^{\frac{1}{2}} = \gamma(\bar{e}p' + \bar{f}q') - \gamma\frac{V_0}{\sigma_D}q' = A_0p' + H_0q',$$

$$p'\bar{Y}\mathfrak{b}^{\frac{1}{2}} + q'\bar{Z}\mathfrak{c}^{\frac{1}{2}} = \gamma(\bar{f}p' + \bar{g}q') + \gamma\frac{V_0}{\sigma_D}p' = H_0p' + B_0q',$$

because of the relations

$$\frac{V_0}{\sigma_D} = \begin{vmatrix} \bar{e}p' + \bar{f}q', & \bar{f}p' + \bar{g}q' \\ A_0p' + H_0q', & H_0p' + B_0q' \end{vmatrix},$$

$$\frac{1}{\gamma} = \bar{e}p'^2 + 2\bar{f}p'q' + \bar{g}q'^2, \quad A_0p'^2 + 2H_0p'q' + B_0q'^2 = 1.$$

Accordingly, we have

$$\begin{vmatrix} \bar{X}\mathfrak{a}^{\frac{1}{2}} - A_0, & \bar{Y}\mathfrak{b}^{\frac{1}{2}} - H_0 \\ \bar{Y}\mathfrak{b}^{\frac{1}{2}} - H_0, & \bar{Z}\mathfrak{c}^{\frac{1}{2}} - B_0 \end{vmatrix} = 0,$$

which is an equation of the second degree in $\bar{x}$ and $\bar{z}$, because there are surviving terms in $(\mathfrak{ac})^{\frac{1}{2}}\bar{X}\bar{Z} - \mathfrak{b}\bar{Y}^2$. It follows that the locus of the orthogonal centre of domainal flexure for the domainal surface is a conic in the orthogonal plane of the surface.

The result is the extension to domainal surfaces of the property that, for surfaces in a plenary homaloidal quadruple space, the locus of the orthogonal

centre of circular curvature is a conic (Kommerell's characteristic conic) in the orthogonal plane.*

In passing, it may be noted that the envelope of the orthogonal plane of a domainal surface is represented by the equations

$$\sum\{(\bar{y}-y)y'\}=0, \quad \sum\{(\bar{y}-y)\lambda_3\}=0,$$
$$\sum\{(\bar{y}-y)l\}=\gamma, \quad \sum\{(\bar{y}-y)m\}=\gamma\frac{\tau_D}{\sigma_D}.$$

The section of this amplitude by the orthogonal plane itself is the foregoing conic-locus of the orthogonal centre of domainal flexure for the surface.

Lemniscate-locus of centres of domainal flexure of concurrent geodesics.

365. The centre of domainal flexure of any superficial geodesic lies in the orthogonal plane of the surface; and it belongs to a locus arising out of the aggregate of those geodesics. Let y_D be the typical space-coordinate of this centre of flexure, so that

$$y_D-y=l\gamma,$$

and therefore

$$\frac{1}{\gamma^2}(y_D-y)=\xi_{11}p'^2+2\xi_{12}p'q'+\xi_{22}q^{2'}.$$

Referring the locus to the same axes of coordinates in the orthogonal plane as have been used for the locus of the orthogonal centre, we take

$$\mathfrak{a}^{-\frac{1}{2}}\sum\{(y_D-y)\xi_{11}\}=\bar{X}_D=\bar{x}_D+\bar{z}_D\cos\mu,$$
$$\mathfrak{b}^{-\frac{1}{2}}\sum\{(y_D-y)\xi_{12}\}-\bar{Y}_D,$$
$$\mathfrak{c}^{-\frac{1}{2}}\sum\{(y_D-y)\xi_{22}\}=\bar{Z}_D=\bar{x}_D\cos\mu+\bar{z}_D,$$

with the corresponding relation

$$\{\mathfrak{a}(\mathfrak{bc}-\mathfrak{f}^2)\}^{\frac{1}{2}}\bar{X}_D-\{\mathfrak{b}(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}\}\bar{Y}_D+\{\mathfrak{c}(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}\}\bar{Z}_D=0.$$

Also, we have

$$\gamma^2=\bar{x}_D{}^2+2\bar{x}_D\bar{z}_D\cos\mu+\bar{z}_D{}^2,$$

so that γ can now be regarded as a function of $\bar{x}_D$ and $\bar{z}_D$; and

$$\frac{1}{\gamma^2}\bar{X}_D\mathfrak{a}^{\frac{1}{2}}=\mathfrak{a}p'^2+2\mathfrak{h}p'q'+\mathfrak{g}q'^2,$$
$$\frac{1}{\gamma^2}\bar{Y}_D\mathfrak{b}^{\frac{1}{2}}=\mathfrak{h}p'^2+2\mathfrak{b}p'q'+\mathfrak{f}q'^2,$$
$$\frac{1}{\gamma^2}\bar{Z}_D\mathfrak{c}^{\frac{1}{2}}=\mathfrak{g}p'^2+2\mathfrak{f}p'q'+\mathfrak{c}q'^2.$$

* Kommerell, *Math. Ann.*, vol. lx (1905), p. 554; also my *G.F.D.*, vol. i, § 251.

Further, we have

$$1 = A_0 p'^2 + 2H_0 p'q' + B_0 q'^2.$$

Now

$$\frac{1}{\gamma^2}\{\mathfrak{a}^{\frac{1}{2}}\bar{X}_D p'^2 + 2\mathfrak{b}^{\frac{1}{2}}\bar{Y}_D p'q' + \mathfrak{c}^{\frac{1}{2}}\bar{Z}_D q'^2\}$$
$$= \mathfrak{a}p'^4 + 4\mathfrak{h}p'^3q' + 6\mathfrak{k}p'^2q'^2 + 4\mathfrak{f}p'q'^3 + \mathfrak{c}q'^4$$
$$= \frac{1}{\gamma^2},$$

so that

$$(\bar{X}_D \mathfrak{a}^{\frac{1}{2}} - A_0)p'^2 + 2(\bar{Y}_D \mathfrak{b}^{\frac{1}{2}} - H_0)p'q' + (\bar{Z}_D \mathfrak{c}^{\frac{1}{2}} - B_0)q'^2 = 0.$$

Another equation, homogeneous and linear in p'^2, $p'q'$, q'^2, is

$$(\bar{X}_D \mathfrak{g}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{a}\mathfrak{c}^{\frac{1}{2}})p'^2 + 2(\bar{X}_D \mathfrak{f}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{h}\mathfrak{c}^{\frac{1}{2}})p'q' + (\bar{X}_D \mathfrak{c}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{g}\mathfrak{c}^{\frac{1}{2}})q'^2 = 0.$$

Let

$$L_0 = (\bar{Y}_D \mathfrak{b}^{\frac{1}{2}} - H_0)(\bar{X}_D \mathfrak{c}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{g}\mathfrak{c}^{\frac{1}{2}}) - (\bar{Z}_D \mathfrak{c}^{\frac{1}{2}} - B_0)(\bar{X}_D \mathfrak{f}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{h}\mathfrak{c}^{\frac{1}{2}}),$$
$$M_0 = (\bar{Z}_D \mathfrak{c}^{\frac{1}{2}} - B_0)(\bar{X}_D \mathfrak{g}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{a}\mathfrak{c}^{\frac{1}{2}}) - (\bar{X}_D \mathfrak{a}^{\frac{1}{2}} - A_0)(\bar{X}_D \mathfrak{c}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{g}\mathfrak{c}^{\frac{1}{2}}),$$
$$N_0 = (\bar{X}_D \mathfrak{a}^{\frac{1}{2}} - A_0)(\bar{X}_D \mathfrak{f}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{h}\mathfrak{c}^{\frac{1}{2}}) - (\bar{Y}_D \mathfrak{b}^{\frac{1}{2}} - H_0)(\bar{X}_D \mathfrak{g}\mathfrak{a}^{\frac{1}{2}} - \bar{Z}_D \mathfrak{a}\mathfrak{c}^{\frac{1}{2}});$$

then

$$\frac{p'^2}{L_0} = \frac{2p'q'}{M_0} = \frac{q'^2}{N_0},$$

and therefore the eliminant is

$$M_0^2 = 4L_0 N_0,$$

which is an initial form of the equation of the locus.

For a developed form, we use the relation

$$\bar{X}_D^2 - 2\bar{X}_D \bar{Z}_D \cos\mu + \bar{Z}_D^2 = \gamma^2 \sin^2\mu;$$

and we find

$$L_0 = \gamma^2 \left(\frac{\mathfrak{b}\mathfrak{c} - \mathfrak{f}^2}{\mathfrak{a}\mathfrak{c} - \mathfrak{g}^2}\right)^{\frac{1}{2}} \mathfrak{a}\mathfrak{c}\sin^2\mu - \mathfrak{a}^{\frac{1}{2}}(\mathfrak{c}H_0 - \mathfrak{f}B_0)\bar{X}_D + \mathfrak{c}^{\frac{1}{2}}(\mathfrak{g}H_0 - \mathfrak{h}B_0)\bar{Z}_D,$$
$$M_0 = \qquad -\gamma^2 \mathfrak{a}\mathfrak{c}\sin^2\mu - \mathfrak{a}^{\frac{1}{2}}(\mathfrak{g}B_0 - \mathfrak{c}A_0)\bar{X}_D + \mathfrak{c}^{\frac{1}{2}}(\mathfrak{a}B_0 - \mathfrak{g}A_0)\bar{Z}_D,$$
$$N_0 = \gamma^2 \left(\frac{\mathfrak{a}\mathfrak{b} - \mathfrak{h}^2}{\mathfrak{a}\mathfrak{c} - \mathfrak{g}^2}\right)^{\frac{1}{2}} \mathfrak{a}\mathfrak{c}\sin^2\mu - \mathfrak{a}^{\frac{1}{2}}(\mathfrak{f}A_0 - \mathfrak{g}H_0)\bar{X}_D + \mathfrak{c}^{\frac{1}{2}}(\mathfrak{h}A_0 - \mathfrak{a}H_0)\bar{Z}_D.$$

Thus the equation of the locus is

$$T_0(\bar{x}_D^2 + 2\bar{x}_D\bar{z}_D\cos\mu + \bar{z}_D^2)^2 + T_1(\bar{x}_D^2 + 2\bar{x}_D\bar{z}_D\cos\mu + \bar{z}_D^2) + T_2 = 0,$$

where

$$T_0 = (\mathfrak{a}\mathfrak{c} - \mathfrak{g}^2)(\mathfrak{a}\mathfrak{c} - \mathfrak{g}^2 - 4\mathfrak{f}\mathfrak{h} + 4\mathfrak{b}\mathfrak{g}) = (\mathfrak{a}\mathfrak{c} - \mathfrak{g}^2)(I_0 - 12\theta^2),$$

T_1 is a linear homogeneous function of $\bar{x}_D$ and $\bar{z}_D$, and T_2 is a quadratic homogeneous function of $\bar{x}_D$ and $\bar{z}_D$.

The locus of the centre of domainal flexure of superficial geodesics is therefore a lemniscate curve, with its (real or imaginary) double point at the origin.

Let Q_D be the centre of domainal flexure for any superficial geodesic, and Q_0 the orthogonal centre for that geodesic, so that the typical space-coordinate of Q_D is

$$\bar{y}_D - y = l\gamma,$$

and the typical space-coordinate of Q_0 is

$$\bar{y}_0 - y = l\gamma + m\gamma \frac{\tau_D}{\sigma_D}.$$

In these expressions, l and m are the typical direction-cosines of two lines in the orthogonal plane, one of them the radius of domainal flexure, the other the domainal trinormal of the superficial geodesic; and the two lines in that plane are at right angles.

The locus of Q_0 is a conic; and the locus of Q_D is a lemniscate. It is a known property that the pedal of a conic is a lemniscate; and thus the preceding forms for $\bar{y}_0$ and $\bar{y}_D$ suggest that the lemniscate-locus of the centre of domainal flexure is the pedal, with respect to the initiating point O on the surface, of the conic-locus of the orthogonal centre. To test this surmise, we note that, if the direction of the tangent to the conic at Q_0 be given by dx_0 and dz_0, and if the conic be represented by $f(x_0, z_0)=0$, we have

$$\frac{\partial f}{\partial x_0} dx_0 + \frac{\partial f}{\partial z_0} dz_0 = 0.$$

Again, the direction of OQ_D is given by x_D and z_D; and the condition that OQ_D should be perpendicular to the direction dx_0, dz_0, is

$$(x_D + z_D \cos\mu)\, dx_0 + (x_D \cos\mu + z_D)\, dz_0 = 0,$$

that is,

$$\overline{X}_D \frac{\partial f}{\partial z_0} - \overline{Z}_D \frac{\partial f}{\partial x_0} = 0.$$

Now as the conic is

$$f(x_0, z_0) = (\overline{X}\mathfrak{a}^{\frac{1}{2}} - A_0)(\overline{Z}\mathfrak{c}^{\frac{1}{2}} - B_0) - (\overline{Y}\mathfrak{b}^{\frac{1}{2}} - H_0)^2 = 0,$$

we have

$$\frac{\partial f}{\partial x_0} = (\overline{Z}\mathfrak{c}^{\frac{1}{2}} - B_0)\mathfrak{a}^{\frac{1}{2}} + (\overline{X}\mathfrak{a}^{\frac{1}{2}} - A_0)\mathfrak{c}^{\frac{1}{2}}\cos\mu$$
$$- 2(\overline{Y}\mathfrak{b}^{\frac{1}{2}} - H_0)\mathfrak{b}^{\frac{1}{2}}\left[\left\{\frac{\mathfrak{a}(\mathfrak{bc} - \mathfrak{f}^2)}{\mathfrak{b}(\mathfrak{ac} - \mathfrak{g}^2)}\right\}^{\frac{1}{2}} + \left\{\frac{\mathfrak{c}(\mathfrak{ab} - \mathfrak{h}^2)}{\mathfrak{b}(\mathfrak{ac} - \mathfrak{g}^2)}\right\}^{\frac{1}{2}}\cos\mu\right],$$

where $(\mathfrak{ac})^{\frac{1}{2}}\cos\mu = \mathfrak{g}$. But the factor of $-2(\overline{Y}\mathfrak{b}^{\frac{1}{2}} - H_0)\mathfrak{b}^{\frac{1}{2}}$

$$= \frac{1}{\{\mathfrak{ab}(\mathfrak{ac} - \mathfrak{g}^2)\}^{\frac{1}{2}}}\{\mathfrak{a}(\mathfrak{bc} - \mathfrak{f}^2)^{\frac{1}{2}} + \mathfrak{g}(\mathfrak{ab} - \mathfrak{h}^2)^{\frac{1}{2}}\} = \frac{\mathfrak{h}}{(\mathfrak{ab})^{\frac{1}{2}}},$$

by the result in § 357, *Ex.* 2; and therefore

$$\frac{\partial f}{\partial x_0}=\mathfrak{a}^{\frac{1}{2}}(\overline{Z}\mathfrak{c}^{\frac{1}{2}}-B_0)-2\frac{\mathfrak{h}}{\mathfrak{a}^{\frac{1}{2}}}(\overline{Y}\mathfrak{b}^{\frac{1}{2}}-H_0)+\frac{\mathfrak{g}}{\mathfrak{a}^{\frac{1}{2}}}(\overline{X}\mathfrak{a}^{\frac{1}{2}}-A_0).$$

Also from the equation of the conic, we have

$$\frac{\overline{X}\mathfrak{a}^{\frac{1}{2}}-A_0}{q'^2}=\frac{\overline{Y}\mathfrak{b}^{\frac{1}{2}}-H_0}{-q'p'}=\frac{\overline{Z}\mathfrak{c}^{\frac{1}{2}}-B_0}{p'^2}=\Theta,$$

where, for the result, the explicit value of Θ is immaterial; thus

$$\frac{\partial f}{\partial x_0}=\mathfrak{a}^{-\frac{1}{2}}\Theta[\mathfrak{a}p'^2+2\mathfrak{h}p'q'+\mathfrak{g}q'^2]=\frac{\Theta}{\gamma^2}\overline{X}_D.$$

Similarly, we find

$$\frac{\partial f}{\partial z_0}=\mathfrak{c}^{-\frac{1}{2}}\Theta[\mathfrak{g}p'^2+2\mathfrak{f}p'q'+\mathfrak{c}q'^2]=\frac{\Theta}{\gamma^2}\overline{Z}_D.$$

Consequently

$$\overline{X}_D\frac{\partial f}{\partial z_0}-\overline{Z}_D\frac{\partial f}{\partial x_0}=0,$$

and the condition is satisfied.

It follows that the lemniscate-locus in the orthogonal plane is the pedal of the conic-locus in that plane with respect to the central point O; and, in the diagram, we have

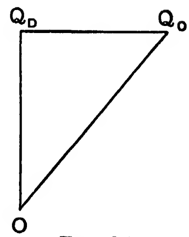

FIG. 35.

$$OQ_D=\gamma,\quad Q_DQ_0=\gamma\frac{\tau_D}{\sigma_D},$$

while

$$\tan Q_0OQ_D=\frac{\tau_D}{\sigma_D}.$$

Various geometrical inferences follow from this relationship between the curves: they will merely be stated, and left without the respective analytical verifications.

(i) From the point O, four normals to the conic can be drawn: at the foot of every such normal, the tangent to the conic being at right angles to the normal, there is a point on the lemniscate; and considerations of normality shew that the line is there a tangent to the lemniscate. Thus the conic and the lemniscate touch in four points.

(ii) At any such point P, the line OP is normal to the lemniscate, and therefore is a maximum or a minimum radius vector, the point being an apse; hence the directions of the four normals to the conic are the directions of the four principal radii of domainal curvature. Consequently, the directions of the curves of domainal flexure on the surface are determined by means of these four normals.

(iii) We know (§ 361) that the domainal torsion $1/\sigma_D$ of a superficial geodesic tangent to a curve of domainal flexure on the surface vanishes, so that

$$\begin{vmatrix} \bar{e}p'+\bar{f}q', & A_0p'+H_0q' \\ \bar{f}p'+\bar{g}q', & H_0p'+G_0q' \end{vmatrix}=0,$$

that is,

$$\begin{vmatrix} \mathfrak{a}p'^3+3\mathfrak{h}p'^2q'+3\mathfrak{k}p'q'^2+\mathfrak{f}q'^3, & A_0p'+H_0q' \\ \mathfrak{h}p'^3+3\mathfrak{k}p'^2q'+3\mathfrak{f}p'q'^2+\mathfrak{c}q'^3, & H_0p'+B_0q' \end{vmatrix}=0,$$

a quartic equation in $p':q'$. This quartic equation gives the directions connected with the four normals drawn from O, alike to the conic and to the lemniscate.

Ex. Prove that the value of the quantity Θ in the preceding investigation

$$=-\frac{(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}B_0-(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}H_0+(\mathfrak{bc}-\mathfrak{f}^2)^{\frac{1}{2}}A_0}{(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}p'^2+(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}p'q'+(\mathfrak{bc}-\mathfrak{f}^2)^{\frac{1}{2}}q'^2},$$

or, what is the same thing,

$$\frac{V_0}{\gamma\tau_D}\Theta=(\mathfrak{ab}-\mathfrak{h}^2)^{\frac{1}{2}}B_0-(\mathfrak{ac}-\mathfrak{g}^2)^{\frac{1}{2}}H_0+(\mathfrak{bc}-\mathfrak{f}^2)^{\frac{1}{2}}A_0.$$

Two propositions on centres of circular curvature of superficial geodesics.

366. Two propositions are stated here, to complete the consideration of the loci of centres of curvature and centres of flexure; they can be established by analysis similar to that employed in § 113. They will be considered in more detail for the special case when the domain is primary (§§ 374-376).

I. The locus of the centre of circular curvature of concurrent geodesics on a surface enclosed in a domain is a skew quartic curve lying in a flat, which is orthogonal to the surface; the curve is the intersection of a sphere and of a quadric cone with its apex on the surface of the sphere, both quadrics lying in the flat.

II. The locus of the centre of circular curvature of the domainal geodesics touching concurrent geodesics on a surface in the domain (or, what is the same thing, of concurrent geodesics lying in any superficial orientation in the domain) is a lemniscate curve in a plane orthogonal to the domain if the plenary homaloidal space is six-dimensional, and is a skew quartic curve in a flat orthogonal to the domain when the plenary space has more than six dimensions.

The relation between corresponding points of these two loci and of the locus of the centre of domainal flexure, arising out of the same superficial geodesic, has been established in the investigation of § 342.

CHAPTER XXXI

Properties of Primary Domains

Principal circular curvatures of geodesics.

367. The different kinds of measures of curvature, characteristic of a domain, are expressible in simple form when the domain is primary, that is, when it exists in quintuple space.

The principal measures of linear curvature are obtained by obtaining the maximum and minimum values of $1/\rho$, the circular curvature of a domainal geodesic. For this purpose, the conditions to be satisfied, in order that

$$\frac{1}{\rho}=\sum \bar{A}p'^2$$

may be a maximum or a minimum subject to the relation

$$1=\sum Ap'^2,$$

are the four critical equations

$$v_1=\lambda u_1, \quad v_2-\lambda u_2, \quad v_3=\lambda u_3, \quad v_4=\lambda u_4.$$

The value of the multiplier λ is obtained, from multiplying the equations by p', q', r', t', respectively, and adding the products : thus

$$\frac{1}{\rho}=\lambda.$$

When this value of λ is inserted, the critical equations become

$$\left.\begin{aligned}
\left(\bar{A}-\frac{A}{\rho}\right)p'+\left(\bar{H}-\frac{H}{\rho}\right)q'+\left(\bar{G}-\frac{G}{\rho}\right)r'+\left(\bar{L}-\frac{L}{\rho}\right)t'&=0\\
\left(\bar{H}-\frac{H}{\rho}\right)p'+\left(\bar{B}-\frac{B}{\rho}\right)q'+\left(\bar{F}-\frac{F}{\rho}\right)r'+\left(\bar{M}-\frac{M}{\rho}\right)t'&=0\\
\left(\bar{G}-\frac{G}{\rho}\right)p'+\left(\bar{F}-\frac{F}{\rho}\right)q'+\left(\bar{C}-\frac{C}{\rho}\right)r'+\left(\bar{N}-\frac{N}{\rho}\right)t'&=0\\
\left(\bar{L}-\frac{L}{\rho}\right)p'+\left(\bar{M}-\frac{M}{\rho}\right)q'+\left(\bar{N}-\frac{N}{\rho}\right)r'+\left(\bar{D}-\frac{D}{\rho}\right)t'&=0
\end{aligned}\right\}$$

Elimination of p', q', r', t', leads to the equation

$$\begin{vmatrix} \bar{A} - \frac{A}{\rho}, & \bar{H} - \frac{H}{\rho}, & \bar{G} - \frac{G}{\rho}, & \bar{L} - \frac{L}{\rho} \\ \bar{H} - \frac{H}{\rho}, & \bar{B} - \frac{B}{\rho}, & \bar{F} - \frac{F}{\rho}, & \bar{M} - \frac{M}{\rho} \\ \bar{G} - \frac{G}{\rho}, & \bar{F} - \frac{F}{\rho}, & \bar{C} - \frac{C}{\rho}, & \bar{N} - \frac{N}{\rho} \\ \bar{L} - \frac{L}{\rho}, & \bar{M} - \frac{M}{\rho}, & \bar{N} - \frac{N}{\rho}, & \bar{D} - \frac{D}{\rho} \end{vmatrix} = 0,$$

which determines the maximum and minimum values in question. Thus there are four principal linear curvatures, being the roots of this quartic which may be written

$$\frac{1}{\rho^4}\Omega - \frac{1}{\rho^3}E_1 + \frac{1}{\rho^2}E_2 - \frac{1}{\rho}E_3 + \Upsilon = 0,$$

with the customary significance for Ω and Υ, while

$$E_1 = \sum \bar{A}\frac{\partial\Omega}{\partial A} = \sum \bar{A}a, \quad E_3 = \sum A\frac{\partial\Upsilon}{\partial\bar{A}} = \sum A\bar{a},$$

$$E_2 = \sum\sum \bar{A}\bar{B}\frac{\partial^2\Omega}{\partial A\partial B} = \sum\sum AB\frac{\partial^2\Upsilon}{\partial\bar{A}\partial\bar{B}}.$$

Consequently, there are four measures of curvature of the domain, originating in the principal linear curvatures; these are

$$\sum\frac{1}{\rho_1} = \frac{E_1}{\Omega}, \quad \sum\frac{1}{\rho_1\rho_2\rho_3} = \frac{E_3}{\Omega},$$

$$\sum\frac{1}{\rho_1\rho_2} = \frac{E_2}{\Omega}, \quad \frac{1}{\rho_1\rho_2\rho_3\rho_4} = \frac{\Upsilon}{\Omega}.$$

Moreover, the same four equations determine the four principal directions corresponding to the four principal linear curvatures, the values of $p' : q' : r' : t'$ being determined for each of these directions by any three of the equations. Also, on the assumption that no two of these curvatures are equal—a contrary assumption would imply relations connecting the fundamental magnitudes—, it is easy to prove, by the usual kind of analysis, that every two of these four principal directions are at right angles to one another. The domainal envelopes of these directions are curves of curvature; and therefore the domain possesses four distinct families of curves of curvature which constitute a quadruply orthogonal system of curves in the domain.

The quantities E_1, E_2, E_3, like Ω and Υ, are invariants of the domainal configuration: and the discriminant of the quaternary quadratic form $\sum(\bar{A} - \theta A)p'^2$ is

$$\Upsilon - \theta E_3 + \theta^2 E_2 - \theta^3 E_1 + \theta^4\Omega.$$

When the two quaternary quadratics $\sum Ap'^2$, $\sum \bar{A}p'^2$, are taken in the canonical forms (obtained from interpreting them by means of two quadrics and referring the quadrics to a common self-conjugate tetrahedron), we denote the direction-variables by p_0', q_0', r_0', t_0', the new parametric curves being the curves of curvature of the primary domain; and the two forms then are

$$\left.\begin{aligned}\frac{1}{\rho}&=\bar{A}_0p_0'^2+\bar{B}_0q_0'^2+\bar{C}_0r_0'^2+\bar{D}_0t_0'^2\\ 1&=A_0p_0'^2+B_0q_0'^2+C_0r_0'^2+D_0t_0'^2\end{aligned}\right\}.$$

The equation for the principal circular curvatures of domainal geodesics then becomes

$$\left(\bar{A}_0-\frac{A_0}{\rho}\right)\left(\bar{B}_0-\frac{B_0}{\rho}\right)\left(\bar{C}_0-\frac{C_0}{\rho}\right)\left(\bar{D}_0-\frac{D_0}{\rho}\right)=0,$$

while the invariants assume the values

$$\Omega=A_0B_0C_0D_0,\qquad Y=\bar{A}_0\bar{B}_0\bar{C}_0\bar{D}_0,$$
$$E_1=B_0C_0D_0\bar{A}_0+C_0D_0A_0\bar{B}_0+D_0A_0B_0\bar{C}_0+A_0B_0C_0\bar{D}_0,$$
$$E_3=A_0\bar{B}_0\bar{C}_0\bar{D}_0+B_0\bar{C}_0\bar{D}_0\bar{A}_0+C_0\bar{D}_0\bar{A}_0\bar{B}_0+D_0\bar{A}_0\bar{B}_0\bar{C}_0,$$
$$E_2=B_0C_0\bar{A}_0\bar{D}_0+A_0D_0\bar{B}_0\bar{C}_0+C_0A_0\bar{B}_0\bar{D}_0+B_0D_0\bar{C}_0\bar{A}_0+A_0B_0\bar{C}_0\bar{D}_0+C_0D_0\bar{A}_0\bar{B}_0.$$

Ex. Denoting by $\bar{a}$, $\bar{h}$, ..., the minors of $\bar{A}$, $\bar{H}$, ..., respectively in Y, analogous to the minors a, h, ..., of the constituents of Ω, establish the results

$$\sum \bar{a}u_1v_1=-\frac{\Omega}{\rho_1\rho_2\rho_3\rho_4},$$
$$\sum \bar{a}v_1^2=\frac{\Omega}{\rho\rho_1\rho_2\rho_3\rho_4};$$

and obtain a geometrical interpretation of the concomitant

$$\sum \bar{a}u_1^2.$$

Volumetric measures of curvature of a primary domain.

368. When a geodesic in any domain touches a region $\epsilon(p, q, r, t)=0$ in that domain, its direction-variables are subject to the two conditions

$$\sum Ap'^2=1,\quad \epsilon_1p'+\epsilon_2q'+\epsilon_3r'+\epsilon_4t'=0;$$

and the direction can be regarded as originating in the region. Among these regional directions, there are principal directions: that is, such as provide a maximum or a minimum among all the values of the circular curvature of the domainal geodesics; and it has been proved (§ 293) that the critical equations for these principal directions in the region are

$$v_i-\frac{u_i}{\rho}=\frac{1}{\sigma}\frac{\epsilon_i}{\epsilon_n},$$

where $i=1, 2, 3, 4$, and $1/\sigma$ is the torsion of the domainal geodesic. (This torsion vanishes for a direction that is principal to the domain, but not for a principal direction in a region of the domain.)

When the domain is primary, its magnitudes $\bar{A}$, $\bar{H}$, ..., are quantities of position, and not of direction. Let the magnitudes p', q', r', t', $1/\sigma$, be eliminated from these four equations and from the condition $\epsilon_1 p' + \epsilon_2 q' + \epsilon_3 r' + \epsilon_4 t' = 0$; then we have

$$\begin{vmatrix} \bar{A} - \frac{A}{\rho}, & \bar{H} - \frac{H}{\rho}, & \bar{G} - \frac{G}{\rho}, & \bar{L} - \frac{L}{\rho}, & \epsilon_1 \\ \bar{H} - \frac{H}{\rho}, & \bar{B} - \frac{B}{\rho}, & \bar{F} - \frac{F}{\rho}, & \bar{M} - \frac{M}{\rho}, & \epsilon_2 \\ \bar{G} - \frac{G}{\rho}, & \bar{F} - \frac{F}{\rho}, & \bar{C} - \frac{C}{\rho}, & \bar{N} - \frac{N}{\rho}, & \epsilon_3 \\ \bar{L} - \frac{L}{\rho}, & \bar{M} - \frac{M}{\rho}, & \bar{N} - \frac{N}{\rho}, & \bar{D} - \frac{D}{\rho}, & \epsilon_4 \\ \epsilon_1, & \epsilon_2, & \epsilon_3, & \epsilon_4, & 0 \end{vmatrix} = 0.$$

The three roots of this eliminant equation are the three principal values. On using $\bar{a}$, $\bar{h}$, ..., to denote first minors of Y as a, h, ..., denote first minors of Ω, and writing

$$\mathfrak{a} = \left(\sum \bar{A} \frac{\partial}{\partial A}\right) a, \quad \mathfrak{h} = \left(\sum \bar{A} \frac{\partial}{\partial A}\right) h, \; \ldots,$$

$$\bar{\mathfrak{a}} = \left(\sum A \frac{\partial}{\partial \bar{A}}\right) \bar{a}, \quad \bar{\mathfrak{h}} = \left(\sum A \frac{\partial}{\partial \bar{A}}\right) \bar{h}, \; \ldots,$$

the eliminant assumes the form

$$\sum \bar{a}\epsilon_1^2 - \frac{1}{\rho} \sum \bar{\mathfrak{a}}\epsilon_1^2 + \frac{1}{\rho^2} \sum \mathfrak{a}\epsilon_1^2 - \frac{1}{\rho^3} \sum a\epsilon_1^2 = 0.$$

Hence, if c_1, c_2, c_3, are the roots of the cubic, we have

$$\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} = V_1 = \frac{\sum \mathfrak{a}\epsilon_1^2}{\sum a\epsilon_1^2},$$

$$\frac{1}{c_2 c_3} + \frac{1}{c_3 c_1} + \frac{1}{c_1 c_2} = V_2 = \frac{\sum \bar{\mathfrak{a}}\epsilon_1^2}{\sum a\epsilon_1^2},$$

$$\frac{1}{c_1 c_2 c_3} = V_3 = \frac{\sum \bar{a}\epsilon_1^2}{\sum a\epsilon_1^2}.$$

Moreover, for each of the three roots, the foregoing critical equations determine a unique set of values for $p' : q' : r' : t'$, and the actual values of the direction-variables satisfy the condition $\sum A p'^2 = 1$; so that there are three principal regional directions for domainal circular curvature.

These three principal regional directions are at right angles to one another. Let p_i', q_i', r_i', t_i', be the direction of the domainal geodesic with the circular curvature $1/c_i$, for $i=1, 2, 3$; then we have

$$\sum v_1^{(1)} p_2' - \frac{1}{c_1} \sum u_1^{(1)} p_2' = \frac{1}{\sigma_1 \epsilon_n} \sum \epsilon_1 p_2' = 0,$$

that is,

$$\sum \bar{A} p_1' p_2' - \frac{1}{c_1} \sum A p_1' p_2' = 0:$$

and similarly

$$\sum \bar{A} p_2' p_1' - \frac{1}{c_2} \sum A p_2' p_1' = 0.$$

For any general region, the quantities c_1, c_2, c_3, are unequal: hence

$$\sum A p_1' p_2' = 0;$$

and, similarly,

$$\sum A p_1' p_3' = 0, \quad \sum A p_2' p_3' = 0.$$

The orthogonality of the directions is thus established.

It follows also that there are three measures V_1, V_2, V_3, of circular curvature of domainal geodesics for directions in the orientation of any region.

369. If, instead of assigning the equation of a region as the means of estimating a domainal orientation, we determine the orientation by volumetric variables P, Q, R, T, as in § 269, the three measures of circular curvature of domainal geodesics, which originate in that orientation, are

$$V_1 = \frac{\sum \mathfrak{a} P^2}{\sum a P^2}, \quad V_2 = \frac{\sum \bar{\mathfrak{a}} P^2}{\sum a P^2}, \quad V_3 = \frac{\sum \bar{a} P^2}{\sum a P^2}.$$

We have $\sum a P^2 = 1$ when the variables are in canonical form; for the values of V_1, V_2, V_3, the ratios of the variables P are sufficient.

When these magnitudes V_1, V_2, V_3, are associated with a specific region $\epsilon = 0$, they are isolated quantities, certainly independent of directions touching the region; but they are functions of position only, and their relation to the region is as intrinsic in quality as is their relation to the domain. When they are associated with the variables P, Q, R, T, in the domain, they become the measures of curvature for any arbitrarily assumed volumetric orientation; as functions of the variables of that orientation, they acquire principal values, being the maximum or the minimum among all the values which arise from all admissible orientations. These principal values are obtained by making the magnitudes V_1, V_2, V_3, subject to the critical equations for maximum or minimum values as functions of P, Q, R, T; but the principal values do not emerge simply when a parametric region is postulated as the origin of the magnitudes.

Accordingly, we take (as in § 269) any three non-complanar directions with direction-variables p_i', q_i', r_i', t_i', for $i=1, 2, 3$; and we regard them as leading lines for the determination of a volumetric orientation, with volumetric variables P, Q, R, T, defined by relations

$$P, Q, R, T = \left\| \begin{matrix} p_1', & q_1', & r_1', & t_1' \\ p_2', & q_2', & r_2', & t_2' \\ p_3', & q_3', & r_3', & t_3' \end{matrix} \right\|.$$

As before, we have

$$\sum aP^2 = \sum \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ M, & N, & D \end{vmatrix} \begin{vmatrix} q_1', & r_1', & t_1' \\ q_2', & r_2', & t_2' \\ q_3', & r_3', & t_3' \end{vmatrix}^2$$

$$= \begin{vmatrix} \sum Ap_1'^2, & \sum Ap_1'p_2', & \sum Ap_1'p_3' \\ \sum Ap_1'p_2', & \sum Ap_2'^2, & \sum Ap_2'p_3' \\ \sum Ap_1'p_3', & \sum Ap_2'p_3', & \sum Ap_3'^2 \end{vmatrix} = \begin{vmatrix} 1, & c_{12}, & c_{13} \\ c_{12}, & 1, & c_{23} \\ c_{13}, & c_{23}, & 1 \end{vmatrix} = \Theta,$$

where c_{ij} is the cosine of the angle between the directions determined by ij. (If it should happen that the three lines are orthogonal, $\Theta=1$; if they are not orthogonal, the canonical variables are $P\Theta^{-\frac{1}{2}}$, $Q\Theta^{-\frac{1}{2}}$, $R\Theta^{-\frac{1}{2}}$, $T\Theta^{-\frac{1}{2}}$; the discrimination between the alternatives is unessential to the results.)

Any direction p', q', r', t', lying in the volumetric orientation is such that

$$z' = \lambda z_1' + \mu z_2' + \nu z_3',$$

for $z=p, q, r, t$, so that

$$\begin{vmatrix} p', & q', & r', & t' \\ p_1', & q_1', & r_1', & t_1' \\ p_2', & q_2', & r_2', & t_2' \\ p_3', & q_3', & r_3', & t_3' \end{vmatrix} = 0,$$

and therefore

$$p'P + q'Q + r'R + t'T = 0.$$

In order to find the principal values of the circular curvature of domainal geodesics originating in the specified volumetric orientation, we have to make $1/\rho$, where

$$\frac{1}{\rho} = \sum \bar{A} p'^2,$$

a maximum or a minimum for values of the variables p', q', r', t', which are subject to the two conditions

$$\sum Ap'^2 = 1, \quad p'P + q'Q + r'R + t'T = 0.$$

The critical equations are

$$v_1 = \kappa u_1 + \xi P, \quad v_2 = \kappa u_2 + \xi Q, \quad v_3 = \kappa u_3 + \xi R, \quad v_4 = \kappa u_4 + \xi T,$$

where κ and ξ are left undetermined in the construction of the equations.

Multiplying by p', q', r', t', and adding, we have

$$\frac{1}{\rho}=\kappa,$$

on account of the two conditions; so that the equations are

$$v_i-\frac{1}{\rho}u_i=\xi X_i, \quad (i=1, 2, 3, 4),$$

where $X_1, X_2, X_3, X_4, =P, Q, R, T$. When we eliminate p', q', r', t', ξ, between these four equations and the second condition, the eliminant is

$$\begin{vmatrix} \bar{A}-\dfrac{A}{\rho}, & \bar{H}-\dfrac{H}{\rho}, & \bar{G}-\dfrac{G}{\rho}, & \bar{L}-\dfrac{L}{\rho}, & P \\ \bar{H}-\dfrac{H}{\rho}, & \bar{B}-\dfrac{B}{\rho}, & \bar{F}-\dfrac{F}{\rho}, & \bar{M}-\dfrac{M}{\rho}, & Q \\ \bar{G}-\dfrac{G}{\rho}, & \bar{F}-\dfrac{F}{\rho}, & \bar{C}-\dfrac{C}{\rho}, & \bar{N}-\dfrac{N}{\rho}, & R \\ \bar{L}-\dfrac{L}{\rho}, & \bar{M}-\dfrac{M}{\rho}, & \bar{N}-\dfrac{N}{\rho}, & \bar{D}-\dfrac{D}{\rho}, & T \\ P, & Q, & R, & T, & 0 \end{vmatrix}=0,$$

a cubic equation which, when the determinant is expanded, acquires the form

$$\sum \bar{a}P^2-\frac{1}{\rho}\sum \bar{\mathrm{a}}P^2+\frac{1}{\rho^2}\sum \mathrm{a}P^2-\frac{1}{\rho^3}\sum aP^2=0,$$

with the former significance for coefficients of the type $\bar{a}$, $\bar{\mathrm{a}}$, a.

Manifestly, there are three principal values; and the simple symmetric functions V_1, V_2, V_3, of these principal values, in the forms

$$V_1=\frac{\sum \mathrm{a}P^2}{\sum aP^2}, \quad V_2=\frac{\sum \bar{\mathrm{a}}P^2}{\sum aP^2}, \quad V_3=\frac{\sum \bar{a}P^2}{\sum aP^2},$$

are three volumetric measures of circular curvature of domainal geodesics originating in the volumetric orientation specified by the orientation-variables P, Q, R, T. As these measures V_1, V_2, V_3, involve only the ratios $P:Q:R:T$ of the orientation-variables, the relation $\sum aP^2=\Theta$ does not affect the values of the measures.

But the critical equations

$$v_i-\frac{1}{\rho}u_i=\xi X_i$$

also determine the ratios (and, with $\sum Ap'^2=1$, determine the values) of the set of direction-variables to be associated with a principal value of ρ; and thus there are three principal directions in any volumetric orientation. As in § 368, these three principal directions are orthogonal to one another.

The value of ξ is determinable as before. Because

$$\sum aP^2 = \Theta,$$

and the quantities X, are the orientation-variables P, we have

$$\begin{aligned} \xi^2\Theta &= \xi^2 \sum aP^2 \\ &= \sum a\left(v_1 - \frac{u_1}{\rho}\right)^2 = \frac{\Omega}{\sigma^2}, \end{aligned}$$

where $1/\sigma$ is the torsion of the domainal geodesic in the direction p', q', r', t'; and thus the critical equations are the four of the type

$$v_1 - \frac{1}{\rho} u_1 = \left(\frac{\Omega}{\Theta}\right)^{\frac{1}{2}} \frac{1}{\sigma} P,$$

the equations referring to the principal directions in the volumetric orientation.

Ex. Show that when the primary domain is referred to its own four curves of curvature as parametric curves, the equation for the principal circular curvatures in the volumetric orientation P, Q, R, T, is

$$\frac{P^2}{\bar{A} - \frac{A}{\rho}} + \frac{Q^2}{\bar{B} - \frac{B}{\rho}} + \frac{R^2}{\bar{C} - \frac{C}{\rho}} + \frac{T^2}{\bar{D} - \frac{D}{\rho}} = 0.$$

370. It thus appears that, for any volumetric orientation in a primary domain, there are three distinct measures of circular curvature of the domain, each of the measures being a homogeneous function (of order zero) of the variables of the orientation. They have been constructed from the circular curvatures of domainal geodesics originating in the orientation. But the results are independent of all such geodesics; they constitute volumetric measures of domainal curvature in any orientation, and they are dependent solely upon the orientation.

Each of these three volumetric measures V_1, V_2, V_3, acquires a range of values when all the possible volumetric orientations in the domain are taken into account. Thus each of them, by itself, has its own principal values—that is, the values which are a maximum or a minimum within its own range; it will appear that, for the primary domain, these principal values are compounded of the principal values of the circular curvature of domainal geodesics. Corresponding to each principal value, there are principal volumetric orientations; it appears that these principal orientations have, as their leading lines, appropriate sets of the principal directions of the circular curvature of domainal geodesics.

The three measures will be considered briefly, in succession.

I. The general value of the measure V_3 is

$$V_3 = \frac{\sum \bar{a}P^2}{\sum aP^2}.$$

Hence, by the customary analysis for the determination of the maximum and the minimum values of V_3, we find that these principal values are the roots of the quartic equation

$$\begin{vmatrix} \bar{a}-aV_3, & \bar{h}-hV_3, & \bar{g}-gV_3, & \bar{l}-lV_3 \\ \bar{h}-hV_3, & \bar{b}-bV_3, & \bar{f}-fV_3, & \bar{m}-mV_3 \\ \bar{g}-gV_3, & \bar{f}-fV_3, & \bar{c}-cV_3, & \bar{n}-nV_3 \\ \bar{l}-lV_3, & \bar{m}-mV_3, & \bar{n}-nV_3, & \bar{d}-dV_3 \end{vmatrix}=0.$$

In this equation, when the determinant is expanded, the term independent of V_3

$$=|\,\bar{a}\bar{b}\bar{c}\bar{d}\,|=Y^3;$$

the coefficient of $-V_3$

$$=\sum\left(a\frac{\partial}{\partial\bar{a}}\right)|\,\bar{a}\bar{b}\bar{c}\bar{d}\,|$$

$$=Y^2\sum a\bar{A}-Y^2E_1,$$

where E_1 is the invariant from § 367; the coefficient of V_3^2

$$=\sum\left(a\frac{\partial}{\partial\bar{a}}\right)^2|\,\bar{a}\bar{b}\bar{c}\bar{d}\,|$$

$$=\Omega Y\left(\sum AB\frac{\partial^2 Y}{\partial\bar{A}\,\partial\bar{B}}\right)=\Omega YE_2,$$

with the same source for E_2; the coefficient of V_3^4

$$=|\,abcd\,|=\Omega^3;$$

and the coefficient of $-V_3^3$

$$=\sum\left(\bar{a}\frac{\partial}{\partial a}\right)|\,abcd\,|$$

$$=\Omega^2\sum\bar{a}A=\Omega^2E_3.$$

Consequently the equation, giving the principal values of the measure V_3 is

$$Y^3-Y^2E_1V_3+Y\Omega E_2V_3^2-\Omega^2E_3V_3^3+\Omega^3V_3^4=0.$$

Having regard to the equation for the principal values of the circular curvature of domainal geodesics on p. 550, we see at once that the principal values of V_3 are

$$\frac{1}{\rho_2\rho_3\rho_4},\quad\frac{1}{\rho_3\rho_4\rho_1},\quad\frac{1}{\rho_4\rho_1\rho_2},\quad\frac{1}{\rho_1\rho_2\rho_3}:$$

that is, the principal values of V_3 are compounded of the principal values of the circular curvature of the domainal geodesics.

II. The general value of the measure V_2 is

$$V_2=\frac{\sum\bar{a}P^2}{\sum aP^2};$$

and the equation, which determines the principal values of V_2 among all the values arising from the variety of orientations, is similarly found to be the quartic equation

$$\begin{vmatrix} \bar{\mathfrak{a}}-aV_2, & \bar{\mathfrak{h}}-hV_2, & \bar{\mathfrak{g}}-gV_2, & \bar{\mathfrak{l}}-lV_2 \\ \bar{\mathfrak{h}}-hV_2, & \bar{\mathfrak{b}}-bV_2, & \bar{\mathfrak{f}}-fV_2, & \bar{\mathfrak{m}}-mV_2 \\ \bar{\mathfrak{g}}-gV_2, & \bar{\mathfrak{f}}-fV_2, & \bar{\mathfrak{c}}-cV_2, & \bar{\mathfrak{n}}-nV_2 \\ \bar{\mathfrak{l}}-lV_2, & \bar{\mathfrak{m}}-mV_2, & \bar{\mathfrak{n}}-nV_2, & \bar{\mathfrak{d}}-dV_2 \end{vmatrix}=0.$$

The coefficient of V_2^4 in this quartic equation

$$=|\,abcd\,|=\Omega^3.$$

The coefficients of the other terms can be evaluated by using the forms connected with the reference of the whole domain to its curves of circular curvature as the parametric curves, as on p. 551. For these forms, the preceding invariantive equation becomes

$$(\bar{\mathfrak{a}}_0-a_0V_2)(\bar{\mathfrak{b}}_0-b_0V_2)(\bar{\mathfrak{c}}_0-c_0V_2)(\bar{\mathfrak{d}}_0-d_0V_2)=0.$$

The roots of the equation in this form are of the type

$$\begin{aligned} V_2&=\frac{\bar{\mathfrak{a}}_0}{a_0} \\ &=\frac{1}{B_0C_0D_0}(B_0\bar{C}_0\bar{D}_0+C_0\bar{D}_0\bar{B}_0+D_0\bar{B}_0\bar{C}_0) \\ &=\frac{\bar{C}_0}{C_0}\frac{\bar{D}_0}{D_0}+\frac{\bar{D}_0}{D_0}\frac{\bar{B}_0}{B_0}+\frac{\bar{B}_0}{B_0}\frac{\bar{C}_0}{C_0} \\ &=\frac{1}{\rho_3\rho_4}+\frac{1}{\rho_4\rho_2}+\frac{1}{\rho_2\rho_3}. \end{aligned}$$

Thus the roots of the general quartic equation are the four quantities of this type, obtained by taking the like combinations of the roots $\rho_1, \rho_2, \rho_3, \rho_4$, of the equation

$$\frac{1}{\rho^4}\Omega-\frac{1}{\rho^3}E_1+\frac{1}{\rho^2}E_2-\frac{1}{\rho}E_3+\Upsilon=0,$$

which give the principal values of the circular curvature of domainal geodesics.

Denoting the four roots of this quartic in $1/\rho$ by $\alpha, \beta, \gamma, \delta$, we take a quantity

$$\xi=\gamma\delta+\delta\beta+\beta\gamma,$$

which can have four values arising out of the various combinations of the four roots $\alpha, \beta, \gamma, \delta$; and the quartic equation satisfied by ξ is

$$\begin{aligned} &\Omega^3\xi^4-2\Omega^2E_2\xi^3+\Omega(E_2^2+E_1E_3+2\Omega\Upsilon)\xi^2 \\ &\qquad -(\Omega E_3^2+E_1E_2E_3+2\Omega E_2\Upsilon-E_1^2\Upsilon)\xi+(E_2E_3^2-E_1E_3\Upsilon+\Omega\Upsilon^2)=0. \end{aligned}$$

Consequently, when we take $\xi=V_2$, we have the quartic equation giving the principal values of V_2; and the roots of this equation are known in terms of the

roots of the equation for the principal values of the circular curvature of domainal geodesics: or the principal values of V_2 are compounded of the principal values of these circular curvatures.

Moreover, as the expressed values of invariantive forms, we have the results:

$$\sum\left\{\left(\bar{a}\frac{\partial}{\partial a}\right)|\,abcd\,|\right\}=2\Omega^2E_2,$$

and therefore

$$\sum \bar{a}A=2E_2;$$

also

$$\sum\left\{\left(\bar{a}\frac{\partial}{\partial a}\right)^2|\,abcd\,|\right\}=\Omega(E_2{}^2+E_1E_3+2\Omega Y),$$

$$|\,\bar{a}\bar{b}\bar{c}\bar{d}\,|=E_2E_3{}^2-E_1E_3Y+\Omega Y^2,$$

$$\sum\left\{\left(a\frac{\partial}{\partial \bar{a}}\right)|\,\bar{a}\bar{b}\bar{c}\bar{d}\,|\right\}=\Omega E_3{}^2+E_1E_2E_3+2\Omega E_2Y-E_1{}^2Y.$$

III. The general value of the measure V_1 is

$$V_1=\frac{\sum \mathrm{a}P^2}{\sum aP^2};$$

and the quartic equation, which determines the principal values of V_1 among all its possible values, is found to be

$$\begin{vmatrix} \mathrm{a}-aV_1, & \mathrm{h}-hV_1, & \mathrm{g}-gV_1, & \mathrm{l}-lV_1 \\ \mathrm{h}-hV_1, & \mathrm{b}-bV_1, & \mathrm{f}-fV_1, & \mathrm{m}-mV_1 \\ \mathrm{g}-gV_1, & \mathrm{f}-fV_1, & \mathrm{c}-cV_1, & \mathrm{n}-nV_1 \\ \mathrm{l}-lV_1, & \mathrm{m}-mV_1, & \mathrm{n}-nV_1, & \mathrm{d}-dV_1 \end{vmatrix}=0,$$

where the term involving $V_1{}^4$ is $\Omega^3V_1{}^4$.

In the same way as for the quartic equation which determines the principal values of V_2, we determine the explicit forms of the coefficients of this equation in terms of the invariants Y, E_3, E_2, E_1, by means of the canonical forms occurring when the whole domain is referred to its curves of curvature. The quartic equation then becomes

$$(\mathrm{a}_0-a_0V_1)(\mathrm{b}_0-b_0V_1)(\mathrm{c}_0-c_0V_1)(\mathrm{d}_0-d_0V_1)=0;$$

and, in this form, its four roots are of the type

$$\begin{aligned} V_1&=\frac{\mathrm{a}_0}{a_0} \\ &=\frac{1}{B_0C_0D_0}(\bar{B}_0C_0D_0+\bar{C}_0D_0B_0+\bar{D}_0B_0C_0) \\ &=\frac{\bar{B}_0}{B_0}+\frac{\bar{C}_0}{C_0}+\frac{\bar{D}_0}{D_0} \\ &=\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}. \end{aligned}$$

Thus the roots of the quartic equation for V_1, in its unspecialised form, are the four quantities of this type constructed from the roots of the equation

$$\frac{1}{\rho^4}\Omega-\frac{1}{\rho^3}E_1+\frac{1}{\rho^2}E_2-\frac{1}{\rho}E_3+Y=0,$$

which determines the principal circular curvatures of the domainal geodesics.

As before, we denote the four roots of this equation by α, β, γ, δ; and we use a (four-valued) magnitude η defined by the relation

$$\eta=\beta+\gamma+\delta.$$

The quartic equation satisfied by η is

$$\begin{aligned}\Omega^3\eta^4-3\Omega^2E_1\eta^3+\Omega(3E_1^2+\Omega E_2)\eta^2\\ -(E_1^3+2\Omega E_1E_2-\Omega^2E_3)\eta+(\Omega^2Y-\Omega E_1E_3+E_1^2E_2)=0.\end{aligned}$$

Consequently, when we take $\eta=V_1$, we have the explicit form of the quartic equation which determines the principal values of V_1; and the roots of this equation are known in terms of the roots of the equation for the principal values of the circular curvature of the domainal geodesics: or the principal values of V_1 are compounded of the principal values of these circular curvatures.

Moreover, we have the following values of the invariantive coefficients in the equation:

$$\sum\left\{\left(\mathrm{a}\frac{\partial}{\partial a}\right)|\,abcd\,|\right\}=3\Omega^2E_1,$$

and therefore

$$\sum \mathrm{a}A=3E_1;$$

also

$$\sum\left\{\left(\mathrm{a}\frac{\partial}{\partial a}\right)^2|\,abcd\,|\right\}=\Omega(3E_1^2+\Omega E_2),$$

$$|\,\mathbf{abcd}\,|=\Omega^2Y-\Omega E_1E_3+E_1^2E_2,$$

$$\sum\left\{\left(a\frac{\partial}{\partial \mathrm{a}}\right)|\,\mathbf{abcd}\,|\right\}=E_1^3+2\Omega E_1E_2-\Omega^2E_3.$$

IV. The principal orientations, for each of the three measures V_1, V_2, V_3, of domainal curvature in volumetric orientation, are compounded from the different sets of three directions, which can be selected from the four directions of curves of circular curvature of the domain.

The result follows at once when the whole domain is referred to its curves of circular curvature as parametric curves. For each of the three principal measures of volumetric curvature of the domain, the principal orientations are seen to be the four of the type

$$Q,\ R,\ T,\ =0;\quad P\neq 0:$$

that is, as

$$P, Q, R, T, = \left\| \begin{matrix} p_1', & q_1', & r_1', & t_1' \\ p_2', & q_2', & r_2', & t_2' \\ p_3', & q_3', & r_3', & t_3' \end{matrix} \right\|,$$

in general, we have, for the particular reference of the selected orientation,

$$P, Q, R, T, = \left\| \begin{matrix} 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{matrix} \right\|.$$

Thus the principal volumetric orientations of a primary domain are compounded by taking, as their leading lines, the sets of three out of the four directions of curves of circular curvature of domainal geodesics.

Superficial measures of curvature of a primary domain.

371. When we consider the aggregate of domainal geodesics, which originate in a superficial orientation with surface-variables s_{12}, s_{23}, s_{31}, s_{14}, s_{24}, s_{34}, that are subject to the relation

$$s_{23}s_{14} + s_{31}s_{24} + s_{12}s_{34} = 0,$$

(or, what is the same thing, the domainal geodesics which touch a surface given as the intersection of two regions $\epsilon = 0$, $\omega = 0$, the orientation-variables then being expressible in terms of parametric derivatives of ϵ and ω, as in § 270), the direction-variables p', q', r', t', of such geodesics are subject to the conditions

$$\sum Ap'^2 = 1, \quad p's_{23} + q's_{31} + r's_{12} = 0, \quad p's_{24} + q's_{41} + t's_{12} = 0,$$

where we retain only two independent relations out of the four interdependent relations of § 272.

Among these orientated directions, there are principal directions: that is, such as provide a maximum or a minimum among all the values of the circular curvature of domainal geodesics for the admissible values of the direction-variables. Such directions are obtained by assigning the conditions for a maximum or a minimum value of $1/\rho$; the equations are

$$\begin{aligned} v_1 &= \kappa u_1 + \lambda s_{23} + \mu s_{24}, \\ v_2 &= \kappa u_2 + \lambda s_{31} + \mu s_{41}, \\ v_3 &= \kappa u_3 + \lambda s_{12} \qquad\quad , \\ v_4 &= \kappa u_4 \qquad\qquad + \mu s_{12}, \end{aligned}$$

where the quantities κ, λ, μ, are left undetermined in the construction of the critical equations. The value of κ is found at once, on multiplying the equations by p', q', r', t', respectively: adding: and using the conditions attaching to the variables: it is

$$\kappa = \frac{1}{\rho}.$$

Thus the four equations are

$$\left(\bar{A}-\frac{A}{\rho}\right)p'+\left(\bar{H}-\frac{H}{\rho}\right)q'+\left(\bar{G}-\frac{G}{\rho}\right)r'+\left(\bar{L}-\frac{L}{\rho}\right)t'=\lambda s_{23}+\mu s_{24},$$
$$\left(\bar{H}-\frac{H}{\rho}\right)p'+\left(\bar{B}-\frac{B}{\rho}\right)q'+\left(\bar{F}-\frac{F}{\rho}\right)r'+\left(\bar{M}-\frac{M}{\rho}\right)t'=\lambda s_{31}+\mu s_{41},$$
$$\left(\bar{G}-\frac{G}{\rho}\right)p'+\left(\bar{F}-\frac{F}{\rho}\right)q'+\left(\bar{C}-\frac{C}{\rho}\right)r'+\left(\bar{N}-\frac{N}{\rho}\right)t'=\lambda s_{12}\qquad,$$
$$\left(\bar{L}-\frac{L}{\rho}\right)p'+\left(\bar{M}-\frac{M}{\rho}\right)q'+\left(\bar{N}-\frac{N}{\rho}\right)r'+\left(\bar{D}-\frac{D}{\rho}\right)t'=\qquad\mu s_{12};$$

and there are also the two conditions

$$s_{23}p'+s_{31}q'+s_{12}r'=0,\quad s_{24}p'+s_{41}q'+s_{12}t'=0.$$

When the quantities p', q', r', t', λ, μ, are eliminated determinantally, there results the equation

$$\begin{vmatrix} \bar{A}-\frac{A}{\rho}, & \bar{H}-\frac{H}{\rho}, & \bar{G}-\frac{G}{\rho}, & \bar{L}-\frac{L}{\rho}, & s_{23}, & s_{24} \\ \bar{H}-\frac{H}{\rho}, & \bar{B}-\frac{B}{\rho}, & \bar{F}-\frac{F}{\rho}, & \bar{M}-\frac{M}{\rho}, & s_{31}, & s_{41} \\ \bar{G}-\frac{G}{\rho}, & \bar{F}-\frac{F}{\rho}, & \bar{C}-\frac{C}{\rho}, & \bar{N}-\frac{N}{\rho}, & s_{12}, & 0 \\ \bar{L}-\frac{L}{\rho}, & \bar{M}-\frac{M}{\rho}, & \bar{N}-\frac{N}{\rho}, & \bar{D}-\frac{D}{\rho}, & 0, & s_{12} \\ s_{23}, & s_{31}, & s_{12}, & 0, & 0, & 0 \\ s_{24}, & s_{41}, & 0, & s_{12}, & 0, & 0 \end{vmatrix}=0,$$

obviously a quadratic in $1/\rho$. In evaluating the determinant, the relation $s_{23}s_{14}+s_{31}s_{24}+s_{12}s_{34}=0$ is used, and an irrelevant factor $s_{12}{}^2$ can be removed. Let

$$S_0=\sum\{(AB-H^2)s_{12}{}^2\}=\sum\{(A_{ik}A_{jl}-A_{il}A_{jk})s_{ij}s_{kl}\},$$
$$S_1=\sum\{(A\bar{B}-2H\bar{H}+B\bar{A})s_{12}{}^2\},$$
$$=\sum\{(A_{ik}\bar{A}_{jl}-A_{il}\bar{A}_{jk}-A_{jk}\bar{A}_{il}+A_{jl}\bar{A}_{ik})s_{ij}s_{kl}\},$$
$$S_2=\sum\{(\bar{A}\bar{B}-\bar{H}^2)s_{12}{}^2\}=\sum\{(\bar{A}_{ik}\bar{A}_{jl}-\bar{A}_{il}\bar{A}_{jk})s_{ij}s_{kl}\},$$

so that the quantities S_0, S_1, S_2, are surface-covariants of the whole system of concomitants of the domain; then the foregoing quadratic equation becomes

$$S_2-\frac{1}{\rho}S_1+\frac{1}{\rho^2}S_0=0.$$

Thus there are two principal values of the circular curvature of domainal geodesics originating in any superficial orientation within the domain; and, in

every such orientation, there are two principal directions of circular curvature. Let these two circular curvatures be denoted by $1/\bar{\rho}_1$ and $1/\bar{\rho}_2$; then there are two superficial measures, $\mathbf{K}_s$ and $\mathbf{H}_s$, of domainal curvature belonging to any superficial orientation, and they are

$$\mathbf{H}_s = \frac{1}{\bar{\rho}_1} + \frac{1}{\bar{\rho}_2} = \frac{S_1}{S_0}, \quad \mathbf{K}_s = \frac{1}{\bar{\rho}_1 \bar{\rho}_2} = \frac{S_2}{S_0}.$$

These two measures correspond to the two measures (K and H, the "specific" curvature and the "additive" curvature) of a Gaussian surface in triple homaloidal space. Of these two superficial measures, $\mathbf{K}_s$ actually is the same as the sphericity (the Riemann measure of curvature) of the domain estimated in the assigned orientation.

Further, the two principal directions of circular curvature of domainal geodesics lying in the assigned orientation are at right angles to one another. Let the direction-variables of these leading directions be p_1', q_1', r_1', t_1', (with quantities $\kappa_1, \lambda_1, \mu_1$, for the critical equations, the value of κ_1 being $1/\bar{\rho}_1$), and p_2', q_2', r_2', t_2', (with quantities $\kappa_2, \lambda_2, \mu_2$, for the critical equations, the value of κ_2 being $1/\bar{\rho}_2$). On multiplying the four critical equations for the direction p_1', q_1', r_1', t_1', by p_2', q_2', r_2', t_2', respectively, and adding the products, we have

$$\sum (\bar{A} p_1' p_2') - \frac{1}{\bar{\rho}_1} (\sum A p_1' p_2')$$
$$= \lambda_1 (s_{23} p_2' + s_{31} q_2' + s_{12} r_2') + \mu_1 (s_{24} p_2' + s_{41} q_2' + s_{12} t_2') = 0;$$

and similarly, on multiplying the four critical equations for the direction p_2', q_2', r_2', t_2', by p_1', q_1', r_1', t_1', respectively, and adding the products, we have

$$\sum (\bar{A} p_2' p_1') - \frac{1}{\bar{\rho}_2} (\sum A p_1' p_2')$$
$$= \lambda_2 (s_{23} p_1' + s_{31} q_1' + s_{12} r_1') + \mu_2 (s_{24} p_1' + s_{41} q_1' + s_{12} t_1') = 0.$$

We assume that the orientation is arbitrarily selected, so that $\bar{\rho}_1$ and $\bar{\rho}_2$ are unequal; then the two relations require the conditions

$$\sum \bar{A} p_1' p_2' = 0, \quad \sum A p_1' p_2' = 0,$$

the second of which shews that the two leading directions in the assigned orientation are at right angles to one another.

Ex. 1. Verify that each of the magnitudes S_0, S_1, S_2, satisfies the two partial differential equations

$$\frac{\partial^2 S}{\partial s_{23}\, \partial s_{14}} + \frac{\partial^2 S}{\partial s_{31}\, \partial s_{24}} + \frac{\partial^2 S}{\partial s_{12}\, \partial s_{34}} = 0,$$

$$\frac{\partial S}{\partial s_{23}} \frac{\partial S}{\partial s_{14}} + \frac{\partial S}{\partial s_{31}} \frac{\partial S}{\partial s_{24}} + \frac{\partial S}{\partial s_{12}} \frac{\partial S}{\partial s_{34}} = 0.$$

Ex. 2. Shew that the quantities

$$\sum \left\{ (ab-h^2) \left(\frac{\partial S_0}{\partial s_{12}} \right)^2 \right\},$$

$$\sum \left\{ (a\bar{b} - 2h\bar{h} + b\bar{a}) \left(\frac{\partial S_1}{\partial s_{12}} \right)^2 \right\},$$

$$\sum \left\{ (\bar{a}\bar{b} - \bar{h}^2) \left(\frac{\partial S_2}{\partial s_{12}} \right)^2 \right\},$$

are concomitants of the whole system of the domain ; and obtain their values.

372. These two measures of superficial curvatures are functions of position in the domain and of the orientation-variables s_{ij}; and therefore, like the circular curvature of geodesics as a function of direction-variables, and like the volumetric curvatures as functions of the regional variables, they also have principal values : that is, values which are a maximum or a minimum among those arising when all the possible superficial orientations are taken into consideration.

Proceeding as in § 313, we find that the critical equations for the principal values of $\mathbf{H}_s$ are the set

$$\frac{\partial S_1}{\partial s_{ij}} - \mathbf{H}_s \frac{\partial S_0}{\partial s_{ij}} = 0,$$

for the six combinations $ij = 23$, 31, 12, 14, 24, 34. These equations are linear and homogeneous in the six linearly independent orientation-variables s_{ij}; and when these variables are eliminated, there results a sextic equation in $\mathbf{H}_s$. Thus there are six principal values for $\mathbf{H}_s$; and also there are six principal orientations for that measure of superficial curvature of the domain.

Similarly, the critical equations for the principal values of $\mathbf{K}_s$ are the set

$$\frac{\partial S_2}{\partial s_{ij}} - \mathbf{K}_s \frac{\partial S_0}{\partial s_{ij}} = 0,$$

for the same six combinations ij. The principal values of $\mathbf{K}_s$, thus determinable, are six in number, being the roots of another sextic equation; and associated with each such principal value, there is a principal superficial orientation.

The forms of the sextic equations are derivable immediately from the respective sets of critical equations; the discussion of their roots is simplified by referring the primary domain to its curves of circular curvature as the parametric curves. When this reference is effected, and when the orientation-variables of the superficial orientation continue to be denoted by s_{ij}, we find

$$\begin{aligned}
S_0 = {} & A_0B_0s_{12}^2 + C_0A_0s_{31}^2 + B_0C_0s_{23}^2 + A_0D_0s_{14}^2 + B_0D_0s_{24}^2 + C_0D_0s_{34}^2, \\
S_1 = {} & (A_0\bar{B}_0 + B_0\bar{A}_0)s_{12}^2 + (C_0\bar{A}_0 + A_0\bar{C}_0)s_{31}^2 + (B_0\bar{C}_0 + C_0\bar{B}_0)s_{23}^2 \\
& + (C_0\bar{D}_0 + D_0\bar{C}_0)s_{34}^2 + (B_0\bar{D}_0 + D_0\bar{B}_0)s_{24}^2 + (A_0\bar{D}_0 + D_0\bar{A}_0)s_{14}^2, \\
S_2 = {} & \bar{A}_0\bar{B}_0s_{12}^2 + \bar{C}_0\bar{A}_0s_{31}^2 + \bar{B}_0\bar{C}_0s_{23}^2 + \bar{A}_0\bar{D}_0s_{14}^2 + \bar{B}_0\bar{D}_0s_{24}^2 + \bar{C}_0\bar{D}_0s_{34}^2.
\end{aligned}$$

I. Consider the principal values of $\mathbf{K}_s$; they are given, together with the ratios of the variables of the associated principal superficial orientations, by the six equations

$$\frac{\partial S_2}{\partial s_{ij}} = \mathbf{K}_s \frac{\partial S_0}{\partial s_{ij}},$$

that is, by the six equations

$$\left.\begin{array}{ll} \bar{A}_0\bar{B}_0 s_{12} = \mathbf{K}_s A_0 B_0 s_{12}, & \bar{A}_0\bar{D}_0 s_{14} = \mathbf{K}_s A_0 D_0 s_{14} \\ \bar{C}_0\bar{A}_0 s_{31} = \mathbf{K}_s C_0 A_0 s_{31}, & \bar{B}_0\bar{D}_0 s_{24} = \mathbf{K}_s B_0 D_0 s_{24} \\ \bar{B}_0\bar{C}_0 s_{23} = \mathbf{K}_s A_0 B_0 s_{23}, & \bar{C}_0\bar{D}_0 s_{34} = \mathbf{K}_s C_0 D_0 s_{34} \end{array}\right\}.$$

Not all the quantities s_{ij} can vanish; but their simultaneous values are subject to the relation

$$s_{23}s_{14} + s_{31}s_{24} + s_{12}s_{34} = 0.$$

In the first place, let s_{12} be distinct from zero; then

$$\bar{A}_0\bar{B}_0 = \mathbf{K}_s A_0 B_0,$$

that is,

$$\mathbf{K}_s = \frac{\bar{A}_0}{A_0}\frac{\bar{B}_0}{B_0} = \frac{1}{\rho_1\rho_2},$$

where ρ_1 and ρ_2 are principal radii of circular curvature of domainal geodesics. With this value of $\mathbf{K}_s$, an equation

$$\bar{C}_0\bar{A}_0 s_{31} = \mathbf{K}_s C_0 A_0 s_{31}$$

becomes

$$\frac{1}{\rho_3\rho_1} s_{31} = \mathbf{K}_s s_{31},$$

which, on the hypothesis that the principal radii of circular curvature are unequal, (that is, a hypothesis of unconditioned generality for the domain), can be satisfied only by

$$s_{31} = 0.$$

Similarly, for this value of $\mathbf{K}_s$, all the other equations can be satisfied only if

$$s_{23} = 0, \quad s_{14} = 0, \quad s_{24} = 0, \quad s_{34} = 0.$$

When the sets of values of the variables s are thus known, and note is taken of their definitions as given in § 272, we see that the two parametric curves p = variable, q, r, t = constants; and q = variable, p, r, t, = constants; that is, the directions of the two curves of curvature, corresponding to ρ_1 and ρ_2, are the guiding lines of the superficial orientation. We thus have a principal value of $\mathbf{K}_s$ given as the product

$$\frac{1}{\rho_1\rho_2}$$

of two principal circular curvatures of domainal geodesics, and the corresponding

principal orientation is compounded of the curves of curvature appertaining to those principal circular curvatures.

Similarly for each of the other possibilities from the critical equations. The final inference is that the six principal values of the superficial measure $\mathbf{K}_s$ are the products of the six several pairs of principal curvatures of domainal geodesics

$$\frac{1}{\rho_2\rho_3}, \quad \frac{1}{\rho_3\rho_1}, \quad \frac{1}{\rho_1\rho_2}, \quad \frac{1}{\rho_1\rho_4}, \quad \frac{1}{\rho_2\rho_4}, \quad \frac{1}{\rho_3\rho_4},$$

and that the principal orientation, appertaining to a principal measure $\mathbf{K}_s$, is compounded of the directions of the two curves of curvature appertaining to those circular curvatures of which the measure $\mathbf{K}_s$ is the product.

II. When the measure $\mathbf{H}_s$ is similarly discussed, the final inference is that its six principal values are the sums of the six several pairs of principal curvatures of domainal geodesic

$$\frac{1}{\rho_2}+\frac{1}{\rho_3}, \quad \frac{1}{\rho_3}+\frac{1}{\rho_1}, \quad \frac{1}{\rho_1}+\frac{1}{\rho_2}, \quad \frac{1}{\rho_1}+\frac{1}{\rho_4}, \quad \frac{1}{\rho_2}+\frac{1}{\rho_4}, \quad \frac{1}{\rho_3}+\frac{1}{\rho_4},$$

and that the principal orientation, appertaining to a principal value of the measure, is compounded of the directions of the two curves of curvature appertaining to those principal circular curves of which that value is the sum.

To obtain the developed expressions of the equations (in their general form) for $\mathbf{K}_s$ and $\mathbf{H}_s$, we proceed as follows. When the roots of the quartic equation

$$a_0x^4 - a_1x^3 + a_2x^2 - a_3x + a_4 = 0$$

are denoted by α, β, γ, δ, and when ξ and ζ denote the six-valued quantities

$$\xi = \alpha\beta, \quad \zeta = \alpha+\beta,$$

then ξ is a root of the sextic equation

$$\begin{aligned} &a_0{}^3\xi^6 - a_0{}^2a_2\xi^5 + (a_0a_1a_3 - a_0{}^2a_4)\xi^4 \\ &\quad - (a_0a_3{}^2 + a_1{}^2a_4 - 2a_0a_2a_4)\xi^3 + (a_1a_3a_4 - a_0a_4{}^2)\xi^2 - a_2a_4{}^2\xi + a_4{}^3 = 0, \end{aligned}$$

and ζ is a root of the sextic equation

$$\begin{aligned} &a_0{}^3\zeta^6 - 3a_0{}^2a_1\zeta^5 + (2a_0{}^2a_2 + 3a_0a_1{}^2)\zeta^4 - (4a_0a_1a_2 + a_1{}^3)\zeta^3 \\ &\quad + (a_0a_1a_3 + a_0a_2{}^2 + 2a_1{}^2a_2 - 4a_0{}^2a_4)\zeta^2 - (a_1a_2{}^2 + a_1{}^2a_3 - 4a_0a_1a_4)\zeta \\ &\qquad\qquad + (a_1a_2a_3 - a_3{}^2 - a_1{}^2a_4) = 0. \end{aligned}$$

The principal values of $\mathbf{K}_s$ are given by the ξ-equation, those of $\mathbf{H}_s$ by the ζ-equation.

As regards the equation for the principal values of $\mathbf{K}_s$, where

$$\mathbf{K}_s = \frac{\sum\{(\bar{A}_{ik}\bar{A}_{jl} - \bar{A}_{il}\bar{A}_{jk})s_{ij}s_{kl}\}}{\sum\{(A_{ik}A_{jl} - A_{il}A_{jk})s_{ij}s_{kl}\}},$$

we use a symbol $[PQ, RS]$ according to the definition

$$[PQ, RS] = \bar{P}\bar{Q} - \bar{R}\bar{S} - \mathbf{K}_s(PQ - RS),$$

where P, Q, R, S, are any four of the quantities A, B, C, D, F, G, H, L, M, N, and similarly for $\bar{P}$, $\bar{Q}$, $\bar{R}$, $\bar{S}$; and then the equation for the principal values for $\mathbf{K}_s$ is

$$\begin{vmatrix} [AB, HH], & [GH, AF], & [HF, BG], & [AM, HL], & [HM, BL], & [GM, FL] \\ [GH, AF], & [CA, GG], & [FG, CH], & [GL, AN], & [FL, HN], & [CL, GN] \\ [HF, BG], & [FG, CH], & [BC, FF], & [HN, GM], & [BN, FM], & [FN, CM] \\ [AM, HL], & [GL, AN], & [HN, GM], & [AD, LL], & [HD, LM], & [GD, LN] \\ [HM, BL], & [PL, HN], & [BN, FM], & [HD, LM], & [BD, MM], & [FD, MN] \\ [GM, FL], & [CL, GN], & [FN, CM], & [GD, LN], & [FD, MN], & [CD, NN] \end{vmatrix} = 0,$$

and its developed form, expressed in terms of the measures of circular curvature of the domainal geodesics E_1, E_2, E_3, Y, is

$$\Omega^3\mathbf{K}_s{}^6 - \Omega^2 E_2\mathbf{K}_s{}^5 + (\Omega E_1 E_3 - \Omega^2 Y)\mathbf{K}_s{}^4$$
$$- (\Omega E_3{}^2 + E_1{}^2 Y - 2\Omega E_2 Y)\mathbf{K}_s{}^3 + (E_1 E_3 Y - \Omega Y^2)\mathbf{K}_s{}^2 - E_2 Y^2 \mathbf{K}_s + Y^3 = 0.$$

The equation for the principal values for $\mathbf{H}_s$ is the same formal determinantal equation as for $\mathbf{K}_s$, when the symbol $[PQ, RS]$ is defined by the relation

$$[PQ, RS] = \bar{P}Q - \bar{R}S - \bar{S}R + \bar{Q}P - \mathbf{H}_s(PQ - RS);$$

its developed form, expressed in terms of the measures of circular curvature of the domainal geodesics E_1, E_2, E_3, Y, is

$$\Omega^3\mathbf{H}_s{}^6 - 3\Omega^2 E_1\mathbf{H}_s{}^5 + (2\Omega^2 E_2 + 3\Omega E_1{}^2)\mathbf{H}_s{}^4 - (4\Omega E_1 E_2 + E_1{}^3)\mathbf{H}_s{}^3$$
$$+ (\Omega E_1 E_3 + \Omega E_2{}^2 + 2E_1{}^2 E_2 - 4\Omega^2 Y)\mathbf{H}_s{}^2 - (E_1 E_2{}^2 + E_1{}^2 E_3 - 4\Omega E_1 Y)\mathbf{H}_s$$
$$+ (E_1 E_2 E_3 - \Omega E_3{}^2 - E_1{}^2 Y) = 0.$$

Also, a comparison of the two forms of each equation leads to the explicit evaluation of invariantive combinations of the two sets of quantities A, H, ..., $\bar{A}$, $\bar{H}$, ..., in terms of the curvature invariants E_1, E_2, E_3, Y, of the domain.

373. When a domain is primary, so that the plenary homaloidal space is quintuple, there are only four kinds of curvature to which any curve in the domain is subject; they are the circular curvature, the torsion, the tilt, and the coil. Hence, for domainal geodesics in particular, the curvatures of higher grade do not exist: that is, a measure of such a curvature is zero.

In § 294, the relation

$$\frac{\Omega^{\frac{1}{2}} l_6}{\sigma^2 \tau \rho_5} = \begin{vmatrix} \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{vmatrix}$$

has been obtained, as typical of the direction-cosines of the fifth normal of a domainal geodesic and the fifth curvature of the geodesic, when the domain exists in any plenary homaloidal space. Accordingly, should the domain be primary, the expression should then provide a zero value for $1/\rho_5$, the typical direction-cosine l_6 then having no significance. For a primary domain, we have

$$\eta_{ij} = y_{ij} - y_1 \Gamma_{ij} - y_2 \Delta_{ij} - y_3 \Theta_{ij} - y_4 \Phi_{ij} = Y \bar{A}_{ij},$$

for all the combinations $i, j, = 1, 2, 3, 4$; and therefore

$$\begin{aligned} \eta_i &= \eta_{i1} p' + \eta_{i2} q' + \eta_{i3} r' + \eta_{i4} t' \\ &= Y(\bar{A}_{i1} p' + \bar{A}_{i2} q' + \bar{A}_{i3} r' + \bar{A}_{i4} t') = Y v_i, \end{aligned}$$

for all the values of i. When these values of the quantities η are inserted, the determinant vanishes, for all the different magnitudes l_6. As these magnitudes do not simultaneously vanish because $\sum l_6^2 = 1$, it follows that $1/\rho_5$ must vanish, the torsion and the tilt not being zero. Thus the necessary requirement is satisfied.

It may be added that all the quantities ξ_i of § 295 vanish for a primary domain because all the magnitudes denoted by m_{ij} vanish; and therefore, as required, the expression for $1/\rho_5$ on p. 343 vanishes for a primary domain.

Centres of circular curvature and of flexure.

374. It has already (§ 255) been proved that the locus of the centre of circular curvature of geodesics of a region existing in a plenary quintuple space is a lemniscate curve which lies in the plane orthogonal to the region.

In connection with this result, it is proper to consider various loci connected with centres of curvature and centres of flexure which appertain to regions and to surfaces in a primary domain. For this purpose we take geodesics, which have a common tangent and belong respectively to a domain, to a region in the domain, and to a surface in the domain: the surface being postulated analytically as the intersection of two regions in the domain. It is not necessary to give further consideration to parametric curves in the domain when it is primary; for the investigation of § 255 is concerned with the quintuple homaloid (that is, with a plenary space for a primary domain) of which the five leading lines are the tangent to the curve, the three domainal normals to the three regions intersecting in the curve, and the normal to the domain.

Accordingly, let two general domainal regions be given by the parametric equations $\epsilon(p, q, r, t) = 0$, $\omega(p, q, r, t) = 0$, as in § 340. Their intersection is a domainal surface. Through any point O on this surface, let a superficial geodesic be drawn; its tangent touches the ϵ-region, the ω-region, and the domain. Let the geodesics in the ϵ-region, the ω-region, and the domain, respectively, be drawn having this direction for their tangent. When the domainal normals to the

ϵ-region and the ω-region are drawn, and also the prime normal to the domainal geodesic (which is the normal to the domain), we have the construction represented in the diagram (§ 342), in which the flat $OF_\epsilon F_\omega Y$ is orthogonal to the tangent plane to the surface at O, this plane not being indicated in the diagram.

Now that the domain is restricted to be primary, the normal OY to the domain is the same for all geodesics through O; that is, for the totality of superficial and other geodesics through O, that normal can be regarded as a fixed line in the plenary space. Also when the ϵ-region is a definite region in the domain (and not merely a member of the set $a_0\epsilon + c_0\omega = 0$, $a_1\epsilon + c_1\omega = 0$, which by their intersection determine the surface), the domainal normal OF_ϵ is a fixed line in the plenary space; and, when the ω-region is similarly regarded, the domainal normal OF_ω is also a fixed line in that space, while OY is at right angles to the plane $F_\epsilon OF_\omega$ which is the domainal plane orthogonal to the surface. We thus have three non-complanar directions OY, OF_ϵ, OF_ω, invariable for all directions through O on the surface: they are convenient leading lines for the flat orthogonal to the surface, and their typical direction-cosines $Y, \dfrac{dy}{dn}, \dfrac{dy}{d\nu}$, do not involve any direction-variables of a superficial geodesic. But the lengths of the lines OY, OF_ϵ, OF_ω, vary from one geodesic direction to another geodesic direction, for they respectively denote the radius of circular curvature of the domainal geodesic, the radius of domainal flexure of the geodesic in the ϵ-region, and the radius of domainal flexure of the geodesic in the ω-region.

The centres of domainal flexure for a geodesic in the ϵ-region, for a geodesic in the ω-region, and for a geodesic on the surface S which is the intersection of the regions, are F_ϵ, F_ω, F_0, respectively. The locus of F_ϵ is the portion of the line OF_ϵ, limited by the principal centres of domainal flexure of the ϵ-region in the orientation of the surface S. Similarly the locus of F_ω is the portion of the line OF_ω, limited by the principal centres of domainal flexure of the ω-region in the same orientation. And it has been proved (§ 365) that the locus of F_0 is a lemniscate curve in the plane $F_\epsilon OF_\omega$, orthogonal to the surface and lying in the tangent block of the domain.

The centres of circular curvature for the four geodesics are: Y, for the domainal geodesic; C_ϵ, the foot of the perpendicular from O on YF_ϵ, for the geodesic in the ϵ-region; C_ω, the foot of the perpendicular from O on YF_ω, for the geodesic in the ω-region; and C_0, the foot of the perpendicular from O on YF_0, for the geodesic on the surface S.

The locus of Y consists of limited portions of the line OY, lying within the ranges between the centres of principal circular curvature of the primary domain: it is not a proper locus in the customary sense of the word, because every point on one of the portions is a centre of circular curvature for an unlimited number of domainal geodesics. The locus of C_ϵ is a lemniscate curve in the plane YOF_ϵ; and, similarly, the locus of C_ω is a lemniscate curve in the plane YOF

Also, we know (§ 141) that the locus of the centre of circular curvature of concurrent geodesics on a surface, existing in a plenary space of more than four dimensions, is a twisted quartic in a flat orthogonal to the surface. In the present instance, when the plenary homaloidal space for S is five-dimensional, this spatial flat orthogonal to the surface is the flat $OF_\epsilon F_\omega Y$; and accordingly the locus of C_0 is a twisted quartic in the flat $OF_\epsilon F_\omega Y$.

These scattered results for circular curvatures can be established together, in relation to the flat, by means of the same set of formulæ when the domain is primary. Let the typical space-coordinates of C_ϵ, C_ω, C_0, be y_ϵ, y_ω, y_0, respectively, so that

$$y_\epsilon - y = Y_\epsilon \rho_\epsilon, \quad y_\omega - y = Y_\omega \rho_\omega, \quad y_0 - y = Y_0 \rho_0,$$

where

$$\frac{Y_\epsilon}{\rho_\epsilon} = \frac{Y}{\rho} + \frac{1}{\gamma_\epsilon}\frac{dy}{dn},$$

$$\frac{Y_\omega}{\rho_\omega} = \frac{Y}{\rho} + \frac{1}{\gamma_\omega}\frac{dy}{d\nu},$$

$$\frac{Y_0}{\rho_0} = \frac{Y}{\rho} + \frac{l}{\gamma} = \frac{Y}{\rho} + g_\epsilon \frac{dy}{dn} + g_\omega \frac{dy}{d\nu}.$$

With the notation of §§ 342, 345, we can take

$$\frac{Y_\epsilon}{\rho_\epsilon} = \eta_{11}{}^{(\epsilon)} p'^2 + 2\eta_{12}{}^{(\epsilon)} p'q' + \eta_{22}{}^{(\epsilon)} q'^2,$$

$$\frac{Y_\omega}{\rho_\omega} = \eta_{11}{}^{(\omega)} p'^2 + 2\eta_{12}{}^{(\omega)} p'q' + \eta_{22}{}^{(\omega)} q'^2,$$

where

$$\left.\begin{aligned}\eta_{11}{}^{(\epsilon)} &= [\eta_{11}] - \frac{1}{\epsilon_n}\bar{E}_{11}\frac{dy}{dn}\\ \eta_{12}{}^{(\epsilon)} &= [\eta_{12}] - \frac{1}{\epsilon_n}\bar{E}_{12}\frac{dy}{dn}\\ \eta_{22}{}^{(\epsilon)} &= [\eta_{22}] - \frac{1}{\epsilon_n}\bar{E}_{22}\frac{dy}{dn}\end{aligned}\right\}, \qquad \left.\begin{aligned}\eta_{11}{}^{(\omega)} &= [\eta_{11}] - \frac{1}{\omega_\nu}\bar{\Omega}_{11}\frac{dy}{d\nu}\\ \eta_{12}{}^{(\omega)} &= [\eta_{12}] - \frac{1}{\omega_\nu}\bar{\Omega}_{12}\frac{dy}{d\nu}\\ \eta_{22}{}^{(\omega)} &= [\eta_{22}] - \frac{1}{\omega_\nu}\bar{\Omega}_{22}\frac{dy}{d\nu}\end{aligned}\right\};$$

also,

$$\frac{Y_0}{\rho_0} = \bar{\eta}_{11} p'^2 + 2\bar{\eta}_{12} p'q' + \bar{\eta}_{22} q'^2,$$

where (§ 345)

$$\bar{\eta}_{11} = [\eta_{11}] + \xi_{11}, \quad \bar{\eta}_{12} = [\eta_{12}] + \xi_{12}, \quad \bar{\eta}_{22} = [\eta_{22}] + \xi_{22}.$$

For the representation of coordinates in the flat, we take the direction OF_ϵ to be the axis of a variable z_1, the direction OF_ω to be the axis of a variable z_2, and

the direction OY to be the axis of a variable z, the last axis being perpendicular to the plane z_1Oz_2 through the other two, and the angle z_1Oz_2 being denoted by ι as before. Then for any current point in the flat, with a typical space-variable $\bar{y}$, we have coordinates z_1, z_2, z, referred to these axes, with the definitions

$$\sum\left\{(\bar{y}-y)\frac{dy}{dn}\right\}=z_1+z_2\cos\iota,\quad \sum\left\{(\bar{y}-y)\frac{dy}{d\nu}\right\}=z_2+z_1\cos\iota,\quad \sum\{(\bar{y}-y)Y\}=z.$$

375. I. We begin with the locus of C_ϵ. For that locus, we have

$$\frac{y_\epsilon-y}{\rho_\epsilon^2}=\frac{Y_\epsilon}{\rho_\epsilon}=\eta_{11}^{(\epsilon)}p'^2+2\eta_{12}^{(\epsilon)}p'q'+\eta_{22}^{(\epsilon)}q'^2.$$

Now

$$\sum Y\eta_{11}^{(\epsilon)}=\sum\{Y[\eta_{11}]\}-\frac{1}{\epsilon_n}\bar{E}_{11}\left\{\sum Y\frac{dy}{dn}\right\}=\bar{A}_0,$$

and similarly

$$\sum Y\eta_{12}^{(\epsilon)}=\bar{H}_0,\quad \sum Y\eta_{22}^{(\epsilon)}=\bar{B}_0;$$

hence

$$\frac{1}{\rho_\epsilon^2}z=\bar{A}_0p'^2+2\bar{H}_0p'q'+\bar{B}_0q'^2.$$

Again,

$$\sum\frac{dy}{dn}\eta_{11}^{(\epsilon)}=\sum\left\{\frac{dy}{dn}[\eta_{11}]\right\}-\frac{1}{\epsilon_n}\bar{E}_{11}\left\{\sum\left(\frac{dy}{dn}\right)^2\right\}=-\frac{1}{\epsilon_n}\bar{E}_{11},$$

$$\sum\frac{dy}{d\nu}\eta_{11}^{(\epsilon)}=\sum\left\{\frac{dy}{d\nu}[\eta_{11}]\right\}-\frac{1}{\epsilon_n}\bar{E}_{11}\left\{\sum\frac{dy}{dn}\frac{dy}{d\nu}\right\}=-\frac{1}{\epsilon_n}\bar{E}_{11}\cos\iota;$$

and similarly

$$\sum\frac{dy}{dn}\eta_{12}^{(\epsilon)}=-\frac{1}{\epsilon_n}\bar{E}_{12},\quad \sum\frac{dy}{d\nu}\eta_{12}^{(\epsilon)}=-\frac{1}{\epsilon_n}\bar{E}_{12}\cos\iota,$$

$$\sum\frac{dy}{dn}\eta_{22}^{(\epsilon)}=-\frac{1}{\epsilon_n}\bar{E}_{22},\quad \sum\frac{dy}{d\nu}\eta_{22}^{(\epsilon)}=-\frac{1}{\epsilon_n}E_{22}\cos\iota.$$

Consequently

$$\frac{1}{\rho_\epsilon^2}(z_1+z_2\cos\iota)=-\frac{1}{\epsilon_n}(\bar{E}_{11}p'^2+2\bar{E}_{12}p'q'+\bar{E}_{22}q'^2),$$

$$\frac{1}{\rho_\epsilon^2}(z_1\cos\iota+z_2)=-\frac{1}{\epsilon_n}(\bar{E}_{11}p'^2+2\bar{E}_{12}p'q'+\bar{E}_{22}q'^2)\cos\iota;$$

and therefore

$$z_2=0,\quad \frac{z_1}{\rho_\epsilon^2}=-\frac{1}{\epsilon_n}(\bar{E}_{11}p'^2+2\bar{E}_{12}p'q'+\bar{E}_{22}q'^2).$$

Moreover, we have

$$\rho_\epsilon^2=z^2+z_1^2.$$

Thus the equations for the locus are

$$\frac{z}{\rho_\epsilon^2}=\bar{A}_0p'^2+2\bar{H}_0p'q'+\bar{B}_0q'^2,$$

$$\frac{z_1}{\rho_\epsilon^2}=-\frac{1}{\epsilon_n}(\bar{E}_{11}p'^2+2\bar{E}_{12}p'q'+\bar{E}_{22}q'^2),\quad z_2=0,$$

while, always,

$$1=A_0p'^2+2H_0p'q'+B_0q'^2.$$

As $z_2=0$, the locus lies in the plane $F_\epsilon OY$, that is, in the plane z_1Oz. To obtain the equation of the plane locus, we eliminate p' and q' between these equations; and we obtain this eliminant in the form

$$4\begin{vmatrix} z, & \bar{H}_0, & \bar{B}_0 \\ -\epsilon_n z_1, & \bar{E}_{12}, & \bar{E}_{22} \\ z^2+z_1^2, & H_0, & B_0 \end{vmatrix}\begin{vmatrix} z, & \bar{A}_0, & \bar{H}_0 \\ -\epsilon_n z_1, & \bar{E}_{11}, & \bar{E}_{12} \\ z^2+z_1^2, & A_0, & H_0 \end{vmatrix}=\begin{vmatrix} z, & \bar{A}_0, & \bar{B}_0 \\ -\epsilon_n z_1, & \bar{E}_{11}, & \bar{E}_{22} \\ z^2+z_1^2, & A_0, & B_0 \end{vmatrix}^2.$$

The plane curve, thus represented, is a lemniscate curve having a double point (real or imaginary) at O, an equivalent form of the equation being

$$(z^2+z_1^2)^2+u_1(z^2+z_1^2)+u_2=0,$$

where u_1 is of the form $\alpha_1z+\beta_1z_1$, and u_2 of the form $\alpha_2z^2+2\beta_2zz_1+\gamma_2z_1^2$.

II. In the same way, we find the locus of C_ω to be a curve in the plane $F_\omega OY$, that is, in the plane z_2Oz, the value of z_1 being zero; and the equation of the locus in that plane is the foregoing equation when we substitute z_2 for z_1, ω_ν for ϵ_n, and $\bar{\Omega}_{11}$, $\bar{\Omega}_{12}$, $\bar{\Omega}_{22}$, for $\bar{E}_{11}$, $\bar{E}_{12}$, $\bar{E}_{22}$, respectively.

III. For the locus of C_0, the centre of circular curvature of the superficial geodesic, we have

$$\frac{y_0-y}{\rho_0^2}=\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+g_\epsilon\frac{dy}{dn}+g_\omega\frac{dy}{d\nu}.$$

Then

$$\frac{z}{\rho_0^2}=\frac{1}{\rho_0^2}\sum\{Y(y_0-y)\}=\frac{1}{\rho}=\bar{A}_0p'^2+2\bar{H}_0p'q'+\bar{B}_0q'^2;$$

$$\frac{1}{\rho_0^2}(z_1+z_2\cos\iota)=\frac{1}{\rho_0^2}\sum\left\{\frac{dy}{dn}(y_0-y)\right\}=g_\epsilon+g_\omega\cos\iota$$

$$=\frac{1}{\gamma_\epsilon}=-\frac{1}{\epsilon_n}(\bar{E}_{11}p'^2+2\bar{E}_{12}p'q'+\bar{E}_{22}q'^2);$$

$$\frac{1}{\rho_0^2}(z_1\cos\iota+z_2)=\frac{1}{\rho_0^2}\sum\left\{\frac{dy}{d\nu}(y_0-y)\right\}=g_\epsilon\cos\iota+g_\omega$$

$$=\frac{1}{\gamma_\omega}=-\frac{1}{\omega_\nu}(\bar{\Omega}_{11}p'^2+2\bar{\Omega}_{12}p'q'+\bar{\Omega}_{22}q'^2);$$

while

$$\rho_0^2 = z^2 + z_1^2 + 2z_1 z_2 \cos\iota + z_2^2,$$

and, always,

$$1 = A_0 p'^2 + 2H_0 p'q' + B_0 q'^2.$$

Thus there are four equations involving p' and q', together with the coordinates of a point on the locus; consequently, the eliminant must consist of a couple of relations between these coordinates, and the locus is a skew curve in the flat.

One of these relations can be obtained as the determinantal eliminant of p'^2, $p'q'$, q'^2, from the four equations, in a form

$$\begin{vmatrix} z & , \bar{A}_0, & \bar{H}_0, & \bar{B}_0 \\ -\epsilon_n(z_1 + z_2\cos\iota), & \bar{E}_{11}, & \bar{E}_{12}, & \bar{E}_{22} \\ -\omega_\nu(z_1\cos\iota + z_2), & \bar{\Omega}_{11}, & \bar{\Omega}_{12}, & \bar{\Omega}_{22} \\ \rho_0^2 & , A_0, & H_0, & B_0 \end{vmatrix} = 0,$$

which, because of the value of ρ_0^2, represents a sphere in the flat passing through the initial origin O of reference.

Further, resolving the first three of the four equations to obtain the values of p'^2, $2p'q'$, q'^2, and again eliminating, we find

$$4\begin{vmatrix} z & , \bar{H}_0, & \bar{B}_0 \\ -\epsilon_n(z_1 + z_2\cos\iota), & \bar{E}_{12}, & \bar{E}_{22} \\ -\omega_\nu(z_1\cos\iota + z_2), & \bar{\Omega}_{12}, & \bar{\Omega}_{22} \end{vmatrix}\begin{vmatrix} z & , \bar{A}_0, & \bar{H}_0 \\ -\epsilon_n(z_1 + z_2\cos\iota), & \bar{E}_{11}, & \bar{E}_{12} \\ -\omega_\nu(z_1\cos\iota + z_2), & \bar{\Omega}_{11}, & \bar{\Omega}_{12} \end{vmatrix}$$
$$= \begin{vmatrix} z & , \bar{A}_0, & \bar{B}_0 \\ -\epsilon_n(z_1 + z_2\cos\iota), & \bar{E}_{11}, & \bar{E}_{22} \\ -\omega_\nu(z_1\cos\iota + z_2), & \bar{\Omega}_{11}, & \bar{\Omega}_{22} \end{vmatrix}^2,$$

which represents a quadric cone in the flat, with its vertex at the initial origin O of reference.

The required locus of the centre of circular curvature of the superficial geodesics through O is the intersection of the sphere and this quadric cone: that is, the locus is a twisted quartic in the flat.

Ex. Verify that the flat-coordinates of the centre of circular curvature of a superficial geodesic can be expressed, in terms of the magnitudes of the domain and of the two regions, by the equations

$$\frac{1}{\rho z} = \frac{1}{\gamma_\epsilon(z_1 + z_2\cos\iota)} = \frac{1}{\gamma_\omega(z_2 + z_1\cos\iota)} = \frac{1}{\rho^2} + \left(\frac{1}{\gamma_\epsilon^2} - \frac{2\cos\iota}{\gamma_\epsilon\gamma_\omega} + \frac{1}{\gamma_\omega^2}\right)\frac{1}{\sin^2\iota}.$$

As before (§ 143), we can associate the curves of circular curvature on the surface (that is, the directions of principal circular curvature at O) with this locus; for the radius vector from O to any point of the locus is the radius of circular curvature of the geodesic in the corresponding direction on the surface, and

therefore a principal direction on the surface is to be associated with a radius vector which is also a normal to the curve.

These normals from O to the locus may be determined as follows. We introduce variables t, t_1, t_2, by the definitions

$$t=\begin{vmatrix} z, & \bar{H}_0, & \bar{B}_0 \\ -\epsilon_n(z_1+z_2\cos\iota), & \bar{E}_{12}, & \bar{E}_{22} \\ -\omega_\nu(z_1\cos\iota+z_2), & \bar{\Omega}_{12}, & \bar{\Omega}_{22} \end{vmatrix}, \quad t_2=\begin{vmatrix} z, & \bar{A}_0, & \bar{H}_0 \\ -\epsilon_n(z_1+z_2\cos\iota), & \bar{E}_{11}, & \bar{E}_{12} \\ -\omega_\nu(z_1\cos\iota+z_2), & \bar{\Omega}_{11}, & \bar{\Omega}_{12} \end{vmatrix},$$

$$t_1=\begin{vmatrix} z, & \bar{A}_0, & \bar{B}_0 \\ -\epsilon_n(z_1+z_2\cos\iota), & \bar{E}_{11}, & \bar{E}_{22} \\ -\epsilon_n(z_1\cos\iota+z_2), & \bar{\Omega}_{11}, & \bar{\Omega}_{22} \end{vmatrix},$$

and we use ∇ to denote the determinant

$$\nabla=\begin{vmatrix} \bar{A}_0, & \bar{H}_0, & \bar{B}_0 \\ \bar{E}_{11}, & \bar{E}_{12}, & \bar{E}_{22} \\ \Omega_{11}, & \bar{\Omega}_{12}, & \bar{\Omega}_{22} \end{vmatrix}.$$

Then

$$\begin{aligned} \nabla z &= \bar{A}_0 t+\bar{H}_0 t_1+\bar{B}_0 t_2, \\ -\epsilon_n\nabla(z_1+z_2\cos\iota) &= \bar{E}_{11}t+\bar{E}_{12}t_1+\bar{E}_{22}t_2, \\ -\omega_\nu\nabla(z_1\cos\iota+z_2) &= \bar{\Omega}_{11}t+\bar{\Omega}_{12}t_1+\bar{\Omega}_{22}t_2, \end{aligned}$$

and

$$\frac{t}{p'^2}=\frac{t_1}{2p'q'}=\frac{t_2}{q'^2}=\nabla\rho_0{}^2;$$

and the equations of the sphere and of the quadric cone, the intersection of which is the locus, become

$$\nabla\rho_0{}^2=A_0t+H_0t_1+B_0t_2, \quad t_1{}^2=4tt_2.$$

Let the flat-coordinates of the foot of a normal from O to the curve be denoted by $\bar{z}$, $\bar{z}_1$, $\bar{z}_2$, (with corresponding quantities $\bar{t}$, $\bar{t}_1$, $\bar{t}_2$); and let the direction-variables of the tangent to the curve-locus be denoted by $\bar{z}'$, $\bar{z}_1'$, $\bar{z}_2'$, (with corresponding quantities $\bar{t}'$, $\bar{t}_1'$, $\bar{t}_2'$); these direction-variables are given, as to their ratios, by the relations

$$2\bar{t}_2\bar{t}'-\bar{t}_1\bar{t}_1'+2\bar{t}\bar{t}_2'=0,$$

$$\tfrac{1}{2}(A_0\bar{t}'+H_0\bar{t}_1'+B_0\bar{t}_2')=\{\bar{z}\bar{z}'+(\bar{z}_1+\bar{z}_2\cos\iota)\bar{z}_1'+(\bar{z}_1\cos\iota+\bar{z}_2)\bar{z}_2'\}\nabla.$$

The condition of normality, being the condition that the radius vector from O to the point $\bar{z}$, $\bar{z}_1$, $\bar{z}_2$, shall be at right angles to the tangent at that point with direction-variables $\bar{z}'$, $\bar{z}_1'$, $\bar{z}_2'$, is

$$\bar{z}\bar{z}'+(\bar{z}_1+\bar{z}_2\cos\iota)\bar{z}_1'+(\bar{z}_1\cos\iota+\bar{z}_2)\bar{z}_2'=0,$$

which, when we substitute for $\bar{z}'$, $\bar{z}_1'$, $\bar{z}_2'$, in terms of $\bar{t}'$, $\bar{t}_1'$, $\bar{t}_2'$, becomes

$$\left(\bar{A}_0\bar{z}-\frac{\bar{E}_{11}}{\epsilon_n}\bar{z}_1-\frac{\bar{\Omega}_{11}}{\omega_\nu}\bar{z}_2\right)\bar{t}'+\left(\bar{H}_0\bar{z}-\frac{\bar{E}_{12}}{\epsilon_n}\bar{z}_1-\frac{\bar{\Omega}_{12}}{\omega_\nu}\bar{z}_2\right)\bar{t}_1'+\left(\bar{B}_0\bar{z}-\frac{\bar{E}_{22}}{\epsilon_n}\bar{z}_1-\frac{\bar{\Omega}_{22}}{\omega_\nu}z_2\right)\bar{t}_2'=0.$$

When the condition of normality is used, the two former equations for the ratios of $\bar{t}'$, $\bar{t}_1'$, $\bar{t}_2'$, become

$$2\bar{t}_2\bar{t}' - \bar{t}_1\bar{t}_1' + 2\bar{t}\bar{t}_2' = 0,$$
$$A_0\bar{t}' + H_0\bar{t}_1' + B_0\bar{t}_2' = 0.$$

In order that the three relations, linear and homogeneous in $\bar{t}'$, $\bar{t}_1'$, $\bar{t}_2'$, may coexist, the quantities $\bar{z}$, $\bar{z}_1$, $\bar{z}_2$, with the associated quantities $\bar{t}$, $\bar{t}_1$, $\bar{t}_2$, must satisfy the relation

$$\begin{vmatrix} \bar{A}_0\bar{z} - \dfrac{\bar{E}_{11}}{\epsilon_n}\bar{z}_1 - \dfrac{\bar{\Omega}_{11}}{\omega_\nu}\bar{z}_2, & 2\bar{t}_2, & A_0 \\ \bar{H}_0\bar{z} - \dfrac{\bar{E}_{12}}{\epsilon_n}\bar{z}_1 - \dfrac{\bar{\Omega}_{12}}{\omega_\nu}\bar{z}_2, & -\bar{t}_1, & H_0 \\ \bar{B}_0\bar{z} - \dfrac{\bar{E}_{22}}{\epsilon_n}\bar{z}_1 - \dfrac{\bar{\Omega}_{22}}{\omega_\nu}\bar{z}_2, & 2\bar{t}, & B_0 \end{vmatrix} = 0.$$

This equation represents a quadric cone, having its vertex at the origin. We therefore infer that the directions of the prime normals of the superficial geodesics in the four principal directions on the surface (that is, at the four curves of curvature) are given by the four generators common to this quadric cone and the former quadric cone $t_1^2 = 4tt_2$, the cones having a common vertex; and the lengths of the four principal radii of curvature are the lengths of these generators, intercepted between the initial point O and the quartic curve which is the locus of the centres of circular curvature. The superficial direction-variables of the curves of curvature are given by the equations

$$\frac{t}{p'^2} = \frac{2t_1}{p'q'} = \frac{t_2}{q'^2}.$$

Orthogonal centre of geodesics on a surface in a primary domain.

376. It was proved (in § 142) that a surface in a plenary quintuple space possesses an orthogonal centre, defined as the ultimate intersection of the orthogonal flat at O with the orthogonal flats at points in the immediate vicinity of O. (The point is a spatial orthogonal centre and is distinct, in significance and position, from the domainal orthogonal centre of § 363, when the surface is a domainal surface.) The five equations for the spatial orthogonal centre are: the pair

$$\sum (\bar{y} - y)\bar{y}_1 = 0, \quad \sum (\bar{y} - y)\bar{y}_2 = 0,$$

(the symbols $\bar{y}_1$ and $\bar{y}_2$ belonging to the parametric leading lines in the tangent plane of the surface), these two being the equations of the orthogonal flat at O: and the trio

$$\sum \{(\bar{y} - y)\bar{\eta}_{11}\} = A_0, \quad \sum \{(\bar{y} - y)\bar{\eta}_{12}\} = H_0, \quad \sum \{(\bar{y} - y)\bar{\eta}_{22}\} = B_0,$$

(with the preceding notation), these three being equations derived in connection with the equations of orthogonal flats at points near O. This orthogonal centre lies in the orthogonal flat of which the leading lines, as selected in § 342, have typical direction-cosines $Y, \frac{dy}{dn}, \frac{dy}{d\nu}$; and therefore, for all the five coordinates, we can take

$$\bar{y}-y=\kappa Y+\lambda\frac{dy}{dn}+\mu\frac{dy}{d\nu}.$$

Also we have

$$\bar{\eta}_{ij}=[\eta_{ij}]-\frac{1}{\epsilon_n}\bar{E}_{ij}\frac{dy}{dn}-\frac{1}{\omega_\nu}\bar{\Omega}_{ij}\frac{dy}{d\nu},$$

for the combinations $ij=11, 12, 22$.

Let ζ, ζ_1, ζ_2 be the flat-coordinates of the spatial orthogonal centre, so that

$$\zeta=\sum Y(\bar{y}-y)=\kappa,$$

$$\zeta_1+\zeta_2\cos\iota=\sum\frac{dy}{dn}(\bar{y}-y)=\lambda+\mu\cos\iota,$$

$$\zeta_1\cos\iota+\zeta_2=\sum\frac{dy}{d\nu}(\bar{y}-y)=\lambda\cos\iota+\mu,$$

and therefore

$$\bar{y}-y=\zeta Y+\zeta_1\frac{dy}{dn}+\zeta_2\frac{dy}{d\nu}.$$

Now

$$\sum Y[\eta_{11}]=\bar{A}_0,\quad \sum Y\frac{dy}{dn}=0,\quad \sum Y\frac{dy}{d\nu}=0,$$

and therefore

$$\sum Y\bar{\eta}_{11}=\bar{A}_0;$$

and similarly,

$$\sum Y\bar{\eta}_{12}=\bar{H}_0,\quad \sum Y\bar{\eta}_{22}=\bar{B}_0.$$

Again

$$\sum\frac{dy}{dn}[\eta_{11}]=0,\quad \sum\frac{dy}{d\nu}[\eta_{11}]=0,$$

so that

$$\sum\frac{dy}{dn}\bar{\eta}_{11}=-\frac{1}{\epsilon_n}(\bar{E}_{11}+\bar{\Omega}_{11}\cos\iota),$$

$$\sum\frac{dy}{d\nu}\bar{\eta}_{11}=-\frac{1}{\epsilon_n}(\bar{E}_{11}\cos\iota+\bar{\Omega}_{12});$$

and similarly for the like combinations.

Of the five equations for the orthogonal centre, the first two are satisfied unconditionally. When substitution of the quantities $\bar{y}-y$, in terms of ζ, ζ_1, ζ_2, is made in the remaining three, we find

$$\left.\begin{aligned}
\zeta\bar{A}_0 - \frac{\bar{E}_{11}}{\epsilon_n}(\zeta_1+\zeta_2\cos\iota) - \frac{\bar{\Omega}_{11}}{\omega_\nu}(\zeta_1\cos\iota+\zeta_2) = A_0 \\
\zeta\bar{H}_0 - \frac{\bar{E}_{12}}{\epsilon_n}(\zeta_1+\zeta_2\cos\iota) - \frac{\bar{\Omega}_{12}}{\omega_\nu}(\zeta_1\cos\iota+\zeta_2) = H_0 \\
\zeta\bar{B}_0 - \frac{\bar{E}_{22}}{\epsilon_n}(\zeta_1+\zeta_2\cos\iota) - \frac{\bar{\Omega}_{22}}{\omega_\nu}(\zeta_1\cos\iota+\zeta_2) = B_0
\end{aligned}\right\}.$$

The equation of the sphere, which contains the quartic curve-locus of the centre of circular curvature of the superficial geodesic, is

$$\nabla(z^2+z_1^2+2z_1z_2\cos\iota+z_2^2) = A_0t + H_0t_1 + B_0t_2\,;$$

and therefore the centre of that sphere is given by

$$2\nabla z^{(0)} = A_0 \begin{vmatrix} \bar{E}_{12}, & \bar{E}_{22} \\ \bar{\Omega}_{12}, & \bar{\Omega}_{22} \end{vmatrix} + H_0 \begin{vmatrix} \bar{E}_{11}, & \bar{E}_{22} \\ \bar{\Omega}_{11}, & \bar{\Omega}_{22} \end{vmatrix} + B_0 \begin{vmatrix} \bar{E}_{11}, & \bar{E}_{12} \\ \bar{\Omega}_{11}, & \bar{\Omega}_{12} \end{vmatrix},$$

that is,

$$2z_0^{(0)} = \zeta\,;$$

and similarly

$$2z_1^{(0)} = \zeta_1, \quad 2z_2^{(0)} = \zeta_2.$$

Hence the spatial orthogonal centre of the superficial geodesic is the further extremity of the diameter, drawn through O, of the sphere on which lies the locus of the centre of circular curvature—in accordance with the result already (§ 142) established for a surface in a plenary quintuple homaloidal space.

CHAPTER XXXII

Geodesic Parallels in a Domain

Levi-Civita parallels.

377. Geodesic parallels in a general domain may be discussed in the same manner as those in a region. One aim, however, is the construction of closed figures of the type of a parallelogram or a parallelepiped (that is, figures bounded by geodesic edges which, when taken in opposite pairs, can be regarded as constituted by conventionally parallel pairs); and consequently the inferences from definitions, such as Levi-Civita's, Severi's, and others indicated in § 232, which do not lead to the construction of such closed figures, are not developed for domains in the same detail as for regions.

As it is incumbent to conserve the primary fundamental conditions of parallelism (either explicitly or implicitly) which are common to all the modes, we begin with the Levi-Civita system, as having been the earliest to be formulated.

At a point O of a domain, let two domainal geodesics OA and OB be drawn: the geodesic OA, with an element ds_1 of arc and with direction-variables p_1', q_1', r_1', t_1', at O: the geodesic OB, with an element ds_2 of arc and with direction-variables p_2', q_2', r_2', t_2', at O. At any point X in the geodesic OA, let a domainal geodesic XU be drawn, having at X a typical spatial direction-cosine c. In the Levi-Civita definition, the law of parallelism is postulated by the property that the domainal geodesics through successive points X along OA make, with a selected aggregate of lines of reference in the plenary space of the domain, the same several angles as are made by the domainal geodesic OB with those lines. Let l denote the typical direction-cosine of any one such line; then we are to have

$$\sum cl = \text{constant},$$

for the different geodesics c; and therefore we have

$$\sum l \frac{dc}{ds_1} = 0,$$

holding at O, for all the lines l, as a first condition (or aggregate of conditions) of the defined parallelism.

The demand, expressed analytically by this condition, could not be exacted for all lines l in the homaloidal plenary space of the domain; for it would require $\dfrac{dc}{ds_1} = 0$, that is, c constant, for the typical direction-cosine, and a constant direction is not possible in a completely general domain. Accordingly, the aggregate of

lines of reference must be selected; and the selection is made by choosing the aggregate of all possible directions in the tangent block at O, so that

$$l = y_1\kappa + y_2\lambda + y_3\mu + y_4\nu,$$

where κ, λ, μ, ν, are arbitrary parameters, linearly independent of one another. In order that the foregoing condition of parallelism may now be satisfied, we must have

$$\sum y_1 \frac{dc}{ds_1} = 0, \quad \sum y_2 \frac{dc}{ds_1} = 0, \quad \sum y_3 \frac{dc}{ds_1} = 0, \quad \sum y_4 \frac{dc}{ds_1} = 0.$$

At O, the typical direction-cosine of OB

$$= y_1 p_2' + y_2 q_2' + y_3 r_2' + y_4 t_2';$$

and therefore, when p_2', q_2', r_2', t_2', are regarded as parametric along OA, so as to represent c at the successive points X, we have

$$\begin{aligned} \frac{dc}{ds_1} = &\; y_1 \frac{dp_2'}{ds_1} + y_2 \frac{dq_2'}{ds_1} + y_3 \frac{dr_2'}{ds_1} + y_4 \frac{dt_2'}{ds_1} \\ &+ p_2' \frac{dy_1}{ds_1} + q_2' \frac{dy_2}{ds_1} + r_2' \frac{dy_3}{ds_1} + t_2' \frac{dy_4}{ds_1}. \end{aligned}$$

But

$$\begin{aligned} \frac{dy_1}{ds_1} &= \eta_1 + y_1\alpha_1 + y_2\xi_1 + y_3\phi_1 + y_4\kappa_1, \\ \frac{dy_2}{ds_1} &= \eta_2 + y_1\beta_1 + y_2\eta_1 + y_3\chi_1 + y_4\lambda_1, \\ \frac{dy_3}{ds_1} &= \eta_3 + y_1\gamma_1 + y_2\zeta_1 + y_3\psi_1 + y_4\mu_1, \\ \frac{dy_4}{ds_1} &= \eta_4 + y_1\delta_1 + y_2\varpi_1 + y_3\omega_1 + y_4\nu_1, \end{aligned}$$

using the symbols of § 306: or, writing

$$\begin{aligned} P_2 &= p_2'\alpha_1 + q_2'\beta_1 + r_2'\gamma_1 + t_2'\delta_1 = \sum \Gamma_{11} p_1' p_2', \\ Q_2 &= p_2'\xi_1 + q_2'\eta_1 + r_2'\zeta_1 + t_2'\varpi_1 = \sum \Delta_{11} p_1' p_2', \\ R_2 &= p_2'\phi_1 + q_2'\chi_1 + r_2'\psi_1 + t_2'\omega_1 = \sum \Theta_{11} p_1' p_2', \\ T_2 &= p_2'\kappa_1 + q_2'\lambda_1 + r_2'\mu_1 + t_2'\nu_1 = \sum \Phi_{11} p_1' p_2', \end{aligned}$$

we have

$$\begin{aligned} \frac{dc}{ds_1} = &\; \eta_1 p_2' + \eta_2 q_2' + \eta_3 r_2' + \eta_4 t_2' \\ &+ y_1 \left(\frac{dp_2'}{ds_1} + P_2\right) + y_2 \left(\frac{dq_2'}{ds_1} + Q_2\right) + y_3 \left(\frac{dr_2'}{ds_1} + R_2\right) + y_4 \left(\frac{dt_2'}{ds_1} + T_2\right). \end{aligned}$$

The conditions are

$$\sum y_\lambda \frac{dc}{ds_1}=0,$$

for $\lambda=1, 2, 3, 4$. When the value of $\dfrac{dc}{ds_1}$ is inserted, and the relations

$$\sum y_\lambda \eta_\mu = 0,$$

for $\lambda, \mu, =1, 2, 3, 4$, in all combinations, are used, we have

$$A_{1\lambda}\left(\frac{dp_2'}{ds_1}+P_2\right)+A_{2\lambda}\left(\frac{dq_2'}{ds_1}+Q_2\right)+A_{3\lambda}\left(\frac{dr_2'}{ds_1}+R_2\right)+A_{4\lambda}\left(\frac{dt_2'}{ds_1}+T_2\right)=0,$$

for $\lambda=1, 2, 3, 4$. The determinant of the quantities $A_{\lambda\mu}$ is Ω, different from zero; and therefore the four magnitudes, occurring linearly, all vanish: that is,

$$\left.\begin{aligned}
\frac{dp_2'}{ds_1} &= -P_2 = -\sum \Gamma_{11} p_1' p_2' \\
\frac{dq_2'}{ds_1} &= -Q_2 = -\sum \Delta_{11} p_1' p_2' \\
\frac{dr_2'}{ds_1} &= -R_2 = -\sum \Theta_{11} p_1' p_2' \\
\frac{dt_2'}{ds_1} &= -T_2 = -\sum \Phi_{11} p_1' p_2'
\end{aligned}\right\}.$$

These relations will be called the *primary* conditions of parallelism for domainal geodesics.

Let the geodesic arc OX along the domainal geodesic OA be small, and let its length be denoted by x. Then, up to the first order of small quantities, the direction-variables of the domainal geodesic XU (drawn, at X, so as to be parallel to the geodesic OB at O) are

$$p_2' - x\sum \Gamma_{11} p_1' p_2', \quad q_2' - x\sum \Delta_{11} p_1' p_2', \quad r_2' - x\sum \Theta_{11} p_1' p_2', \quad t_2' - x\sum \Phi_{11} p_1' p_2'.$$

If these direction-variables are required to the closer approximation represented by the retention of the second power of x, their formal values are

$$p_2' + x\frac{dp_2'}{ds_1} + \tfrac{1}{2}x^2 \frac{d^2 p_2'}{ds_1^2},$$

with three like values for the other three direction-variables; we therefore require the values of the second arc-derivatives of p_2', q_2', r_2', t_2'.

From the primary conditions, we have

$$\begin{aligned}
\frac{dp_2'}{ds_1} &= -\sum \Gamma_{11} p_1' p_2' \\
&= -(\alpha_1 p_2' + \beta_1 q_2' + \gamma_1 r_2' + \delta_1 t_2'),
\end{aligned}$$

with corresponding values for derivatives of q_2', r_2', t_2'; and therefore

$$\frac{d^2p_2'}{ds_1^2} = -\left(p_2'\frac{d\alpha_1}{ds_1}+q_2'\frac{d\beta_1}{ds_1}+r_2'\frac{d\gamma_1}{ds_1}+t_2'\frac{d\delta_1}{ds_1}\right)$$
$$+p_2'(\alpha_1^2+\beta_1\xi_1+\gamma_1\phi_1+\delta_1\kappa_1)$$
$$+q_2'(\alpha_1\beta_1+\beta_1\eta_1+\gamma_1\chi_1+\delta_1\lambda_1)$$
$$+r_2'(\alpha_1\gamma_1+\beta_1\zeta_1+\gamma_1\psi_1+\delta_1\mu_1)$$
$$+t_2'(\alpha_1\delta_1+\beta_1\varpi_1+\gamma_1\omega_1+\delta_1\nu_1).$$

Now

$$\frac{d\alpha_1}{ds_1}=p_1'\frac{d\Gamma_{11}}{ds_1}+q_1'\frac{d\Gamma_{12}}{ds_1}+r_1'\frac{d\Gamma_{13}}{ds_1}+t_1'\frac{d\Gamma_{14}}{ds_1}+\Gamma_{11}p_1''+\Gamma_{12}q_1''+\Gamma_{13}r_1''+\Gamma_{14}t_1'',$$

and similarly for the arc-derivatives of β_1, γ_1, δ_1, along OA; and therefore, arising out of the first line in the second derivative of p_2', there will be two aggregates of terms. One of these aggregates

$$= -\left(\frac{d\Gamma}{ds_1}\between 1 \between 2\right),$$

in the notation of § 307; the other

$$= -(\alpha_2p_1''+\beta_2q_1''+\gamma_2r_1''+\delta_2t_1''),$$

while

$$p_1'' = -(\alpha_1p_1'+\beta_1q_1'+\gamma_1r_1'+\delta_1t_1'),$$

with like values for q_1'', r_1'', t_1''.

Let the value of the first of these aggregates, as obtained in § 306 (p. 357), be substituted, and the whole expression be reduced. There are three sets of terms, one involving the symbols of the type Γ_{ijk}, one involving the four-index Riemann symbols of the type (ij, kl), and one free from all the symbols of these two types. We take them in turn.

(i) The full set of terms involving Γ_{ijk} is found to be

$$-(\Gamma_{300}p_1'^2p_2'),$$

with the former notation.

(ii) The full set of terms involving the Riemann four-index symbols is found to be

$$\frac{1}{3\Omega}\sum_{\mu}\{a_{1\mu}K_\mu(1, 12)\},$$

where the μ-summation is for the values $\mu=1, 2, 3, 4$, while

$$K_\mu(1, 12)=\sum_{\alpha}\sum_{\beta}[\{p_1'(1\mu, \alpha\beta)+q_1'(2\mu, \alpha\beta)+r_1'(3\mu, \alpha\beta)+t_1'(4\mu, \alpha\beta)\}s_{\alpha\beta}],$$

the α, β, summation being for the set of values 23, 31, 12, 14, 24, 34, with

$$\left.\begin{aligned} s_{23}&=q_1'r_2'-r_1'q_2', & s_{14}&=p_1't_2'-t_1'p_2' \\ s_{31}&=r_1'p_2'-p_1'r_2', & s_{24}&=q_1't_2'-t_1'q_2' \\ s_{12}&=p_1'q_2'-q_1'p_2', & s_{34}&=r_1't_2'-t_1'r_2' \end{aligned}\right\}.$$

(iii) The full set of terms, free from the symbols Γ_{ijk} and from the four-index symbols, reduces to zero.

Hence, finally,

$$\frac{d^2p_2'}{ds_1^2}=-(\Gamma_{300}p_1'^2p_2')+\frac{1}{3\Omega}\sum_\mu\{a_{1\mu}K_\mu(1,\,12)\}.$$

Similarly we find

$$\frac{d^2q_2'}{ds_1^2}=-(\varDelta_{300}p_1'^2p_2')+\frac{1}{3\Omega}\sum_\mu\{a_{2\mu}K_\mu(1,\,12)\},$$

$$\frac{d^2r_2'}{ds_1^2}=-(\Theta_{300}p_1'^2p_2')+\frac{1}{3\Omega}\sum_\mu\{a_{3\mu}K_\mu(1,\,12)\},$$

$$\frac{d^2t_2'}{ds_1^2}=-(\Phi_{300}p_1'^2p_2')+\frac{1}{3\Omega}\sum_\mu\{a_{4\mu}K_\mu(1,\,12)\}.$$

Ex. The value of $\dfrac{dc}{ds_1}$, for the parallelism, is

$$\eta_1p_2'+\eta_2q_2'+\eta_3r_2'+\eta_4t_2'.$$

Find the value of $\dfrac{d^2c}{ds_1^2}$; and verify that the relation

$$\sum y_\lambda\frac{d^2c}{ds_1^2}=0$$

is satisfied for the values $\lambda=1,\,2,\,3,\,4$.

Second-order approximation in the permanent arc-relation.

378. It is convenient to establish two general results at this stage, noting their effect upon the preceding investigation.

Consider, at the point X in OA, a domainal direction, with direction-variables P_2', Q_2', R_2', T_2', given by values

$$\begin{aligned} P_2'&=p_2'-x\sum\Gamma_{11}p_1'p_2'-\tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2')+\tfrac{1}{2}\bar{P}_2, \\ Q_2'&=q_2'-x\sum\varDelta_{11}p_1'p_2'-\tfrac{1}{2}x^2(\varDelta_{300}p_1'^2p_2')+\tfrac{1}{2}\bar{Q}_2, \\ R_2'&=r_2'-x\sum\Theta_{11}p_1'p_2'-\tfrac{1}{2}x^2(\Theta_{300}p_1'^2p_2')+\tfrac{1}{2}\bar{R}_2, \\ T_2'&=t_2'-x\sum\Phi_{11}p_1'p_2'-\tfrac{1}{2}x^2(\Phi_{300}p_1'^2p_2')+\tfrac{1}{2}\bar{T}_2, \end{aligned}$$

where $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$, are magnitudes of the second order of small quantities (that is, of the order x^2, though they may arise in other associations and not be dependent upon x alone). It is to be noted that these expressions conform to the primary conditions of parallelism, and that they contain those portions of the terms such

as $\frac{1}{2}x^2\dfrac{d^2p_2'}{ds_1^2}$ which do not involve the four-index symbols.* As these are to be domainal directions at X in OA, the permanent arc-relation

$$\sum A_X P_2'^2 = 1$$

must be satisfied: accordingly, some relation must be satisfied by the quantities $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$.

To obtain the relation indicated, we must take approximations up to the second order of small quantities inclusive. Thus, using the notation of p. 357, we have

$$P_2'^2 = p_2'^2 - 2x(p_2'\bar{\gamma}_{12}) + x^2\{\bar{\gamma}_{12}^2 - p_2'(\Gamma_{300}p_1'^2p_2')\} + p_2'\bar{P}_2,$$

with like expressions for all the two-dimensional combinations of P_2', Q_2', R_2', T_2': also, the values of the primary magnitudes at X are given by expressions of the form

$$A_X = A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2},$$

up to the second order. When these values are substituted in the permanent arc-relation, it must be satisfied for all values of x: or, what is the same thing, the finite terms (free from x) must balance and the aggregate terms of each successive order must vanish.

The finite terms in $\sum A_X P_2'^2$

$$= \sum A p_2'^2 = 1;$$

and thus they balance the right-hand side in the relation.

The aggregate of terms of the first order may be denoted by xS_1, where

$$\begin{aligned} S_1 &= \sum \frac{dA}{ds_1}p_2'^2 - 2\sum A p_2'\bar{\gamma}_{12} \\ &= \sum \frac{dA}{ds_1}p_2'^2 - 2\{u_1^{(2)}\bar{\gamma}_{12} + u_2^{(2)}\bar{\delta}_{12} + u_3^{(2)}\bar{\theta}_{12} + u_4^{(2)}\bar{\phi}_{12}\} \\ &= 0, \end{aligned}$$

on using the results of § 307: that is, the terms of the first order of small quantities disappear from the equation, without any residual condition.

The aggregate of terms of the second order

$$\begin{aligned} = \quad & \tfrac{1}{2}x^2\sum \frac{d^2A}{ds_1^2}p_2'^2 + \{u_1^{(2)}\bar{P}_2 + u_2^{(2)}\bar{Q}_2 + u_3^{(2)}\bar{R}_2 + u_4^{(2)}\bar{T}_2\} \\ & - 2x^2\sum \frac{dA}{ds_1}p_2'\bar{\gamma}_{12} - x^2\sum\{u_1^{(2)}(\Gamma_{300}p_1'^2p_2')\} + x^2\sum A\bar{\gamma}_{12}^2; \end{aligned}$$

* The aggregate of terms in P_2', Q_2', R_2', T_2', other than $\frac{1}{2}\bar{P}_2$, $\frac{1}{2}\bar{Q}_2$, $\frac{1}{2}\bar{R}_2$, $\frac{1}{2}\bar{T}_2$, is the same as would occur if the domain were developable into a block; its geodesics, parallel to OB, would develop into straight lines parallel to the development of OB; and the said aggregates would give, up to the second order, the direction-variables of the parallel geodesics in the developable domain.

and this aggregate is to vanish. The expression for the coefficient of $\frac{1}{2}x^2$ in the first term is derivable from the result in § 308 by taking $k=1$, $i=j=2$. The combinations of the type $\sum \frac{dA}{ds_1} p_2'$, being the coefficients of the quantities $\bar{\gamma}_{12}$, $\bar{\delta}_{12}$, $\bar{\theta}_{12}$, $\bar{\phi}_{12}$, in the first term in the second line, are similarly derivable from the results in § 307. When these values are substituted, and reduction is effected, the condition is found to be

$$u_1^{(2)}\bar{P}_2+u_2^{(2)}\bar{Q}_2+u_3^{(2)}\bar{R}_2+u_4^{(2)}\bar{T}_2=\tfrac{1}{3}x^2\sum(\alpha\beta,\,\gamma\delta)\,s_{\alpha\beta}s_{\gamma\delta},$$

where the orientation-variables s have the meanings assigned on p. 582, and where the summation extends over the combinations of α, β, γ, δ, which have significance for these orientation-variables.

When this condition is satisfied by $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$, the arc-relation is satisfied, so far as approximations up to the second order (inclusive) are concerned. Manifestly, it is a purely analytical condition affecting all directions at X inclusible under the forms adopted for the postulated direction-variables P_2', Q_2', R_2', T_2'.

Ex. The application to the Levi-Civita parallels is immediate. They are included in the postulated form, by taking

$$\bar{P}_2=\frac{x^2}{3\Omega}\sum_\mu\{a_{1\mu}(K_\mu(1,12)\},$$

$$\bar{Q}_2=\frac{x^2}{3\Omega}\sum_\mu\{a_{2\mu}K_\mu(1,\,12)\},$$

$$\bar{R}_2=\frac{x^2}{3\Omega}\sum_\mu\{a_{3\mu}K_\mu(1,\,12)\},$$

$$\bar{T}_2=\frac{x^2}{3\Omega}\sum_\mu\{a_{4\mu}K_\mu(1,\,12)\}.$$

When these values are substituted in the left-hand side of the relation, the coefficient of $K_1(1,\,12)$

$$=\frac{x^2}{3\Omega}\{u_1^{(2)}a_{11}+u_2^{(2)}a_{12}+u_3^{(2)}a_{13}+u_4^{(2)}a_{14}\}=\tfrac{1}{3}x^2p_2',$$

and so for the others ; so that the relation becomes equivalent to

$$p_2'K_1(1,\,12)+q_2'K_2(1,\,12)+r_2'K_3(1,\,12)+t_2'K_4(1,\,12)=\sum(\alpha\beta,\,\gamma\delta)s_{\alpha\beta}s_{\gamma\delta},$$

which is an identity on the substitution of the values of the quantities K_μ.

379. In the next place, we consider the relation of the angle AXU, between the domainal geodesics XA and XU at X, to the angle AOB, between the domainal geodesics OA and OB at O. In the Levi-Civita definition, and in the Severi definition, of parallel geodesics the two angles are definitely equal ; in alternative

definitions, such as that which facilitates the construction of a regional cell (§ 240), the difference of the angles is a magnitude of the second order of the small arc-length OX. We proceed to find the value of the difference up to the second order of small quantities, when the direction-variables of the domainal geodesic are the quantities P_2', Q_2', R_2', T_2', of § 378.

At X, the direction-variables of the domainal geodesic XA in the direction XA are the four magnitudes of the type

$$p_1' + xp_1'' + \tfrac{1}{2}x^2p_1''',$$

up to the second order; and therefore we have

$$\begin{aligned}\cos AXU &= \sum \{A_X P_2'(p_1' + xp_1'' + \tfrac{1}{2}x^2p_1''')\}\\ &= \sum \left\{\left(A + x\frac{dA}{ds_1} + \tfrac{1}{2}x^2\frac{d^2A}{ds_1^2}\right)(p_1' + xp_1'' + \tfrac{1}{2}x^2p_1''')P_2'\right\}.\end{aligned}$$

When the postulated values of P_2', Q_2', R_2', T_2', are substituted, we range the terms on the right-hand side in aggregates, of successive orders of powers of the small quantities.

The aggregate of finite terms

$$= \sum Ap_1'p_2' = \cos AOB.$$

The aggregate of terms of the first order of small quantities

$$\begin{aligned}&= x\sum \frac{dA}{ds_1}p_1'p_2' - x\sum Ap_1'\bar{\gamma}_{12} + x\sum Ap_2'p_1''\\ &= x\sum \frac{dA}{ds_1}p_1'p_2' - x\{\sum (u_1^{(1)}\bar{\gamma}_{12}) + \sum (u_2^{(2)}\bar{\gamma}_{11})\}\\ &= 0,\end{aligned}$$

when the value of the sum in the first term is substituted from the result in § 307.

The aggregate of terms of the second order of small quantities

$$\begin{aligned}= \tfrac{1}{2}x^2\sum \frac{d^2A}{ds_1^2}p_1'p_2' + \tfrac{1}{2}x^2\sum\{u_1^{(2)}p_1'''\} + \tfrac{1}{2}\sum\{u_1^{(1)}\bar{P}_2\} - \tfrac{1}{2}x^2\sum\{u_1^{(1)}(\Gamma_{300}p_1'^2p_2')\}\\ - x^2\sum\frac{dA}{ds_1}\gamma_{12}p_1' + x^2\sum\frac{dA}{ds_1}p_1''p_2' - x^2\sum Ap_1''\bar{\gamma}_{12}.\end{aligned}$$

We substitute, in the first term, the value of the expression (§ 308) containing the second arc-derivatives of the quantities A, when $k=1$, $i=1$, $j=2$, so that all the variables $s_{\alpha\beta}$ vanish: thus all the terms which could involve the four-index symbols disappear. We substitute also the values of the expressions containing the first arc-derivatives of the quantities A. Then, on reduction, it is found that the total aggregate of surviving terms consists only of those contained in the third summation in the first line, being

$$= \tfrac{1}{2}\sum\{u_1^{(1)}\bar{P}_2\}.$$

Hence we have

$$\cos AXU - \cos AOB = \tfrac{1}{2}\{u_1^{(1)}\bar{P}_2 + u_2^{(1)}\bar{Q}_2 + u_3^{(1)}\bar{R}_2 + u_4^{(1)}\bar{T}_2\},$$

up to the second order of small quantities. If therefore

$$AXU = AOB + \nabla,$$

we have

$$\nabla \sin AOB = -\tfrac{1}{2}\{u_1^{(1)}\bar{P}_2 + u_2^{(1)}\bar{Q}_2 + u_3^{(1)}\bar{R}_2 + u_4^{(1)}\bar{T}_2\},$$

∇ being a small quantity, the lowest part of which is of the second order; and the value of that lowest part is given by the foregoing relation.

As in the preceding investigation, the result is a purely analytical relation affecting all directions at X inclusible under the forms adopted for the postulated variables P_2', Q_2', R_2', T_2'.

Ex. Again the application to the Levi-Civita parallels is immediate. On substitution, the right-hand side

$$= -\tfrac{1}{6}x^2\{p_1'K_1(1, 12) + q_1'K_2(1, 12) + r_1'K_3(1, 12) + t_1'K_4(1, 12)\} = 0:$$

and thus, verified up to the second order, $\nabla = 0$. Up to the second order, the angle at X is equal to the angle at O; the fundamental property of the Levi-Civita definition ultimately is the full equality of the two angles.

Severi parallels.

380. The Severi definition of a domainal geodesic XU, which at X is geodesically parallel to the domainal geodesic OB at O, requires that the direction of XU at X shall be contained in the surface which, at O, is geodesic to the domain: this geodesic surface being determinate by the two domainal geodesics OA and OB. It also requires that the angle AXU shall be equal to the angle AOB.

Accordingly, let the direction-variables of the geodesic XU at X in the direction XU be P_2', Q_2', R_2', T_2', as postulated formally in § 378. The four quantities $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$, in the expressions of these variables have to be determined so as to conform to the foregoing requirements, which are:

(i) the angles AXU and AOB shall be equal, and

(ii) the initiating direction of XU at X shall lie in the surface which at O is geodesic to the domain constituted by the domainal geodesics OA and OB.

Precedent to all imposed conditions, the permanent arc-relation $\sum A_X P_2'^2 = 1$, for the direction XU at X, must be satisfied. Hence, after the result in § 378, we must have the general condition

$$u_1^{(2)}\bar{P}_2 + u_2^{(2)}\bar{Q}_2 + u_3^{(2)}\bar{R}_2 + u_4^{(2)}\bar{T}_2 = \tfrac{1}{3}x^2 \sum (\alpha\beta, \gamma\delta) s_{\alpha\beta} s_{\gamma\delta},$$

where the $\alpha\beta$ and $\gamma\delta$ summations are for the orientation-magnitudes $s_{\lambda\mu}$, as defined in § 377 and constructed from the direction-variables at O of the domainal geodesics OA and OB.

The requirement of equality between the angles AXU and AOB annihilates the quantity ∇ of § 379; and therefore, as a special limiting condition,

$$u_1^{(1)}\bar{P}_2 + u_2^{(1)}\bar{Q}_2 + u_3^{(1)}\bar{R}_2 + u_4^{(1)}\bar{T}_2 = 0.$$

Next, we must obtain the special limiting conditions expressing the requirement that the direction of XU at X shall lie in the surface which is geodesic at O to the domain. Let the surface be represented by the parametric equations $\epsilon = 0$ and $\omega = 0$; the relations, which (up to the second order of approximation inclusive) secure its geodesic quality, have been obtained in § 344. Then the special limiting conditions on the geodesic, which arise from the requirement that it shall lie in this surface, are

$$\epsilon_1^{(X)}P_2' + \epsilon_2^{(X)}Q_2' + \epsilon_3^{(X)}R_2' + \epsilon_4^{(X)}T_2' = 0,$$
$$\omega_1^{(X)}P_2' + \omega_2^{(X)}Q_2' + \omega_3^{(X)}R_2' + \omega_4^{(X)}T_2' = 0.$$

Up to the second order of small quantities inclusive, we have

$$\epsilon_1^{(X)} = \epsilon_1 + x\frac{d\epsilon_1}{ds_1} + \tfrac{1}{2}x^2\frac{d^2\epsilon_1}{ds_1^2},$$

with like values for $\epsilon_2^{(X)}$, $\epsilon_3^{(X)}$, $\epsilon_4^{(X)}$; and therefore the first of the superficial relations is

$$\sum\left[\left(\epsilon_1 + x\frac{d\epsilon_1}{ds_1} + \tfrac{1}{2}x^2\frac{d^2\epsilon_1}{ds_1^2}\right)\{p_2' - x\bar{\gamma}_{12} - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2') + \tfrac{1}{2}\bar{P}_2\}\right] = 0,$$

the summation on the left-hand side extending over the four sets of terms associated with the parameters of the domain. The second of the superficial relations has the similar form

$$\sum\left[\left(\omega_1 + x\frac{d\omega_1}{ds_1} + \tfrac{1}{2}x^2\frac{d^2\omega_1}{ds_1^2}\right)\{p_2' - x\bar{\gamma}_{12} - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2') + \tfrac{1}{2}\bar{P}_2\}\right] = 0.$$

These relations must be developed up to the second order of small quantities inclusive.

In the first of the relations, the aggregate of the finite terms

$$= \sum \epsilon_1 p_2' = 0,$$

because the direction-variables of OA and OB satisfy the equations

$$\left.\begin{matrix}\sum \epsilon_1 p_1' = 0\\ \sum \omega_1 p_1' = 0\end{matrix}\right\},\qquad \left.\begin{matrix}\sum \epsilon_1 p_2' = 0\\ \sum \omega_1 p_2' = 0\end{matrix}\right\}.$$

In that same relation, the aggregate of terms of the first order is xS_1, where

$$S_1 = \sum\frac{d\epsilon_1}{ds_1}p_2' - \sum \epsilon_1\bar{\gamma}_{12}.$$

Now we have

$$\begin{aligned}\frac{d\epsilon_1}{ds_1} &= \epsilon_{11}p_1' + \epsilon_{12}q_1' + \epsilon_{13}r_1' + \epsilon_{14}t_1'\\ &= \bar{\epsilon}_{11}p_1' + \bar{\epsilon}_{12}q_1' + \bar{\epsilon}_{13}r_1' + \bar{\epsilon}_{14}t_1' + \epsilon_1\alpha_1 + \epsilon_2\xi_1 + \epsilon_3\phi_1 + \epsilon_4\kappa_1.\end{aligned}$$

To evaluate the first summation in S_1, we take separately the aggregate of terms which involve the quantities $\bar{\epsilon}_{ij}$: this aggregate

$$=\sum \bar{\epsilon}_{11} p_1' p_2'.$$

When the umbral notation of § 344 is used, so that we have $\bar{\epsilon}_{ij}=m_i m_j$, the foregoing expression is changed to the form

$$\sum m_{p_1'} m_{p_2'}.$$

Because the direction p_1', q_1', r_1', t_1', lies in the superficial orientation given by the equations $R_0=0$, $T_0=0$, we have

$$m_{p_1'}=M_1 p_1'+M_2 q_1';$$

and because the direction p_2', q_2', r_2', t_2', lies in the same orientation, we have

$$m_{p_2'}=M_1 p_2'+M_2 q_2'.$$

Consequently the foregoing aggregate

$$\begin{aligned}&=(M_1 p_1'+M_2 q_1')(M_1 p_2'+M_2 q_2')\\ &=M_1^2 p_1' p_2'+M_1 M_2 (p_1' q_2'+q_1' p_2')+M_2^2 q_1' q_2'.\end{aligned}$$

The surface under consideration is geodesic to the domain, so that the relations (in umbral expression)

$$M_1^2=0, \quad M_1 M_2=0, \quad M_2^2=0,$$

are satisfied (§ 344); and therefore the foregoing aggregate vanishes.

Thus the first summation in S_1

$$\begin{aligned}&=\sum p_2'(\epsilon_1 a_1+\epsilon_2 \xi_1+\epsilon_3 \phi_1+\epsilon_4 \kappa_1)\\ &=\epsilon_1 \sum a_1 p_2'+\epsilon_2 \sum \xi_1 p_2'+\epsilon_3 \sum \phi_1 p_2'+\epsilon_4 \sum \kappa_1 p_2'\\ &=\sum \epsilon_1 \bar{\gamma}_{12}.\end{aligned}$$

Hence

$$S_1=0:$$

that is, the aggregate of terms of the first order of small quantities disappears from the first of the surface-relations.

The aggregate of terms of the second order in that same relation

$$=\tfrac{1}{2}(\epsilon_1 \bar{P}_2+\epsilon_2 \bar{Q}_2+\epsilon_3 \bar{R}_2+\epsilon_4 \bar{T}_2)+\tfrac{1}{2}x^2 S_2,$$

where

$$S_2=\sum \frac{d^2 \epsilon_1}{ds_1^2} p_2'-2\sum \frac{d\epsilon_1}{ds_1} \bar{\gamma}_{12}-\sum \{\epsilon_1 (\Gamma_{300} p_1' p_2'^2)\}.$$

In the first term of this expression for S_2, we have

$$\frac{d^2 \epsilon_1}{ds_1^2}=\sum_j \sum_k \epsilon_{1jk} x_j' x_k'+\epsilon_{11} p_1''+\epsilon_{12} q_1''+\epsilon_{13} r_1''+\epsilon_{14} t_1'',$$

with the convention $x_1'=p_1'$, $x_2'=q_1'$, $x_3'=r_1'$, $x_4'=t_1'$; and therefore, with the further convention $z_1'=p_2'$, $z_2'=q_2'$, $z_3'=q_2'$, $z_4'=t_2'$, the first term in S_2

$$=\sum_j\sum_k\sum_l \epsilon_{jkl}x_j'x_k'z_l' + \sum_l \{(\epsilon_{l1}p_1''+\epsilon_{l2}q_1''+\epsilon_{l3}r_1''+\epsilon_{l4}t_1'')z_l'\}.$$

The complete coefficient of the combination $x_j'x_k'z_l'$ is

$$2(\epsilon_{jkl}-\epsilon_{l1}\Gamma_{jk}-\epsilon_{l2}\Delta_{jk}-\epsilon_{l3}\Theta_{jk}-\epsilon_4\Phi_{jk}),$$

or, when the quantities $\bar{\epsilon}_{jkl}$ are introduced from § 325, this complete coefficient

$$\begin{aligned}=2(\bar{\epsilon}_{jkl}+\epsilon_1\Gamma_{jkl}+\epsilon_2\Delta_{jkl}+\epsilon_3\Theta_{jkl}+\epsilon_4\Phi_{jkl})\\ +2(\epsilon_{1j}\Gamma_{kl}+\epsilon_{2j}\Delta_{kl}+\epsilon_{3j}\Theta_{kl}+\epsilon_{4j}\Phi_{kl})\\ +2(\epsilon_{1k}\Gamma_{jl}+\epsilon_{2k}\Delta_{jl}+\epsilon_{3k}\Theta_{jl}+\epsilon_{4k}\Phi_{jl}).\end{aligned}$$

Hence

$$\sum \frac{d^2\epsilon_1}{ds_1^2}p_2' = \sum_j\sum_k\sum_l \bar{\epsilon}_{jkl}x_j'x_k'z_l' + \sum\{\epsilon_1(\Gamma_{300}p_1'^2p_2')\}+2\sum\frac{d\epsilon_1}{ds_1}\bar{\gamma}_{12};$$

and therefore

$$S_2=\sum_j\sum_k\sum_l \bar{\epsilon}_{jkl}x_j'x_k'z_l'.$$

To evaluate this form of S_2, we use the umbral representation of the magnitude $\sum \bar{\epsilon}_{111}p'^3$ in § 344 as given by $g_{p'}{}^3$, so that

$$S_2=g_{p_1'}{}^2g_{p_2'}.$$

Because the direction p_1', q_1', r_1', t_1', lies in the superficial orientation $R_0=0$, $T_0=0$, we have

$$g_{p_1'}=G_1p_1'+G_2q_1';$$

and because the direction p_2', q_2', r_2', t_2', lies in the same orientation, we have

$$g_{p_2'}=G_1p_2'+G_2q_2'.$$

Thus

$$\begin{aligned}S_2&=(G_1p_1'+G_2q_1')^2(G_1p_2'+G_2q_2')\\ &=G_1^3p_1'^2p_2'+G_1^2G_2(p_1'^2q_2'+2p_1'q_1'p_2')+G_1G_2^2(2p_1'q_1'q_2'+q_1'^2p_2')+G_2^3q_1'q_2'.\end{aligned}$$

The surface is geodesic to the domain, so that the relations (in umbral expression)

$$G_1^3=0,\quad G_1^2G_2=0,\quad G_1G_2^2=0,\quad G_2^3=0,$$

are satisfied (§ 344); and therefore

$$S_2=0.$$

Hence the aggregate of terms of the second order

$$=\tfrac{1}{2}(\epsilon_1\bar{P}_2+\epsilon_2\bar{Q}_2+\epsilon_3\bar{R}_2+\epsilon_4\bar{T}_2);$$

and consequently the first of the two relations, securing that the direction XU at X lies in the geodesic surface, becomes

$$\epsilon_1\bar{P}_2+\epsilon_2\bar{Q}_2+\epsilon_3\bar{R}_2+\epsilon_4\bar{T}_2=0.$$

Similarly the second of the two relations, for the same purpose, is

$$\omega_1\bar{P}_2+\omega_2\bar{Q}_2+\omega_3\bar{R}_2+\omega_4\bar{T}_2=0.$$

These are the relations which must be satisfied if the direction P_2', Q_2', R_2', T_2', as formulated in § 378, lies in the geodesic surface. The relation, which expresses the equality of the angles at X and O, has already been stated (p. 587).

Thus the four equations for $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$, are

$$\begin{aligned} u_1^{(2)}\bar{P}_2+u_2^{(2)}\bar{Q}_2+u_3^{(2)}\bar{R}_2+u_4^{(2)}\bar{T}_2 &= \tfrac{1}{3}x^2\sum(\alpha\beta,\gamma\delta)s_{\alpha\beta}s_{\gamma\delta} \\ &= \tfrac{1}{3}x^2K_{12}\sin^2\widehat{12}, \end{aligned}$$

where K_{12} is the sphericity of the domain in the orientation at O constituted by the two directions p_1', q_1', r_1', t_1', and p_2', q_2', r_2', t_2';

$$\begin{aligned} u_1^{(1)}\bar{P}_2+u_2^{(1)}\bar{Q}_2+u_3^{(1)}\bar{R}_2+u_4^{(1)}\bar{T}_2&=0, \\ \epsilon_1\bar{P}_2+\epsilon_2\bar{Q}_2+\epsilon_3\bar{R}_2+\epsilon_4\bar{T}_2&=0, \\ \omega_1\bar{P}_2+\omega_2\bar{Q}_2+\omega_3\bar{R}_2+\omega_4\bar{T}_2&=0; \end{aligned}$$

and therefore

$$\left.\begin{aligned} \bar{P}_2&=\tfrac{1}{3}x^2K_{12}(p_2'-p_1'\cos\widehat{12}) \\ \bar{Q}_2&=\tfrac{1}{3}x^2K_{12}(q_2'-q_1'\cos\widehat{12}) \\ \bar{R}_2&=\tfrac{1}{3}x^2K_{12}(r_2'-r_1'\cos\widehat{12}) \\ \bar{T}_2&=\tfrac{1}{3}x^2K_{12}(t_2'-t_1'\cos\widehat{12}) \end{aligned}\right\}.$$

To compare the directions of the two geodesics which are drawn through the point X, one according to the Levi-Civita definition of parallelism, the other according to the Severi definition, it is sufficient to note their direction-variables. For the Levi-Civita parallel, the p'-variable is

$$p_2'-x\gamma_{12}-\tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2')+\frac{x^2}{6\Omega}\sum_\mu\{a_{1\mu}K_\mu(1,12)\};$$

for the Severi parallel, it is

$$p_2'-x\gamma_{12}-\tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2')+\frac{x^2}{6}K_{12}(p_2'-p_1'\cos\widehat{12}).$$

Manifestly these are not the same as one another; the angle between the two directions is, in general, a small quantity of the second order. But if the domain be of constant sphericity*, the two directions coincide—as was first pointed out by Severi†: the verification of the statement is simple.

* Throughout this treatise, only general amplitudes are discussed, whatever be their dimensions. There are many investigations concerned with specialised amplitudes, in particular, amplitudes having a constant Riemann measure of curvature; the earliest of these appear in the memoir by Beltrami, " Teoria fondamentale degli spazii di curvatura costante ", *Ann. di. Mat.*, Ser. 2, t. ii (1868), pp. 232-255.

† See § 228; Severi's statement was made for an amplitude of any number of dimensions, and not solely for a region or a domain. When the amplitude is a surface, there is no scope for differences of definition.

Third side of a domainal geodesic triangle and the surface geodesic at the vertex.

381. The preceding investigations relate to a direction XU, through a point X on the geodesic OA, the direction being postulated as providing a domainal geodesic parallel to the geodesic OB under the Levi-Civita definition and under the Severi definition respectively. The results, with the interchanges of symbols proper to a change from a point X on OA to a point Y on OB, furnish the variables for a direction YV that provides a domainal geodesic parallel to the geodesic OA under the respective definitions.

Let the domainal geodesic XY be drawn. Its direction-variables at X are (§§ 310, 311) p', q', r', t', where

$$p' = p_0' - x\sum \Gamma_{11}p_1'p_0' - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_0') + \bar{P}_0,$$

with like values for q', r', t', with definitely known values for the second-order magnitudes $\bar{P}_0$, the direction p_0', q_0', r_0', t_0', being a direction in the geodesic surface at O such that

$$zp_0' = yp_2' - xp_1', \quad zq_0' = yq_2' - xq_1', \quad zr_0' = yr_2' - xr_1', \quad zt_0' = yt_2' - xt_1',$$

where $OY = y$, a small quantity of the same order as x, but independent of x. Obviously we have

$$\sum \epsilon_1 p_0' = 0, \quad \sum \omega_1 p_0' = 0.$$

If this domainal geodesic XY should lie in the surface which is geodesic to the domain at O, the two tangential conditions

$$\sum \epsilon_1^{(X)} p' = 0, \quad \sum \omega_1^{(X)} q' = 0,$$

at X should be satisfied. We proceed to prove that the conditions are not satisfied, so that the domainal geodesic XY does not even touch the surface at X.

The first of these conditions is, up to the second order of small quantities,

$$\sum \left[\left(\epsilon_1 + x\frac{d\epsilon_1}{ds_1} + \tfrac{1}{2}x^2 \frac{d^2\epsilon_1}{ds^2} \right) \{ p_0' - x\sum \Gamma_{11}p_1'p_0' - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_0') + \bar{P}_0 \} \right] = 0.$$

For analytical purposes, we may substitute the direction p_0', q_0', r_0', t_0', for the direction p_2', q_2', r_2', t_2', without affecting the geodesic surface; and the analysis of § 380 will apply to the substituted direction. Thus we have

$$\sum \epsilon_1 p_0' = 0,$$

$$\sum \frac{d\epsilon_1}{ds_1} p_0' - \sum (\epsilon_1 \bar{\gamma}_{10}) = 0,$$

$$\sum \frac{d^2\epsilon_1}{ds_1^2} p_0' - 2\sum \frac{d\epsilon_1}{ds_1} \gamma_{10} - \sum \{\epsilon_1(\Gamma_{300}p_1'^2p_0')\} = 0\,;$$

and therefore the first condition becomes

$$\sum \epsilon_1 \bar{P}_0 = 0.$$

Similarly the second condition becomes

$$\sum \omega_1 \bar{P}_0 = 0.$$

Now (§ 311)

$$\bar{P}_0 = \tfrac{1}{6} p_0' K \frac{x^2 y^2}{z^2} \sin^2 \widehat{12} + \frac{xy}{3\Omega} \sum_{\mu} \{a_{1\mu} K_\mu(0, 12)\},$$

with corresponding values of $\bar{Q}_0$, $\bar{R}_0$, $\bar{T}_0$, where the quantity K denotes the sphericity of the domain at O, and the magnitudes $K_\mu(0, 12)$ are defined in § 306. Hence, as $\sum \epsilon_1 p_0' = 0$, $\sum \omega_1 p_0' = 0$, so that the terms in K disappear from the conditions, the two conditions become

$$\sum_\lambda \sum_\mu \{\epsilon_\lambda a_{\lambda\mu} K_\mu(0, 12)\} = 0, \quad \sum_\lambda \sum_\mu \{\omega_\lambda a_{\lambda\mu} K_\mu(0, 12)\} = 0.$$

These conditions are not satisfied for a domain which is of a quite general character: and therefore the domainal geodesic XY, joining the two points X and Y near O all in the surface geodesic at O, does not lie in the geodesic surface.

Consequently the surface is geodesic to the domain at O in all directions; but, at any point other than O in its range, it is geodesic to the domain, only in the direction of a geodesic joining the point to O, such geodesic being common to the domain and the surface.

The conditions, however, are satisfied when the sphericity of the domain is constant—a statement verifiable at once from the forms of the quantities $K_\mu(0, 12)$.

The direction XU, when drawn through X parallel to the geodesic OB under the Severi definition, is tangential to the surface which is geodesic at O to the domain; and its direction-variables at X are the four magnitudes of the type

$$P_2' = p_2' - x\gamma_{12} - \tfrac{1}{2} x^2 (\Gamma_{300} p_1'^2 p_2') + \tfrac{1}{6} x^2 K_{12} (p_2' - p_1' \cos \widehat{12}).$$

Similarly, at a point Y on OB near O such that arc-distance $OY = y$ where y is small, there is a surface-direction YV, parallel to the geodesic OA under the same Severi definition; and its direction-variables at Y are the four magnitudes of the type

$$P_1' = p_1' - y\gamma_{12} - \tfrac{1}{2} y^2 (\Gamma_{300} p_1' p_2'^2) + \tfrac{1}{6} y^2 K_{12} (p_1' - p_2' \cos \widehat{12}).$$

The superficial geodesic, through X in the direction XU, and the superficial geodesic, through Y in the direction YV, intersect because they lie in the surface; and they constitute a Pérès parallelogram (as in §§ 124-126).

Ex. Prove that, as in § 124, the lengths of the third and the fourth sides of this Pérès parallelogram are

$$x - \tfrac{1}{2} K_{12} xy (y + x \cos \widehat{12}), \quad y - \tfrac{1}{2} K_{12} xy (x + y \cos \widehat{12}),$$

and that the fourth angle of the parallelogram exceeds the angle AOB by

$$xy K_{12} \sin \widehat{12}.$$

But a domainal geodesic through X, initiated in the superficial direction XU, and a domainal geodesic through Y, initiated in the superficial direction YV,

both directions being tangential to the surface which is geodesic at O to the domain, do not intersect in general; an assumption, that they do intersect, leads as follows to conditions that are not satisfied.

On the assumption that such domainal geodesics, with the direction-variables P_1', Q_1', R_1', T_1', and P_2', Q_2', R_2', T_2', do intersect at a point O', let the length of the arc YO' be denoted by $x+X$ and that of the arc XO' by $y+Y$, where X and Y are certainly of order higher than the first: an initial supposition, that they are of the third order at least, will be verified in the course of the analysis. The values of the domainal parameters at O' must be the same by the broken geodesic paths OXO' and OYO' in the domain. Accordingly, from these equivalent values of the p-parameter at O' estimated up to the third order of small quantities inclusive, we have a relation

$$\begin{aligned} p+xp_1'+\tfrac{1}{2}x^2p_1''+\tfrac{1}{6}x^3p_1'''+(y+Y)P_2'+\tfrac{1}{2}(y+Y)^2P_2''+\tfrac{1}{6}(y+Y)^3P_2''' \\ =p+yp_2'+\tfrac{1}{2}y^2p_2''+\tfrac{1}{6}y^3p_2'''+(x+X)P_1'+\tfrac{1}{2}(x+X)^2P_1''+\tfrac{1}{6}(x+X)^3P_1''', \end{aligned}$$

the left-hand side being attained by the path OXO' so that P_2'' and P_2''' must be estimated at X, and the right-hand side being attained by the path OYO' so that P_1'' and P_1''' must be estimated at Y.

The values of P_2'' and P_1'' will be required up to the first order of small quantities inclusive, as they are multiplied by quantities already of the second order at least. We have

$$P_2''=-\sum \Gamma_{11}{}^{(X)}P_2'^2$$

always; for the required approximation, we take

$$\begin{aligned} P_2''&=-\sum\left\{\left(\Gamma_{11}+x\frac{d\Gamma_{11}}{ds_1}\right)(p_2'^2-2xp_2'\gamma_{12})\right\} \\ &=-\sum \Gamma_{11}p_2'^2-x\left\{\sum\frac{d\Gamma_{11}}{ds_1}p_2'^2-2(a_2\gamma_{12}+\beta_2\delta_{12}+\gamma_2\theta_{12}+\delta_2\phi_{12})\right\}. \end{aligned}$$

The value of the first summation in the coefficient of x is given by the result in § 306, when $k=1$, $i=2, j=2$; and therefore

$$P_2''-p_2''=-x\left[(\Gamma_{300}p_1'p_2'^2)+\frac{2}{3\Omega}\sum\{a_{1\mu}K_\mu(2,12)\}\right].$$

Similarly

$$P_1''-p_1''=-y\left[(\Gamma_{300}p_1'^2p_2')+\frac{2}{3\Omega}\sum\{a_{1\mu}K_\mu(1,21)\}\right].$$

As P_2''' and P_1''' already are multiplied by magnitudes of the third order at least, the approximation is attained by taking

$$P_2'''=p_2'''=-(\Gamma_{300}p_2'^3),\quad P_1'''=p_1'''=-(\Gamma_{300}p_1'^3).$$

The p-relation thus becomes

$$\begin{aligned}
&p+xp_1'+\tfrac{1}{2}x^2p_1''+\tfrac{1}{6}x^3p_1''' \\
&\quad +yp_2'-xy\bar{\gamma}_{12}-\tfrac{1}{2}x^2y(\Gamma_{300}p_1'^2p_2')+\tfrac{1}{6}x^2yK_{12}(p_2'-p_1'\cos\widehat{12})+Yp_2' \\
&\qquad +\tfrac{1}{2}y^2p_2''-\tfrac{1}{2}xy^2(\Gamma_{300}p_1'p_2'^2)-\frac{1}{3\Omega}xy^2\{\textstyle\sum a_{1\mu}K_\mu(2,12)\}+\tfrac{1}{6}y^3p_2''' \\
&=p+yp_2'+\tfrac{1}{2}y^2p_2''+\tfrac{1}{6}y^3p_2''' \\
&\quad +xp_1'-xy\bar{\gamma}_{12}-\tfrac{1}{2}xy^2(\Gamma_{300}p_1'p_2'^2)+\tfrac{1}{6}xy^2K_{12}(p_1-p_2'\cos\widehat{12})+Xp_1' \\
&\qquad +\tfrac{1}{2}x^2p_1''-\tfrac{1}{2}x^2y(\Gamma_{300}p_1'^2p_2')-\frac{1}{3\Omega}x^2y\{\textstyle\sum a_{1\mu}K_\mu(1,21)\}+\tfrac{1}{6}x^3p_1''';
\end{aligned}$$

and therefore

$$\begin{aligned}
&Yp_2'+\tfrac{1}{6}x^2yK_{12}(p_2'-p_1'\cos\widehat{12})-\frac{1}{3\Omega}xy^2\{\textstyle\sum a_{1\mu}K_\mu(2,12)\} \\
&\qquad =Xp_1'+\tfrac{1}{6}xy^2K_{12}(p_1'-p_2'\cos\widehat{12})-\frac{1}{3\Omega}x^2y\{\textstyle\sum a_{1\mu}K_\mu(1,21)\}.
\end{aligned}$$

There are three similar relations, arising out of the parameters q, r, t, respectively.

In the first place, it is clear that X and Y cannot be of only the second order of small quantities. As the other retained terms are of the third order (and the unretained terms are of order higher than three), we then should have

$$Yp_2'=Xp_1',\quad Yq_2'=Xq_1',\quad Yr_2'=Xr_1',\quad Yt_2'=Xt_1',$$

manifestly an incongruous set. Thus X and Y, if not zero, must be of the third order at least.

In the second place, if these four relations coexist as determining two magnitudes X and Y, there must be conditions

$$\left\|\begin{matrix} J_1, & J_2, & J_3, & J_4 \\ p_1', & q_1', & r_1', & t_1' \\ p_2', & q_2', & r_2', & t_2' \end{matrix}\right\|=0,$$

where, for $i=1, 2, 3, 4$,

$$J_i=y\sum\{a_{i\mu}K_\mu(2,12)\}-x\sum\{a_{i\mu}K_\mu(1,21)\},$$

the quantities x and y being arbitrary. Such conditions are not satisfied. The four relations are incongruous, so that the fundamental assumption is untenable: domainal geodesics, drawn in the Severi directions XU and YV, do not intersect.

Domainal geodesics, under primary conditions of parallelism, drawn to form a quadrilateral.

382. Much of the foregoing analysis can be applied for a different and distinct investigation.

Consider a direction XU at X with direction-variables P_2', Q_2', R_2', T_2', having the values postulated in § 378 in the form

$$P_2'=p_2'-x\sum\Gamma_{11}p_1'p_2'-\tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2')+\tfrac{1}{2}\bar{P}_2,$$

and three similar expressions, the magnitudes $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$, being of the second order of small quantities but otherwise left for determination. Consider also a direction YV at Y with similar direction-variables of like form

$$P_1' = p_1' - y \sum \Gamma_{11} p_1' p_2' - \tfrac{1}{2} y^2 (\Gamma_{300} p_1' p_2'^2) + \tfrac{1}{2} \bar{P}_1,$$

and three similar expressions, the magnitudes $\bar{P}_1$, $\bar{Q}_1$, $\bar{R}_1$, $\bar{T}_1$, being of the second order but otherwise also left for determination. Both these directions XU and YV satisfy the primary conditions (§ 377) of geodesic parallelism (that is, XU to OB, and YV to OA); their form also contains the second-order term common to the direction-variables of a Levi-Civita parallel and a Severi parallel. As there are two sets of quantities $\bar{P}_1$, $\bar{Q}_1$, $\bar{R}_1$, $\bar{T}_1$, and $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$, left for determination, let a limitation be imposed which provides them with the same interchangeability as is possessed by the other respective parts of P_1', Q_1', R_1', T_1', and P_2', Q_2', R_2', T_2', when the symbols for the two geodesics OA and OB are interchanged.

It is required to determine the disposable magnitudes of the type $\bar{P}$ so that, if possible, the domainal geodesic initiated in the direction XU and the domainal geodesic initiated in the direction YV shall intersect. (It will be noted that neither the fundamental property of the Levi-Civita definition nor that of the Severi definition has been developed here.) If O' be the point of intersection, let the geodesic arcs YO' and XO' be denoted by $x+X$ and $y+Y$ respectively, where X and Y will be of at least the second order of small quantities. Proceeding in the same way as in the earlier investigation, the value of the p-parameter at O' provides a relation which, up to the third-order approximation, has the form

$$p_2' Y + \tfrac{1}{2} y \bar{P}_2 - \frac{xy^2}{3\Omega} \sum \{a_{1\mu} K_\mu(2, 12)\} = p_1' X + \tfrac{1}{2} x \bar{P}_1 - \frac{x^2 y}{3\Omega} \sum \{a_{1\mu} K_\mu(1, 21)\};$$

and there are three similar relations which arise from the parameters q, r, t, respectively.

Moreover, there are the permanent arc-relations at X and at Y, which must be satisfied, being $\sum A_X P_2'^2 = 1$ and $\sum A_Y P_1'^2 = 1$ respectively. By § 378, these are

$$u_1^{(2)} \bar{P}_2 + u_2^{(2)} \bar{Q}_2 + u_3^{(2)} \bar{R}_2 + u_4^{(2)} \bar{T}_2 = \tfrac{1}{3} x^2 K_{12} \sin^2 \widehat{12},$$
$$u_1^{(1)} \bar{P}_1 + u_2^{(1)} \bar{Q}_1 + u_3^{(1)} \bar{R}_1 + u_4^{(1)} \bar{T}_1 = \tfrac{1}{3} y^2 K_{12} \sin^2 \widehat{12}.$$

The assumptions, as to formal interchange of $\bar{P}_1$, $\bar{Q}_1$, $\bar{R}_1$, $\bar{T}_1$, and $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$, with the interchange of OA and OB, are in accord with these relations.

Thus far in the analysis, we have assigned no special discriminating characteristic of parallelism which might affect the second-order terms in the direction-variables of XU at X and of YV at Y. In the case of the Levi-Civita definition, when the appropriate values of $\bar{P}_1$ and $\bar{P}_2$ are inserted, the p-parameter relation becomes

$$p_2' Y - p_1' X = \frac{2xy^2}{3\Omega} \sum \{a_{1\mu} K_\mu(2, 12)\} - \frac{2x^2 y}{3\Omega} \sum \{a_{1\mu} K_\mu(1, 21)\};$$

and there are similar forms for the other three relations: the whole set of relations is not satisfied, and so the Levi-Civita domainal parallels through X to OB and through Y to OA do not meet. Similarly, by substituting the values of $\bar{P}_1$ and $\bar{P}_2$ (and the other quantities) appropriate to the Severi definition of parallelism, we find that the Severi domainal parallels through X to OB and through Y to OA do not meet. (The corresponding Severi parallels, belonging to the geodesic surface at X and at Y respectively, do intersect. But at X, there is the deviation between the superficial geodesic and the domainal geodesic, measured by the domainal flexure of the superficial geodesic, a multiple of x and therefore a magnitude of the first order: and the similar consideration holds at Y.)

The equations thus far obtained, even though analytical expression of the interchangeability of the magnitudes $\bar{P}_1$, $\bar{Q}_1$, $\bar{R}_1$, $\bar{T}_1$, and $\bar{P}_2$, $\bar{Q}_2$, $\bar{R}_2$, $\bar{T}_2$, has not been formulated, are inadequate for the precise determination of these magnitudes. As the definitions propounded by Levi-Civita and by Severi do not lead to parallels at X and at Y which intersect, a property of intersection of such parallels requires some alternative definition. Instead of assuming (as is assumed in both those definitions) that the angles at X and at Y are equal to the angle at O, we consider the two equivalent assumptions that the opposite sides of the quadrilateral are equal: so that $YO'=OX$, $XO'=OY$; and therefore, in the preceding analysis, we take $X=0$, $Y=0$.

The p-parameter relation now can be expressed in the form

$$y\left[\bar{P}_2-\frac{2xy}{3\Omega}\sum\{a_{1\mu}K_\mu(2,12)\}\right]=x\left[\bar{P}_1-\frac{2xy}{3\Omega}\sum\{a_{1\mu}K_\mu(1,21)\}\right].$$

The quantity $\bar{P}_2$ is of the second order of small quantities; and it must vanish when $x=0$, because X then coincides with O and P_2' becomes p_2', that is, $\bar{P}_2$ must contain x as a factor. Similarly, $\bar{P}_1$ must contain y as a factor. We therefore can denote the common value of the two sides of the relation by $xy\bar{P}$, where $\bar{P}$ is a small quantity of the first order: and now

$$\bar{P}_2=\frac{2xy}{3\Omega}\sum\{a_{1\mu}K_\mu(2,12)\}+x\bar{P},$$

$$\bar{P}_1=\frac{2xy}{3\Omega}\sum\{a_{1\mu}K_\mu(1,21)\}+y\bar{P};$$

and there are corresponding expressions for $\bar{Q}_2$ and $\bar{Q}_1$ in terms of a quantity $\bar{Q}$, for $\bar{R}_2$ and $\bar{R}_1$ in terms of a quantity $\bar{R}$, and for $\bar{T}_2$ and $\bar{T}_1$ in terms of a quantity $\bar{T}$. Now, by direct calculation, we find

$$\sum_\mu\sum_\lambda u_\lambda^{(2)}a_{\lambda\mu}K_\mu(2,12)=0,\quad \sum_\mu\sum_\lambda u_\lambda^{(1)}a_{\lambda\mu}K_\mu(2,12)=\Omega K_{12}\sin^2\widehat{12},$$

$$\sum_\mu\sum_\lambda u_\lambda^{(1)}a_{\lambda\mu}K_\mu(1,21)=0,\quad \sum_\mu\sum_\lambda u_\lambda^{(2)}a_{\lambda\mu}K_\mu(1,21)=\Omega K_{12}\sin^2\widehat{12};$$

and now the permanent arc-relations become

$$u_1{}^{(2)}\bar{P}+u_2{}^{(2)}\bar{Q}+u_3{}^{(2)}\bar{R}+u_4{}^{(2)}\bar{T}=\tfrac{1}{3}xK_{12}\sin^2\widehat{12},$$
$$u_1{}^{(1)}\bar{P}+u_2{}^{(1)}\bar{Q}+u_3{}^{(1)}\bar{R}+u_4{}^{(1)}\bar{T}=\tfrac{1}{3}yK_{12}\sin^2\widehat{12}.$$

We thus obtain two equations, of explicit analytical form, for the determination of $\bar{P}$, $\bar{Q}$, $\bar{R}$, $\bar{T}$.

But, owing to the interchangeability of $\bar{P}_2$ and $\bar{P}_1$ concurrently with the interchangeability of the sides, the quantity $\bar{P}$ must be symmetrical (or unchangeable) as regards the variables of the sides. Also, it is a small quantity of the first order in small quantities; and therefore we can take

$$\bar{P}=x(Ap_1'+Bp_2')+y(Cp_1'+Dp_2'),$$

with the expectation that $A=D$, $B=C$. Similarly

$$\bar{Q}=x(Aq_1'+Bq_2')+y(Cq_1'+Dq_2'),$$
$$\bar{R}=x(Ar_1'+Br_2')+y(Cr_1'+Dr_2'),$$
$$\bar{T}=x(At_1'+Bt_2')+y(Ct_1'+Dt_2').$$

When these values are substituted in the latest forms of the arc-relations, these become

$$x(A\cos\epsilon+B)+y(C\cos\epsilon+D)=\tfrac{1}{3}xK_{12}\sin^2\widehat{12},$$
$$x(A+B\cos\epsilon)+y(C+D\cos\epsilon)=\tfrac{1}{3}yK_{12}\sin^2\widehat{12}.$$

Now x and y are independent of one another; hence

$$A\cos\epsilon+B=\tfrac{1}{3}K_{12}\sin^2\widehat{12},\qquad C\cos\epsilon+D=0,$$
$$A+B\cos\epsilon=0,\qquad C+D\cos\epsilon=\tfrac{1}{3}K_{12}\sin^2\widehat{12},$$

and therefore

$$A=D=-\tfrac{1}{3}K_{12}\cos\widehat{12},\quad B=C=\tfrac{1}{3}K_{12}.$$

Consequently, we have

$$\bar{P}=\tfrac{1}{3}K_{12}\{x(p_2'-p_1'\cos\widehat{12})+y(p_1'-p_2'\cos\widehat{12})\},$$

with corresponding values for $\bar{Q}$, $\bar{R}$, $\bar{S}$.

To complete the knowledge of the quadrilateral, we require the angles at X and Y. We had $AXU=AOB+\nabla_X$, where (p. 586)

$$\begin{aligned}\nabla_X\sin\widehat{12}&=-\tfrac{1}{2}[u_1{}^{(1)}\bar{P}_2+u_2{}^{(1)}\bar{Q}_2+u_3{}^{(1)}\bar{R}_2+u_4{}^{(1)}\bar{T}_2]\\&=-\tfrac{1}{2}[\tfrac{2}{3}xyK_{12}\sin^2\widehat{12}+\tfrac{1}{3}xyK_{12}\sin^2\widehat{12}]\\&=-\tfrac{1}{2}xyK_{12}\sin^2\widehat{12},\end{aligned}$$

so that

$$\nabla_X=-\tfrac{1}{2}xyK_{12}\sin\widehat{12}.$$

Similarly, if $BYU=AOB+\nabla_Y$, we have

$$\nabla_Y=-\tfrac{1}{2}xyK_{12}\sin\widehat{12},$$

so that the angles at X and Y are equal to one another, though neither of them is equal to AOB. Finally, we determine the angle at O' by estimating the area of the quadrilateral: we have

$$AOB + (\pi - AXU) + (\pi - BYV) + XO'Y - 2\pi = K_{12}xy \sin \widehat{12},$$

that is,

$$XO'Y = AOB.$$

Hence in the quadrilateral thus defined, the opposite angles are equal to one another.

Domainal parallelogram.

383. Gathering together all the results, we can summarize them in the statement:

When two domainal geodesics OXA, OYB, are drawn at a point O in a domain, and small arcs $OX=x$ and $OY=y$ are measured along them respectively, a quadrilateral $OXO'YO$ can be completed by drawing a domainal geodesic through X in a direction with variables P_2', Q_2', R_2', T_2', and a domainal geodesic through Y in a direction with variables P_1', Q_1', R_1', T_1', where

$$\left.\begin{aligned}
P_2' &= p_2' - x\sum \Gamma_{11}p_1'p_2' - \tfrac{1}{2}x^2(\Gamma_{300}p_1'^2p_2') + \tfrac{1}{3}\frac{xy}{\Omega}\sum\{a_{1\mu}K_\mu(2, 12)\} + \tfrac{1}{6}xP\\
Q_2' &= q_2' - x\sum \Delta_{11}p_1'p_2' - \tfrac{1}{2}x^2(\Delta_{300}p_1'^2p_2') + \tfrac{1}{3}\frac{xy}{\Omega}\sum\{a_{2\mu}K_\mu(2, 12)\} + \tfrac{1}{6}xQ\\
R_2' &= r_2' - x\sum \Theta_{11}p_1'p_2' - \tfrac{1}{2}x^2(\Theta_{300}p_1'^2p_2') + \tfrac{1}{3}\frac{xy}{\Omega}\sum\{a_{3\mu}K_\mu(2, 12)\} + \tfrac{1}{6}xR\\
T_2' &= t_2' - x\sum \Phi_{11}p_1'p_2' - \tfrac{1}{2}x^2(\Phi_{300}p_1'^2p_2') + \tfrac{1}{3}\frac{xy}{\Omega}\sum\{a_{4\mu}K_\mu(2, 12)\} + \tfrac{1}{6}xT
\end{aligned}\right\},$$

$$\left.\begin{aligned}
P_1' &= p_1' - y\sum \Gamma_{11}p_1'p_2' - \tfrac{1}{2}y^2(\Gamma_{300}p_1'p_2'^2) + \tfrac{1}{3}\frac{xy}{\Omega}\sum\{a_{1\mu}K_\mu(1, 21)\} + \tfrac{1}{6}yP\\
Q_1' &= q_1' - y\sum \Delta_{11}p_1'p_2' - \tfrac{1}{2}y^2(\Delta_{300}p_1'p_2'^2) + \tfrac{1}{3}\frac{xy}{\Omega}\sum\{a_{2\mu}K_\mu(1, 21)\} + \tfrac{1}{6}yQ\\
R_1' &= r_1' - y\sum \Theta_{11}p_1'p_2' - \tfrac{1}{2}y^2(\Theta_{300}p_1'p_2'^2) + \tfrac{1}{3}\frac{xy}{\Omega}\sum\{a_{3\mu}K_\mu(1, 21)\} + \tfrac{1}{6}yR\\
T_1' &= t_1' - y\sum \Phi_{11}p_1'p_2' - \tfrac{1}{2}y^2(\Phi_{300}p_1'p_2'^2) + \tfrac{1}{3}\frac{xy}{\Omega}\sum\{a_{4\mu}K_\mu(1, 21)\} + \tfrac{1}{6}yT
\end{aligned}\right\},$$

the quantities P, Q, R, T, being given by

$$\left.\begin{aligned}
P &= K_{12}\{x(p_2' - p_1'\cos\widehat{12}) + y(p_1' - p_2'\cos\widehat{12})\}\\
Q &= K_{12}\{x(q_2' - q_1'\cos\widehat{12}) + y(q_1' - q_2'\cos\widehat{12})\}\\
R &= K_{12}\{x(r_2' - r_1'\cos\widehat{12}) + y(r_1' - r_2'\cos\widehat{12})\}\\
T &= K_{12}\{x(t_2' - t_1'\cos\widehat{12}) + y(t_1' - t_2'\cos\widehat{12})\}
\end{aligned}\right\}.$$

(The values of the direction-variables are accurate for approximations of the second order of small quantities inclusive.) The domainal geodesics drawn through X and Y in the specified directions intersect in a point O', completing the quadrilateral. The opposite geodesic arc-sides of the quadrilateral are equal, so that $YO'=OX$, $XO'=OY$. The opposite angles of the quadrilateral are equal, so that $XO'Y=XOY$; and, finally,

$$OXO'=OYO'=\pi-XOY+\tfrac{1}{2}K_{12}xy\sin\widehat{12}.$$

The geodesic quadrilateral thus obtained will be called a *geodesic parallelogram* in the domain.

Moreover, it is convenient to have, at the point O' of intersection of the domainal geodesics through X and Y, the final values of the parameters of the domain. The value of the p-parameter at O'

$$=p+xp_1'+\tfrac{1}{2}x^2p_1''+\tfrac{1}{6}x^3p_1'''+yP_2'+\tfrac{1}{2}y^2P_2''+\tfrac{1}{6}y^3P_2''',$$

the values of P_2'' and P_2''' being taken at X: that is, the value * of the p-parameter at O'

$$\begin{aligned}=p+(xp_1'+yp_2')-\tfrac{1}{2}\textstyle\sum\Gamma_{11}(xp_1'+yp_2')^2-\tfrac{1}{6}\{\Gamma_{300}(xp_1'+yp_2')^3\}\\ +\tfrac{1}{6}xy\{x(p_2'-p_1'\cos\widehat{12})+y(p_1'-p_2'\cos\widehat{12})\}K_{12},\end{aligned}$$

where K_{12} denotes the sphericity of the domain at O estimated for the orientation determined by the two geodesics OA and OB. The value at O' of the q-parameter

$$\begin{aligned}=q+(xq_1'+yq_2')-\tfrac{1}{2}\textstyle\sum\Delta_{11}(xp_1'+yp_2')^2-\tfrac{1}{6}\{\Delta_{300}(xp_1'+yp_2')^3\}\\ +\tfrac{1}{6}xy\{x(q_2'-q_1'\cos\widehat{12})+y(q_1'-q_2'\cos\widehat{12})\}K_{12};\end{aligned}$$

and similarly for the other parameters r and t.

Ex. Show that, up to the third order of small quantities inclusive, the length of the geodesic diagonal OO' of the foregoing geodesic parallelogram

$$=l+\frac{x^2y^2}{3l}K_{12}\sin^2\widehat{12},$$

where $l^2=x^2+y^2+2xy\cos\widehat{12}$.

Domainal cell: domainal paralleloid.

384. The diagram represents a paralleloid in the domain, determined by four small conterminous geodesic edges OA, OB, OC, OD, no three of which lie in a superficial orientation at O, and the whole set not lying in a regional orientation at O. To construct the configuration, we complete the geodesic parallelograms, $OAHB$,

* The results should be compared with the corresponding values in the like investigation (§ 233) for a region.

$OAGC$, $OAF'D$, $OBFC$, $OBG'D$, $OCH'D$, each having as a pair of adjacent edges two of the preceding set of four domainal geodesics. Then, with $OAHB$, $OBFC$, $OCGA$, as three adjacent parallelogrammic faces, we frame a domainal cell (similar to the small regional cell of §§ 240-243) having D' for its vertex diagonally opposite to O; a similar domainal cell having $OBFC$, $OCH'D$, $ODG'B$, as adjacent faces, with A' as its vertex diagonally opposite to O; a similar cell, having $OCH'D$, $ODF'A$, $OAGC$, as adjacent faces, with B' as its vertex diagonally opposite to O; and a fourth similar cell, having $ODF'A$, $OAHB$, $OBG'D$, as adjacent faces, with C' as its vertex diagonally opposite to O.

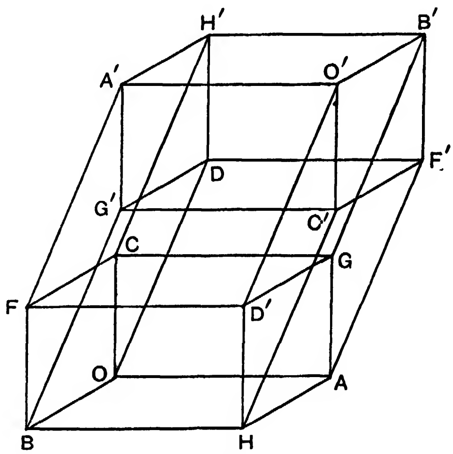

FIG. 36.

Finally, through A', B', C', D', we draw domainal geodesics respectively parallel to OA, OB, OC, OD, (in the sense of parallelism as used in § 382), and of lengths equal to those four edges; and we shall require them to meet in a point O', the vertex of the paralleloid opposite to O.

In this figure, we denote the small lengths of OA, OB, OC, OD, by x, y, z, w, respectively, and the direction-variables at O of these domainal-geodesic edges by p_1', p_2', p_3', p_4', respectively, in each instance with like variables q', r', t'.

Let x, y, z, w, $=l_1$, l_2, l_3, l_4, respectively, and let

$$P_{ij}=\{l_i(p_j'-p_i'\cos\widehat{ij})+l_j(p_i'-p_j'\cos\widehat{ij})\}K_{ij},$$

for all the six combinations $ij=1, 2, 3, 4$, in pairs. Then the direction-variables p' for twelve edges of the paralleloid are known, after § 383, as follows:

$$\left.\begin{aligned}BH:\ & p_1' - y\bar{\gamma}_{12} - \tfrac{1}{2}y^2[\Gamma_{300}p_1'p_2'^2]+\tfrac{1}{3}\frac{xy}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(1, 21)\}+\tfrac{1}{6}yP_{12}\\ AH:\ & p_2' - x\bar{\gamma}_{12} - \tfrac{1}{2}x^2[\Gamma_{300}p_1'^2p_2']+\tfrac{1}{3}\frac{xy}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(2, 12)\}+\tfrac{1}{6}xP_{12}\end{aligned}\right\},$$

$$\left.\begin{aligned}CF:\ & p_2' - z\bar{\gamma}_{23} - \tfrac{1}{2}z^2[\Gamma_{300}p_2'p_3'^2]+\tfrac{1}{3}\frac{yz}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(2, 32)\}+\tfrac{1}{6}zP_{23}\\ BF:\ & p_3' - y\bar{\gamma}_{23} - \tfrac{1}{2}y^2[\Gamma_{300}p_2'^2p_3']+\tfrac{1}{3}\frac{yz}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(3, 23)\}+\tfrac{1}{6}yP_{23}\end{aligned}\right\},$$

$$\left.\begin{aligned}AG:\ & p_3' - x\bar{\gamma}_{31} - \tfrac{1}{2}x^2[\Gamma_{300}p_3'p_1'^2]+\tfrac{1}{3}\frac{xz}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(3, 13)\}+\tfrac{1}{6}xP_{13}\\ CG:\ & p_1' - z\bar{\gamma}_{31} - \tfrac{1}{2}z^2[\Gamma_{300}p_3'^2p_1']+\tfrac{1}{3}\frac{xz}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(1, 31)\}+\tfrac{1}{6}zP_{13}\end{aligned}\right\},$$

$$\left.\begin{aligned}
DF' : \;& p_1' - w\bar{\gamma}_{14} - \tfrac{1}{2}w^2[\Gamma_{300}p_1'p_4'^2] + \tfrac{1}{3}\frac{xw}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(1, 41)\} + \tfrac{1}{6}wP_{14} \\
AF' : \;& p_4' - x\bar{\gamma}_{14} - \tfrac{1}{2}x^2[\Gamma_{300}p_1'^2p_4'] + \tfrac{1}{3}\frac{xw}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(4, 14)\} + \tfrac{1}{6}xP_{14}
\end{aligned}\right\},$$

$$\left.\begin{aligned}
DG' : \;& p_2' - w\bar{\gamma}_{24} - \tfrac{1}{2}w^2[\Gamma_{300}p_2'p_4'^2] + \tfrac{1}{3}\frac{yw}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(2, 42)\} + \tfrac{1}{6}wP_{24} \\
BG' : \;& p_4' - y\bar{\gamma}_{24} - \tfrac{1}{2}y^2[\Gamma_{300}p_2'^2p_4'] + \tfrac{1}{3}\frac{yw}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(4, 24)\} + \tfrac{1}{6}yP_{24}
\end{aligned}\right\},$$

$$\left.\begin{aligned}
DH' : \;& p_3' - w\bar{\gamma}_{34} - \tfrac{1}{2}w^2[\Gamma_{300}p_3'p_4'^2] + \tfrac{1}{3}\frac{zw}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(3, 43)\} + \tfrac{1}{6}wP_{34} \\
CH' : \;& p_4' - z\bar{\gamma}_{34} - \tfrac{1}{2}z^2[\Gamma_{300}p_3'^2p_4'] + \tfrac{1}{3}\frac{zw}{\Omega}\{\textstyle\sum a_{1\mu}K_\mu(4, 34)\} + \tfrac{1}{6}zP_{34}
\end{aligned}\right\}.$$

The q', r', t', direction-variables for these geodesic arcs at their initial points have forms corresponding to the respective p'-variables, as given for AH and for BH in the results in § 383.

Characteristic quantities for a cell.

385. To complete the domainal cell having OA, OB, OC, for its conterminous edges at O, we draw domainal geodesics, one through F parallel and equal to OA, one through G parallel and equal to OB, and one through H parallel and equal to OC; and we make these three geodesics meet in a point D'. As p' direction-variables for these geodesics at F, G, H, respectively, we postulate

$$\begin{aligned}
FD' : \;& p_{23}'^{(1)} = p_1' - (y\bar{\gamma}_{12} + z\bar{\gamma}_{13}) - \tfrac{1}{2}[\Gamma_{300}p_1'(yp_2' + zp_3')^2] + P_{23}^{(1)}, \\
GD' : \;& p_{31}'^{(1)} = p_2' - (z\bar{\gamma}_{23} + x\bar{\gamma}_{21}) - \tfrac{1}{2}[\Gamma_{300}p_2'(zp_3' + xp_1')^2] + P_{31}^{(2)}, \\
HD' : \;& p_{12}'^{(1)} = p_3' - (x\bar{\gamma}_{13} + y\bar{\gamma}_{23}) - \tfrac{1}{2}[\Gamma_{300}p_3'(xp_1' + yp_2')^2] + P_{12}^{(3)},
\end{aligned}$$

with like values for the respective q', r', t', variables; the second-order parts of the type $P_{23}^{(1)}$, $P_{31}^{(2)}$, $P_{12}^{(3)}$, remain for determination by the conditions of concurrence in a point D' with $FD'=x$, $GD'=y$, $HD'=z$. In addition to the conditions of concurrence, there are limitations of analytical symmetry between $P_{23}^{(1)}$, $P_{31}^{(2)}$, $P_{12}^{(3)}$, arising when the variables of the three edges are analytically interchanged. There are, moreover, the permanent arc-relations of the domain to be satisfied, at F for the direction FD', at G for the direction GD', at H for the direction HD'.

We begin with the arc-relations. At F, we must have

$$\textstyle\sum A_F\{p_{23}'^{(1)}\}^2 = 1,$$

and we take approximations up to the second order of small quantities inclusive: we therefore substitute *

* An equivalent form for A_F is $A(p_F, q_F, r_F, t_F)$, where

$$p_F = p + (yp_2' + zp_3') - \tfrac{1}{2}\textstyle\sum \Gamma_{11}(yp_2' + zp_3')^2$$

with like values for q_F, r_F, t_F, up to the second order of small quantities.

$$A_F = A + y\frac{dA}{ds_2} + z\frac{dA}{ds_3} + \tfrac{1}{2}\left(y^2\frac{d^2A}{ds_2{}^2} + 2yz\frac{d^2A}{ds_2ds_3} + z^2\frac{d^2A}{ds_3{}^2}\right),$$

$$\{p_{23}'^{(1)}\}^2 = p_1'^2 - 2p_1'(y\bar{\gamma}_{12} + z\bar{\gamma}_{13}) - p_1'[\Gamma_{300}p_1'(yp_2' + zp_3')^2] + (y\bar{\gamma}_{12} + z\bar{\gamma}_{13})^2 + 2p_1'P_{23}^{(1)},$$

with the adopted convention (§§ 213, 237) as to the second derivatives of the quantities A along different geodesics; and so for the other terms. In the arc-relation, thus modified, the finite terms give the condition

$$\sum Ap_1'^2 = 1,$$

satisfied without any residue. The terms of the first order, which should vanish,

$$= \sum\left\{p_1'^2\left(y\frac{dA}{ds_2} + z\frac{dA}{ds_3}\right)\right\} - 2\sum[u_1^{(1)}\{y\bar{\gamma}_{12} + z\bar{\gamma}_{13}\}];$$

by the results of § 307, these vanish; and thus there is no residuary condition from the first-order terms. The terms of the second order must vanish; and their aggregate

$$\begin{aligned} &= 2\sum u_1^{(1)}P_{23}^{(1)} - \sum[u_1^{(1)}\{\Gamma_{300}p_1'(yp_2' + zp_3')^2\}] + \sum[A(y\bar{\gamma}_{12} + z\bar{\gamma}_{13})^2] \\ &\quad - 2\sum\left\{p_1'(y\bar{\gamma}_{12} + z\bar{\gamma}_{13})\left(y\frac{dA}{ds_2} + z\frac{dA}{ds_3}\right)\right\} \\ &\quad + \tfrac{1}{2}y^2\sum\frac{d^2A}{ds_2{}^2}p_1'^2 + yz\sum\frac{d^2A}{ds_2ds_3}p_1'^2 + \tfrac{1}{2}z^2\sum\frac{d^2A}{ds_3{}^2}p_1'^2. \end{aligned}$$

When we substitute from § 307 for the combinations which involve first arc-derivatives of the primary magnitudes A of the domain, and from §§ 308, 309 for the combinations which involve the second arc-derivatives of the same magnitudes, this aggregate becomes

$$\begin{aligned} &2\sum u_1^{(1)}P_{23}^{(1)} \\ &- \tfrac{1}{3}y^2\sum(\alpha\beta, \gamma\delta)s_{\alpha\beta}s_{\gamma\delta} - \tfrac{2}{3}yz\sum(\alpha\beta, \gamma\delta)\{s_{\alpha\beta}t_{\gamma\delta} + s_{\gamma\delta}t_{\alpha\beta}\} - \tfrac{1}{3}z^2\sum(\alpha\beta, \gamma\delta)t_{\alpha\beta}t_{\gamma\delta}, \end{aligned}$$

where $s_{\alpha\beta}$ are the orientation-variables of p. 582, formed with the direction-variables of OA and OB, and $t_{\alpha\beta}$ are the like orientation-magnitudes formed with the variables of OA and OC. We write, for all the combinations $\alpha\beta$,

$$ys_{\alpha\beta} + zt_{\alpha\beta} = \sigma_{\alpha\beta}(1, 23);$$

and then the necessarily vanishing second-order terms provide a residuary relation

$$\sum u_1^{(1)}P_{23}^{(1)} = \tfrac{1}{3}\sum[(\alpha\beta, \gamma\delta)\sigma_{\alpha\beta}(1, 23)\sigma_{\gamma\delta}(1, 23)],$$

as arising out of the domainal arc-relation at F for the domainal geodesic FD'.

Similarly, the domainal arc-relation at G for the domainal geodesic GD' leaves a residuary second-order condition

$$\sum u_2^{(1)}P_{31}^{(2)} = \tfrac{1}{3}\sum[(\alpha\beta, \gamma\delta)\sigma_{\alpha\beta}(2, 31)\sigma_{\gamma\delta}(2, 31)];$$

and the like arc-relation at H for HD' leaves a residual condition

$$\sum u_3^{(1)} P_{12}^{(3)} = \tfrac{1}{3} \sum [(\alpha\beta, \gamma\delta)\sigma_{\alpha\beta}(3, 12)\sigma_{\gamma\delta}(3, 12)].$$

In the next place, the conditions of concurrence of FD', GD', HD', must be obtained. They are obtainable from the property that the values of the domainal parameters at D', in relation to their values at O, must be the same whatever be the path from O to D'. One such path finishes with FD'; the equivalence of the paths OBF and OCF has already been used in forming the values of the parameters at F, so that (after the result on p. 599) we have

$$p_F = p + yp_2' - zp_3' - \tfrac{1}{2}\sum \Gamma_{11}(yp_2' + zp_3')^2 - \tfrac{1}{6}[\Gamma_{300}(yp_2' + zp_3')^3] + \tfrac{1}{6}yzP_{23},$$

where

$$P_{23} = \{y(p_3' - p_2' \cos \widehat{23}) + z(p_2' - p_3' \cos \widehat{23})\} K_{23},$$

with corresponding values for q_F, r_F, t_F. As we choose a length x along FD' equal to the arc-length x along its parallel OA, the value $p_{D'}$ of the p-parameter, by the broken path ending in FD', is

$$p_{D'} = p_F + xP_{23}'^{(1)} + \tfrac{1}{2}x^2 P_{23}''^{(1)} + \tfrac{1}{6}x^3 P_{23}'''^{(1)},$$

up to the third order inclusive, the values of $P_{23}''^{(1)}$ and $P_{23}'''^{(1)}$ being taken at F. As $P_{23}'''^{(1)}$ is multiplied by x^3, we need only its finite part for this approximation: that is, we take

$$\tfrac{1}{6}x^3 P_3'''^{(1)} = \tfrac{1}{6}x^3 p_1'''.$$

As $P_{23}''^{(1)}$ is multiplied by x^2, we need its value up to the first order inclusive; and therefore

$$\begin{aligned}
P_{23}''^{(1)} &= -\sum \Gamma_{11}^{(F)}\{P_{23}'^{(1)}\}^2 \\
&= -\sum \left[\left(\Gamma_{11} + y\frac{d\Gamma_{11}}{ds_2} + z\frac{d\Gamma_{11}}{ds_3}\right)\{p_1'^2 - 2p_1'(y\bar{\gamma}_{12} + z\bar{\gamma}_{13})\}\right] \\
&= p_1'' - y\left\{\sum \frac{d\Gamma_{11}}{ds_2} p_1'^2 - 2\sum \Gamma_{11}p_1'\bar{\gamma}_{12}\right\} - z\left\{\sum \frac{d\Gamma_{11}}{ds_3} p_1'^2 - 2\sum \Gamma_{11}p_1'\bar{\gamma}_{13}\right\} \\
&= p_1'' - [\Gamma_{300}p_1'^2(yp_2 + zp_3)] - \frac{2}{3\Omega}\sum [a_{1\mu}\{yK_\mu(1, 21) + zK_\mu(1, 31)\}].
\end{aligned}$$

When these values are substituted in $p_{D'}$, as obtained by the indicated path to D', we find

$$\begin{aligned}
p_{D'} = p + xp_1' + yp_2' + zp_3' &- \tfrac{1}{2}\sum \Gamma_{11}(xp_1' + yp_2' + zp_3')^2 - \tfrac{1}{6}[\Gamma_{300}(xp_1' + yp_2' + zp_3')^2] \\
&+ xP_{23}^{(1)} + \tfrac{1}{6}yzP_{23} - \frac{x^2}{3\Omega}[\sum a_{1\mu}\{yK_\mu(1, 21) + zK_\mu(1, 31)\}].
\end{aligned}$$

When we write

$$P_{23}^{(1)} - \frac{1}{3\Omega}\sum [a_{1\mu}\{xyK_\mu(1, 21) + xzK_\mu(1, 31)\}] = [P_{23}^{(1)}],$$

the second line in $p_{D'}$ is

$$x[P_{23}^{(1)}] + \tfrac{1}{6}yzP_{23};$$

and the first line is completely symmetrical in the quantities connected with the directions OA, OB, OC.

A similar expression for $p_{D'}$ is obtainable by the broken geodesic path ending in GD', the first line of it being the same as the foregoing first line; and a third similar expression for $p_{D'}$ is obtainable by the broken geodesic path ending in HD', the first line again being the same as the foregoing first line. The three values, thus obtained for $p_{D'}$, must be equal; and therefore we have the conditions

$$x[P_{23}^{(1)}]+\tfrac{1}{6}yzP_{23}=y[P_{31}^{(2)}]+\tfrac{1}{6}zxP_{31}=z[P_{12}^{(3)}]+\tfrac{1}{6}xyP_{12}.$$

We denote the common value of these three quantities by the symmetric expression

$$xyzP_{123}+\tfrac{1}{6}(yzP_{23}+zxP_{31}+xyP_{12}),$$

which is necessarily of the third order of small quantities, and therefore P_{123} must be a finite magnitude; then

$$[P_{23}^{(1)}]=yzP_{123}+\tfrac{1}{6}(yP_{12}+zP_{13}),$$
$$[P_{31}^{(2)}]=zxP_{123}+\tfrac{1}{6}(zP_{23}+xP_{21}),$$
$$[P_{12}^{(3)}]=xyP_{123}+\tfrac{1}{6}(xP_{31}+yP_{32}).$$

Now we have

$$\sum u_1^{(1)}\{\sum_\mu a_{1\mu}K_\mu(1, 21)\}$$
$$=\Omega\{p_1'K_1(1, 21)+q_1'K_2(1, 21)+r_1'K_3(1, 21)+t_1'K_4(1, 21)\}=0,$$

and likewise

$$\sum u_1^{(1)}\{\sum_\mu a_{1\mu}K_\mu(1, 31)\}=0;$$

so that

$$\sum u_1^{(1)}P_{23}^{(1)}=\sum u_1^{(1)}[P_{23}^{(1)}].$$

Also

$$\sum u_1^{(1)}P_{12}=yK_{12}\sin^2\widehat{12}=y\sum(\alpha\beta, \gamma\delta)s_{\alpha\beta}s_{\gamma\delta},$$
$$\sum u_1^{(1)}P_{13}=zK_{13}\sin^2\widehat{13}=z\sum(\alpha\beta, \gamma\delta)\, t_{\alpha\beta}\,t_{\gamma\delta},$$

with the foregoing notation; and therefore

$$\sum u_1^{(1)}[P_{23}^{(1)}]=yz\sum u_1^{(1)}P_{123}+\tfrac{1}{6}y^2\sum(\alpha\beta, \gamma\delta)s_{\alpha\beta}s_{\gamma\delta}+\tfrac{1}{6}z^2\sum(\alpha\beta, \gamma\delta)t_{\alpha\beta}t_{\gamma\delta}.$$

When substitution is made in the second-order condition from the domainal arc-relation at F for the geodesic FD', we have

$$\sum u_1^{(1)}P_{123}=\tfrac{1}{6}\sum(\alpha\beta, \gamma\delta)[s_{\alpha\beta}t_{\gamma\delta}+s_{\gamma\delta}t_{\alpha\beta}];$$

or, as the variables $s_{\alpha\beta}$ arise out of the directions OA and OB, while the variables $t_{\alpha\beta}$ arise out of the directions OA and OC, we shall express the right-hand in the form

$$\tfrac{1}{6}\sum\begin{vmatrix}\alpha\beta, & \gamma\delta\\ 12, & 13\end{vmatrix}$$

and the equation now is

$$\sum u_1^{(1)} P_{123} = \tfrac{1}{6} \sum \begin{vmatrix} \alpha\beta, & \gamma\delta \\ 12 & 13 \end{vmatrix}.$$

Similarly the arc-relation at G for GD' and the arc-relation at H for HD' lead to the respective equations

$$\sum u_1^{(2)} P_{123} = \tfrac{1}{6} \sum \begin{vmatrix} \alpha\beta, & \gamma\delta \\ 23, & 21 \end{vmatrix},$$

and

$$\sum u_1^{(3)} P_{123} = \tfrac{1}{6} \sum \begin{vmatrix} \alpha\beta, & \gamma\delta \\ 31, & 32 \end{vmatrix}.$$

The quantities P_{123}, Q_{123}, R_{123}, T_{123}, are symmetrical as regards the three directions OA, OB, OC, so that, having regard to the dimensions in the parametric direction-variables, we shall have

$$\begin{aligned} P_{123} &= U_{123} p_1' + V_{123} p_2' + W_{123} p_3', \\ Q_{123} &= U_{123} q_1' + V_{123} q_2' + W_{123} q_3', \\ R_{123} &= U_{123} r_1' + V_{123} r_2' + W_{123} r_3', \\ T_{123} &= U_{123} t_1' + V_{123} t_2' + W_{123} t_3'. \end{aligned}$$

Then the three equations, arising from the second-order conditions surviving from the arc-relations at F, G, H, respectively, become

$$\begin{aligned} U_{123} \qquad + V_{123} \cos \widehat{12} + W_{123} \cos \widehat{13} &= \tfrac{1}{6} \sum \begin{vmatrix} \alpha\beta, & \gamma\delta \\ 12, & 13 \end{vmatrix}, \\ U_{123} \cos \widehat{12} + V_{123} \qquad + W_{123} \cos \widehat{23} &= \tfrac{1}{6} \sum \begin{vmatrix} \alpha\beta, & \gamma\delta \\ 23, & 21 \end{vmatrix}, \\ U_{123} \cos \widehat{13} + V_{123} \cos \widehat{23} + W_{123} \qquad &= \tfrac{1}{6} \sum \begin{vmatrix} \alpha\beta, & \gamma\delta \\ 31, & 32 \end{vmatrix}. \end{aligned}$$

The values of U_{123}, V_{123}, W_{123}, can be regarded as known.

When we summarize the results, the p'-variable for the domainal geodesic FD' at F in the direction FD', geodesically parallel to OA, is

$$\begin{aligned} p_1' - (y\bar{\gamma}_{12} + z\bar{\gamma}_{13}) - \tfrac{1}{2}[\Gamma_{300} p_1' (yp_2' + zp_3')^2] \\ + \frac{1}{3\Omega} \sum_\mu [a_{1\mu} \{xyK_\mu(1, 21) + xzK_\mu(1, 31)\}] + \tfrac{1}{6}(yP_{12} + zP_{13}) + yzP_{123}; \end{aligned}$$

the p'-variable for the domainal geodesic GD' at G in the direction GD', geodesically parallel to OB, is

$$\begin{aligned} p_2' - (z\bar{\gamma}_{23} + x\bar{\gamma}_{21}) - \tfrac{1}{2}[\Gamma_{300} p_2' (zp_3' + xp_1')^2] \\ + \frac{1}{3\Omega} \sum_\mu [a_{1\mu} \{yzK_\mu(2, 32) + yxK_\mu(2, 12)\}] + \tfrac{1}{6}(zP_{23} + xP_{21}) + zxP_{123}; \end{aligned}$$

and the p'-variable for the domainal geodesic HD' at H in the direction HD' geodesically parallel to OC, is

$$p_3' - (x\bar{\gamma}_{31} + y\bar{\gamma}_{32}) - \tfrac{1}{2}[\Gamma_{300}p_3'(xp_1' + yp_2')^2] \\ + \frac{1}{3\Omega}\sum_{\mu}[a_{1\mu}\{zxK_\mu(3, 13) + zyK_\mu(3, 23)\}] + \tfrac{1}{6}(xP_{31} + yP_{32}) + xyP_{123}.$$

Also the value of p at D' is

$$p_{D'} = p + xp_1' + yp_2' + zp_3' - \tfrac{1}{2}\sum \Gamma_{11}(xp_1' + yp_2' + zp_3')^2 - \tfrac{1}{6}[\Gamma_{300}(xp_1' + yp_2' + zp_3')^3] \\ + \tfrac{1}{6}(yzP_{23} + zxP_{31} + xyP_{12}) + xyzP_{123}.$$

By appropriate changes and interchanges of directions and variables, the direction-variables of sets of three geodesic lines terminating in A', in B', and in C', respectively, can be deduced from the direction-variables of the three geodesic lines FD', GD', HD', terminating in D'. Thus the direction-variables p' for FA', $G'A'$, $H'A'$, are

$$\left.\begin{aligned} H'A':\ & p_2' - (z\bar{\gamma}_{23} + w\bar{\gamma}_{24}) - \tfrac{1}{2}[\Gamma_{300}p_2'(zp_3' + wp_4')^2] \\ & + \frac{1}{3\Omega}\sum_{\mu}[a_{1\mu}\{yzK_\mu(2, 32) + ywK_\mu(2, 42)\}] + \tfrac{1}{6}(zP_{23} + wP_{24}) + zwP_{234} \\ G'A':\ & p_3' - (w\bar{\gamma}_{34} + y\bar{\gamma}_{32}) - \tfrac{1}{2}[\Gamma_{300}p_3'(wp_4' + yp_2')^2] \\ & + \frac{1}{3\Omega}\sum_{\mu}[a_{1\mu}\{zwK_\mu(3, 43) + zyK_\mu(3, 23)\}] + \tfrac{1}{6}(wP_{34} + yP_{32}) + wyP_{234} \\ FA':\ & p_4' - (y\bar{\gamma}_{24} + z\bar{\gamma}_{34}) - \tfrac{1}{2}[\Gamma_{300}p_4'(yp_2' + zp_3')^2] \\ & + \frac{1}{3\Omega}\sum_{\mu}[a_{1\mu}\{wyK_\mu(4, 24) + wzK_\mu(4, 34)\}] + \tfrac{1}{6}(yP_{42} + zP_{43}) + yzP_{234} \end{aligned}\right\},$$

and the value of p at A' is

$$p_{A'} = p + yp_2' + zp_3' + wp_4' - \tfrac{1}{2}\sum \Gamma_{11}(yp_2' + zp_3' + wp_4')^2 - \tfrac{1}{6}[\Gamma_{300}(yp_2' + zp_3' + wp_4')^3] \\ + \tfrac{1}{6}(zwP_{34} + wyP_{42} + yzP_{23}) + yzwP_{234}.$$

The direction-variables p' for $F'B'$, GB', $H'B'$, are

$$\left.\begin{aligned} H'B':\ & p_1' - (z\bar{\gamma}_{13} + w\bar{\gamma}_{14}) - \tfrac{1}{2}[\Gamma_{300}p_1'(zp_3' + wp_4')^2] \\ & + \frac{1}{3\Omega}\sum_{\mu}[a_{1\mu}\{xzK_\mu(1, 31) + xwK_\mu(1, 41)\}] + \tfrac{1}{6}(zP_{13} + wP_{14}) + zwP_{341} \\ F'B':\ & p_3' - (w\bar{\gamma}_{34} + x\bar{\gamma}_{31}) - \tfrac{1}{2}[\Gamma_{300}p_3'(wp_4' + xp_1')^2] \\ & + \frac{1}{3\Omega}\sum_{\mu}[a_{1\mu}\{zwK_\mu(3, 43) + zxK_\mu(3, 13)\}] + \tfrac{1}{6}(wP_{34} + xP_{31}) + xwP_{341} \\ GB':\ & p_4' - (x\bar{\gamma}_{41} + z\bar{\gamma}_{43}) - \tfrac{1}{2}[\Gamma_{300}p_4'(xp_1' + zp_3')^2] \\ & + \frac{1}{3\Omega}\sum_{\mu}[a_{1\mu}\{wxK_\mu(4, 14) + wzK_\mu(4, 34)\}] + \tfrac{1}{6}(xP_{41} + zP_{43}) + xzP_{341} \end{aligned}\right\},$$

and the value of p at B' is

$$p_{B'} = p + zp_3' + wp_4' + xp_1' - \tfrac{1}{2}\sum \Gamma_{11}(zp_3' + wp_4' + xp_1')^2 - \tfrac{1}{6}[\Gamma_{300}(zp_3' + wp_4' + xp_1')^3 + \tfrac{1}{6}(zwP_{34} + wxP_{41} + xzP_{13}) + xzwP_{341}.$$

Finally the direction-variables p' for $F'C'$, $G'C'$, $H'C'$, are

$$\left.\begin{aligned}
G'C' :\ & p_1' - (y\bar{\gamma}_{12} + w\bar{\gamma}_{14}) - \tfrac{1}{2}[\Gamma_{300}p_1'(yp_2' + wp_4')^2] \\
& + \frac{1}{3\Omega}\sum_\mu [a_{1\mu}\{xyK_\mu(1, 21) + xwK_\mu(1, 41)\}] + \tfrac{1}{6}(yP_{12} + wP_{14}) + ywP_{412} \\
F'C' :\ & p_2' - (w\bar{\gamma}_{24} + x\bar{\gamma}_{21}) - \tfrac{1}{2}[\Gamma_{300}p_2'(wp_4' + xp_1')^2] \\
& + \frac{1}{3\Omega}\sum_\mu [a_{1\mu}\{ywK_\mu(2, 42) + yxK_\mu(2, 12)\}] + \tfrac{1}{6}(wP_{24} + xP_{21}) + wxP_{412} \\
HC' :\ & p_4' - (x\bar{\gamma}_{41} + y\bar{\gamma}_{42}) - \tfrac{1}{2}[\Gamma_{300}p_4'(xp_1' + yp_2')^2] \\
& + \frac{1}{3\Omega}\sum_\mu [a_{1\mu}\{wxK_\mu(4, 14) + wyK_\mu(4, 24)\}] + \tfrac{1}{6}(xP_{41} + yP_{42}) + xyP_{412}
\end{aligned}\right\}$$

and the value of p at C' is

$$p_{C'} = p + wp_4' + xp_1' + yp_2' - \tfrac{1}{2}\sum \Gamma_{11}(wp_4' + xp_1' + yp_2')^2 - \tfrac{1}{6}[\Gamma_{300}(wp_4' + xp_1' + yp_2')^2] + \tfrac{1}{6}(ywP_{24} + wxP_{41} + xyP_{12}) + xywP_{412}.$$

The values of the direction-variables q', r', t', for each of the lines, are deduced from those of the direction-variables p' of the line by the customary changes of the variable p_i' into q_i', r_i', t_i', respectively (for $i = 1, 2, 3, 4$), and of the minors $a_{1\mu}$ into $a_{2\mu}$, $a_{3\mu}$, $a_{4\mu}$, respectively.

Characteristic quantities for a paralleloid.

386. To complete the paralleloid, having a vertex O' diagonally opposite to O, it is necessary to draw domainal geodesics, one through A' parallel to OA (in the sense of parallelism stated in § 382) and equal in length to OA, another through B' similarly related to OB, another through C' similarly related to OC, and a fourth through D' similarly related to OD. All these four geodesics are to meet in O'. We postulate direction-variables p' for the four geodesics, at A' in the direction $A'O'$, at B' in the direction $B'O'$, at C' in the direction $C'O'$, and at D' in the direction $D'O'$, as follows:

$$\begin{aligned}
A'O' &: P_1' = p_1' - (y\bar{\gamma}_{12} + z\bar{\gamma}_{13} + w\bar{\gamma}_{14}) - \tfrac{1}{2}[\Gamma_{300}p_1'(yp_2' + zp_3' + wp_4')^2] + P_1, \\
B'O' &: P_2' = p_2' - (z\bar{\gamma}_{23} + w\bar{\gamma}_{24} + x\bar{\gamma}_{21}) - \tfrac{1}{2}[\Gamma_{300}p_2'(zp_3' + wp_4' + xp_1')^2] + P_2, \\
C'O' &: P_3' = p_3' - (w\bar{\gamma}_{34} + x\bar{\gamma}_{31} + y\bar{\gamma}_{32}) - \tfrac{1}{2}[\Gamma_{300}p_3'(wp_4' + xp_1' + yp_2')^2] + P_3, \\
D'O' &: P_4' = p_4' - (x\bar{\gamma}_{41} + y\bar{\gamma}_{42} + z\bar{\gamma}_{43}) - \tfrac{1}{2}[\Gamma_{300}p_4'(xp_1' + yp_2' + zp_3')^2] + P_4,
\end{aligned}$$

where the determinable magnitudes P_1, P_2, P_3, P_4, are of the second order of small quantities. The direction-variables q', r', t', for the respective directions at A', B', C', D', are deduced from the direction-variables p', by the customary

changes of parameters by the association of determinable magnitudes Q_i, R_i, T_i, for $i=1, 2, 3, 4$. In order to determine the magnitudes P_i, Q_i, R_i, T_i, we have

(i) the permanent arc-relations at A', B', C', D'; and

(ii) the conditions which arise by equating, at O', the values of the parameters attained at O' by the broken geodesic paths ending in $A'O'$, $B'O'$, $C'O'$, $D'O'$, respectively.

We begin with the residuary conditions from the arc-relations at A', B', C', D', the approximations in these relations being taken up to small quantities of the second order (or, what is effectively the same range, up to third-order arc-derivatives of domainal parameters). At D', this relation is

$$\sum A_{D'}\, P_4'^2 = 1.$$

Up to the second order of small quantities inclusive, we have, as in §§ 213, 237,

$$A_{D'} = A + \left(x\frac{d}{ds_1} + y\frac{d}{ds_2} + z\frac{d}{ds_3}\right)A + \tfrac{1}{2}\left(x\frac{d}{ds_1} + y\frac{d}{ds_2} + z\frac{d}{ds_3}\right)^2 A,$$

$$P_4'^2 = p_4'^2 - 2p_4'(x\bar{\gamma}_{14} + y\bar{\gamma}_{24} + z\bar{\gamma}_{34}) + (x\bar{\gamma}_{14} + y\bar{\gamma}_{24} + z\bar{\gamma}_{34})^2$$
$$- p_4'[\Gamma_{300}p_4'(xp_1' + yp_2' + zp_3')^2] + 2p_4'P_4,$$

with corresponding values for the other primary magnitudes at D' and for the other combinations of the direction-variables P_4', Q_4', R_4', T_4'.

When these values are substituted in the arc-relation, the finite terms on the left-hand side $=\sum Ap_4'^2$, and therefore balance the right-hand side. The aggregate, of the terms of the first order of small quantities, is found to be zero: so that there is no residual first-order condition. The aggregate, of the terms of the second order of small quantities, must vanish; and, as a consequence, there is a residual second-order condition, which ultimately reduces to

$$\begin{aligned}\sum \{u_1^{(4)}P_4\} = \tfrac{1}{6}\sum (\alpha\beta, \gamma\delta)[\ & x^2 s_{\alpha\beta}^{(14)} s_{\gamma\delta}^{(14)} + yz\{s_{\alpha\beta}^{(24)} s_{\gamma\delta}^{(34)} + s_{\gamma\delta}^{(24)} s_{\alpha\beta}^{(34)}\} \\ & + y^2 s_{\alpha\beta}^{(24)} s_{\gamma\delta}^{(24)} + zx\{s_{\alpha\beta}^{(14)} s_{\gamma\delta}^{(34)} + s_{\gamma\delta}^{(14)} s_{\alpha\beta}^{(34)}\} \\ & + z^2 s_{\alpha\beta}^{(34)} s_{\gamma\delta}^{(34)} + xy\{s_{\alpha\beta}^{(14)} s_{\gamma\delta}^{(24)} + s_{\gamma\delta}^{(14)} s_{\alpha\beta}^{(24)}\}],\end{aligned}$$

where an index in a symbol $s_{\alpha\beta}^{(14)}$ implies that the variable $s_{\alpha\beta}$ is formed from the direction-variables of OA and OD; or, if we write

$$xs_{\alpha\beta}^{(14)} + ys_{\alpha\beta}^{(24)} + zs_{\alpha\beta}^{(31)} = \sigma_{\alpha\beta}(4, 123),$$

the condition becomes

$$\sum \{u_1^{(4)}P_4\} = \tfrac{1}{6}\sum [(\alpha\beta, \gamma\delta)\sigma_{\alpha\beta}(4, 123)\sigma_{\gamma\delta}(4, 123)].$$

Similarly, the arc-relations at A', B', C', respectively, produce the residual arc-relations

$$\sum \{u_1^{(1)}P_1\} = \tfrac{1}{6}\sum [(\alpha\beta, \gamma\delta)\sigma_{\alpha\beta}(1, 234)\sigma_{\gamma\delta}(1, 234)],$$

with y, z, w, in the composition of its quantities $\sigma_{\alpha\beta}$;

$$\sum \{u_1^{(2)}P_2\} = \tfrac{1}{6}\sum [(\alpha\beta, \gamma\delta)\sigma_{\alpha\beta}(2, 341)\sigma_{\gamma o}(2, 341)],$$

with z, w, x, in the composition of its quantities $\sigma_{\alpha\beta}$; and

$$\sum\{u_1^{(3)}P_3\}=\tfrac{1}{6}\sum[(\alpha\beta,\gamma\delta)\sigma_{\alpha\beta}(3,412)\sigma_{\gamma\delta}(3,412)],$$

with w, x, y, in the composition of its quantities $\sigma_{\alpha\beta}$.

Next, we have conditions arising out of the values of the parameters at O', which (relative to O) are the same whatever be the broken geodesic path from O to O'. Consider the geodesic path which ends with the geodesic portion $D'O'$; in the value of a parameter at D', the equivalence of broken geodesic paths from O to D' has been used; consequently the single value of the parameter p at O', when O' is attained by a path ending with $D'O'$, is

$$=P_{D'}+wP_4'+\tfrac{1}{2}w^2P_4''+\tfrac{1}{6}w^3P_4''',$$

up to the third order of small quantities, the length of $D'O'$ being w, the same as OD, and the values of P_4'' and P_4''' being taken at D'.

As P_4''' is multiplied by w^3, and as the whole approximation does not proceed beyond the third order, we have $P_4'''=p_4'''$, so that

$$\tfrac{1}{6}w^3P_4'''=\tfrac{1}{6}w^3p_4'''.$$

As P_4'' is multiplied by w^2, we shall require the value of P_4'' up to the first order inclusive; accordingly, we take

$$\begin{aligned}P_4''&=-\sum\Gamma_{11}^{(D')}P_4'^2\\&=-\sum\left[\left(\Gamma_{11}+x\frac{d\Gamma_{11}}{ds_1}+y\frac{d\Gamma_{11}}{ds_2}+z\frac{d\Gamma_{11}}{ds_3}\right)\{p_4'^2-2p_4'(x\bar{\gamma}_{14}+y\bar{\gamma}_{24}+z\bar{\gamma}_{34})\}\right]\\&=p_4''-xE_1-yE_2-zE_3,\end{aligned}$$

up to this order, where

$$\begin{aligned}E_1&=\sum\frac{d\Gamma_{11}}{ds_1}p_4'^2-2\sum\Gamma_{11}p_4'\bar{\gamma}_{14}=[\Gamma_{300}p_1\,p_4'^2]+\frac{2}{3\Omega}\sum_\mu\{a_{1\mu}K_\mu(4,14)\},\\E_2&=\sum\frac{d\Gamma_{11}}{ds_2}p_4'^2-2\sum\Gamma_{11}p_4'\bar{\gamma}_{24}=[\Gamma_{300}p_2'p_4'^2]+\frac{2}{3\Omega}\sum_\mu\{a_{1\mu}K_\mu(4,24)\},\\E_3&=\sum\frac{d\Gamma_{11}}{ds_3}p_4'^2-2\sum\Gamma_{11}p_4'\bar{\gamma}_{34}=[\Gamma_{300}p_3\,p_4'^2]+\frac{2}{3\Omega}\sum_\mu\{a_{1\mu}K_\mu(4,34)\},\end{aligned}$$

by the results of § 306.

When the values of P_4', P_4'', P_4''', are substituted, and like terms are collected, we find

$$\begin{aligned}p_{O'}=p+xp_1'+yp_2'+zp_3'+wp_4'&-\tfrac{1}{2}\sum\Gamma_{11}(xp_1'+yp_2'+zp_3'+wp_4')^2\\&-\tfrac{1}{6}[\Gamma_{300}(xp_1'+yp_2'+zp_3'+wp_4')^3]\\+\tfrac{1}{6}(yzP_{23}+zxP_{31}+xyP_{12})&+xyzP_{123}\\+wP_4-\frac{w^2}{3\Omega}&\sum[a_{1\mu}\{xK_\mu(4,14)+yK_\mu(4,24)+zK_\mu(4,34)\}],\end{aligned}$$

up to the third order of small quantities inclusive. In this expression, the total of the first two lines is actually and formally symmetrical in magnitudes connected with the four directions OA, OB, OC, OD; the total of the last two lines is formally special to the mode of approach to O' by $D'O'$. Let

$$P_4 - \frac{w}{3\Omega} \sum [a_{1\mu}\{xK_\mu(4, 14) + yK_\mu(4, 24) + zK_\mu(4, 34)\}] = \bar{P}_4,$$

so that the total of the last two lines

$$= w\bar{P}_4 + \tfrac{1}{6}(yzP_{23} + zxP_{31} + xyP_{12}) + xyzP_{123}.$$

Similarly, values of $p_{O'}$ are obtainable by the other methods of approach to O', ending in portions $A'O'$, $B'O'$, $C'O'$, respectively. As the four values thus obtained must be the same, we have

$$\begin{aligned} & w\bar{P}_4 + \tfrac{1}{6}(yzP_{23} + zxP_{31} + xyP_{12}) + xyzP_{123} \\ &= x\bar{P}_1 + \tfrac{1}{6}(zwP_{34} + wyP_{42} + yzP_{23}) + yzwP_{234} \\ &= y\bar{P}_2 + \tfrac{1}{6}(wxP_{41} + xzP_{13} + zwP_{34}) + zwxP_{341} \\ &= z\bar{P}_3 + \tfrac{1}{6}(xyP_{12} + ywP_{24} + wxP_{41}) + wxyP_{412}, \end{aligned}$$

each expression being of the third order of small quantities. The common value of the four expressions is therefore symmetrical in magnitudes and direction-variables appertaining to the four initiating directions OA, OB, OC, OD; and therefore, having regard to the terms in the four forms, we could take this common value to be

$$\begin{aligned} & xyzwP + \tfrac{1}{6}(yzP_{23} + zxP_{31} + xyP_{12} + wxP_{14} + wyP_{24} + wzP_{34}) \\ & \quad + yzwP_{234} + zwxP_{341} + wxyP_{412} + xyzP_{123}. \end{aligned}$$

But our approximation has extended only so far as to include terms of the third order in the values of the parameters, and terms of the second order in the values of the direction-variables. A term $xyzwP$ is of the fourth order, and it is not the sole term which would be of that fourth order in the value of $p_{O'}$: accordingly, in the development of the approximation actually retained, consideration of the order of the term $xyzwP$ requires its exclusion. We thus have

$$\left.\begin{aligned} \bar{P}_1 &= \tfrac{1}{6}(\qquad yP_{12} + zP_{13} + wP_{14}) + \qquad zwP_{341} + wyP_{412} + yzP_{123} \\ \bar{P}_2 &= \tfrac{1}{6}(xP_{12} \qquad + zP_{23} + wP_{24}) + wzP_{234} \qquad + xwP_{412} + zxP_{123} \\ \bar{P}_3 &= \tfrac{1}{6}(xP_{13} + yP_{23} \qquad + wP_{34}) + ywP_{234} + wxP_{341} \qquad + xyP_{123} \\ \bar{P}_4 &= \tfrac{1}{6}(xP_{14} + yP_{24} + zP_{34} \qquad) + zyP_{234} + xzP_{341} + yxP_{412} \end{aligned}\right\},$$

all of them quantities of the second order.

These inferences from the community of value in the various expressions for $p_{O'}$ leave no magnitudes for determination; and the same conclusion holds for the expressions for $q_{O'}$, $r_{O'}$, $t_{O'}$. Thus the direction-variables P_ι', Q_ι', R_ι', T_ι', are determinate up to the second order of small quantities inclusive; and, in these

forms, they must satisfy the residual conditions from the four arc-relations at A', B', C', D', which also have been taken to the same second order of approximation. These arc-relations are inclusible in the single form

$$u_1^{(\lambda)}P_\lambda + u_2^{(\lambda)}Q_\lambda + u_3^{(\lambda)}R_\lambda + u_4^{(\lambda)}T_\lambda$$
$$= \tfrac{1}{6}\sum (\alpha\beta,\ \gamma\delta)[\sigma_{\alpha\beta}(\lambda,\ \mu\nu\rho)\sigma_{\gamma\delta}(\lambda,\ \mu\nu\rho)],$$

where λ, μ, ν, ρ, are the integers 1, 2, 3, 4, in cyclical order, and $\lambda=1, 2, 3, 4$, for the four relations.

The verification, that the relations are satisfied, is obtained by direct substitution. We have

$$P_1 = \bar{P}_1 + \frac{x}{3\Omega}\sum_\mu [a_{1\mu}\{yK_\mu(1,\ 21) + zK_\mu(1,\ 31) + wK_\mu(1,\ 41)\}],$$

and

$$\sum u_1^{(1)}\Big[\sum_\mu a_{1\mu}K_\mu(1,\ m1)\Big]$$
$$= \Omega\{p_1'K_1(1,\ m1) + q_1'K_2(1,\ m1) + r_1'K_3(1,\ m1) + t_1'K_4(1,\ m1)\} = 0,$$

for the values $m=2, 3, 4$; and so

$$\sum u_1^{(1)}P_1 = \sum u_1^{(1)}\bar{P}_1.$$

We therefore have to calculate the right-hand side. Now

$$\sum u_1^{(1)}P_{12} = K_{12}\sum [u_1^{(1)}\{x(p_2' - p_1'\cos\widehat{12}) + y(p_1' - p_2'\cos\widehat{12})\}]$$
$$= yK_{12}\sin^2\widehat{12} = y\sum (\alpha\beta,\ \gamma\delta)s_{\alpha\beta}^{(12)}s_{\gamma\delta}^{(12)},$$

and similarly for the terms in P_{13} and P_{14}; hence

$$\sum \{u_1^{(1)}(yP_{12} + zP_{13} + wP_{14})\}$$
$$= \sum [(\alpha\beta,\ \gamma\delta)\{y^2 s_{\alpha\beta}^{(12)}s_{\gamma\delta}^{(12)} + z^2 s_{\alpha\beta}^{(13)}s_{\gamma\delta}^{(13)} + w^2 s_{\alpha\beta}^{(14)}s_{\gamma\delta}^{(14)}\}].$$

Again, as on p. 604,

$$\sum u_1^{(1)}P_{1mn} = \tfrac{1}{6}\sum [(\alpha\beta,\ \gamma\delta)\{s_{\alpha\beta}^{(1m)}s_{\gamma\delta}^{(1n)} + s_{\alpha\beta}^{(1n)}s_{\gamma\delta}^{(1m)}\}]$$

for all the pairs of values for m, $n=2, 3, 4$; and therefore

$$\sum \{u_1^{(1)}(zwP_{341} + wyP_{412} + yzP_{123})\}$$
$$= \tfrac{1}{6}\sum [(\alpha\beta,\ \gamma\delta)\{\ zw(s_{\alpha\beta}^{(13)}s_{\gamma\delta}^{(14)} + s_{\alpha\beta}^{(14)}s_{\gamma\delta}^{(13)})$$
$$+ wy(s_{\alpha\beta}^{(14)}s_{\gamma\delta}^{(12)} + s_{\alpha\beta}^{(12)}s_{\gamma\delta}^{(14)})$$
$$+ yz(s_{\alpha\beta}^{(12)}s_{\gamma\delta}^{(13)} + s_{\alpha\beta}^{(13)}s_{\gamma\delta}^{(12)})\}].$$

Consequently,

$$\sum \{u_1^{(1)}P_1\} = \sum \{u_1^{(1)}\bar{P}_1\} = \tfrac{1}{6}\sum \{(\alpha\beta,\ \gamma\delta)S\},$$

where

$$\begin{aligned} S = &\quad y^2 s_{\alpha\beta}^{(12)} s_{\gamma\delta}^{(12)} + zw\left(s_{\alpha\beta}^{(13)} s_{\gamma\delta}^{(14)} + s_{\alpha\beta}^{(14)} s_{\gamma\delta}^{(13)}\right) \\ &+ z^2 s_{\alpha\beta}^{(13)} s_{\gamma\delta}^{(13)} + wy\left(s_{\alpha\beta}^{(14)} s_{\gamma\delta}^{(12)} + s_{\alpha\beta}^{(12)} s_{\gamma\delta}^{(14)}\right) \\ &+ w^2 s_{\alpha\beta}^{(14)} s_{\gamma\delta}^{(14)} + yz\left(s_{\alpha\beta}^{(12)} s_{\gamma\delta}^{(13)} + s_{\alpha\beta}^{(13)} s_{\gamma\delta}^{(12)}\right) \\ = &\quad \{ys_{\alpha\beta}^{(12)} + zs_{\alpha\beta}^{(13)} + ws_{\alpha\beta}^{(14)}\}\{ys_{\gamma\delta}^{(12)} + zs_{\gamma\delta}^{(13)} + ws_{\gamma\delta}^{(14)}\} \\ = &\quad \sigma_{\alpha\beta}(1, 234)\sigma_{\gamma\delta}(1, 234)\,; \end{aligned}$$

and therefore the second-order condition

$$\sum \{u_1{}^{(1)}P_1\} = \tfrac{1}{6}\sum [(\alpha\beta, \gamma\delta)\sigma_{\alpha\beta}(1, 234)\sigma_{\gamma\delta}(1, 234)]$$

is satisfied.

In the same way, we can verify the other three second-order conditions which assign the obtained values for

$$\sum \{u_1{}^{(2)}P_2\}, \quad \sum \{u_1{}^{(3)}P_3\}, \quad \sum \{u_1{}^{(4)}P_4\},$$

respectively.

All the required conditions are satisfied, by the values obtained; and therefore we have, as the p' direction-variable of $A'O'$ at A',

$$\begin{aligned} P_1' = p_1' - (y\bar{\gamma}_{12} + z\bar{\gamma}_{13} + w\bar{\gamma}_{14}) - \tfrac{1}{2}[\Gamma_{300} p_1'(yp_2' + zp_3' + wp_4')^2] \\ + \frac{x}{3\Omega}\sum_{\mu} [a_{1\mu}\{yK_\mu(1, 21) + zK_\mu(1, 31) + wK_\mu(1, 41)\}] \\ + \tfrac{1}{6}(yP_{12} + zP_{13} + wP_{14}) + zwP_{134} + wyP_{142} + yzP_{123}, \end{aligned}$$

up to the second order of small quantities inclusive. There are similar values for the direction-variables p' of $B'O'$ at B', of $C'O'$ at C', and of $D'O'$ at D'. Further, the direction-variables q', r', t', of the same four dimensional geodesics at the same points are derivable from respective direction-variables p', by appropriate changes (i) of p' into q', r', t', successively, with simultaneous changes (ii) of $\bar{\gamma}_{ij}$ into $\bar{\delta}_{ij}$, $\bar{\theta}_{ij}$, $\bar{\phi}_{ij}$, (iii) of Γ_{ijk} into Δ_{ijk}, Θ_{ijk}, Φ_{ijk}, and (iv) of $a_{1\mu}$ into $a_{2\mu}$, $a_{3\mu}$, $a_{4\mu}$.

Finally, the value of the p-parameter at O' is

$$\begin{aligned} P = p + xp_1' + yp_2' + zp_3' + wp_4' - \tfrac{1}{2}\sum \Gamma_{11}(xp_1' + yp_2' + zp_3' + wp_4')^2 \\ - \tfrac{1}{6}[\Gamma_{300}(xp_1' + yp_2' + zp_3' + wp_4')^3] \\ + \tfrac{1}{6}(yzP_{23} + zxP_{31} + xyP_{12} + wxP_{14} + wyP_{24} + wzP_{34}) \\ + yzwP_{234} + zwxP_{341} + wxyP_{412} + xyzP_{123}\,; \end{aligned}$$

and there are corresponding values at O' for the other three parameters.

The essential magnitudes of the paralleloid are therefore known, up to the retained orders of approximation which are the second for direction-variables and the third for the parameters.

CHAPTER XXXIII

PARAMETRIC CURVES IN A DOMAIN

387. A curve in a domain can be represented analytically as the intersection of three parametric regions

$$\theta(p, q, r, t)=0, \quad \phi(p, q, r, t)=0, \quad \chi(p, q, r, t)=0.$$

The direction-variables p', q', r', t', of its tangent are such that

$$\begin{aligned}\theta_1 p' + \theta_2 q' + \theta_3 r' + \theta_4 t' &= 0,\\ \phi_1 p' + \phi_2 q' + \phi_3 r' + \phi_4 t' &= 0,\\ \chi_1 p' + \chi_2 q' + \chi_3 r' + \chi_4 t' &= 0;\end{aligned}$$

and therefore

$$\frac{p'}{(2, 3, 4)}=\frac{q'}{-(3, 4, 1)}=\frac{r'}{(4, 1, 2)}=\frac{t'}{-(1, 2, 3)}=\nabla,$$

where the denominator symbols are defined by the relation

$$(i, j, k)=\begin{vmatrix} \theta_i, & \theta_j, & \theta_k \\ \phi_i, & \phi_j, & \phi_k \\ \chi_i, & \chi_j, & \chi_k \end{vmatrix}.$$

The common value ∇ of the fractions is given by

$$\frac{1}{\nabla^2}=\sum A\left(\frac{p'}{\nabla}\right)^2=\sum A(2, 3, 4)^2:$$

or, as

$$\begin{vmatrix} b, & f, & m \\ f, & c, & n \\ m, & n, & d \end{vmatrix}=\Omega^2 A,$$

and so for the other combinations of minors in Ω, we have

$$\begin{aligned}\frac{\Omega^2}{\nabla^2} &= \sum \begin{vmatrix} b, & f, & m \\ f, & c, & n \\ m, & n, & d \end{vmatrix} (2, 3, 4)^2 \\ &= \begin{vmatrix} \sum a\theta_1^2, & \sum a\theta_1\phi_1, & \sum a\theta_1\chi_1 \\ \sum a\theta_1\phi_1, & \sum a\phi_1^2, & \sum a\phi_1\chi_1 \\ \sum a\theta_1\chi_1, & \sum a\phi_1\chi_1, & \sum a\chi_1^2 \end{vmatrix} \\ &= \Omega^3\theta_N^2\phi_n^2\chi_\nu^2 \begin{vmatrix} 1, & \cos\omega_3, & \cos\omega_2 \\ \cos\omega_3, & 1, & \cos\omega_1 \\ \cos\omega_2, & \cos\omega, & 1 \end{vmatrix} \\ &= \Omega^3\theta_N^2\phi_n^2\chi_\nu^2\Xi,\end{aligned}$$

where θ_N, ϕ_n, χ_ν, are the domainal dilatations of the regions θ, ϕ, χ, respectively; and where ω_1, ω_2, ω_3, are the angles between the ϕ-region and the χ-region (as measured by the angle between their domainal normals), between the χ-region and the θ-region, and between the θ-region and the ϕ-region, respectively. Thus

$$\frac{1}{\nabla}=\Omega^{\frac{1}{2}}\theta_N\phi_n\chi_\nu\Xi^{\frac{1}{2}}.$$

The direction-variables of the curve of intersection are thus expressible in terms of the first derivatives of the parametric functions.

It is convenient to have the direction-variables of the domainal normals to the regions. We denote by

$$\frac{dp}{dn_\psi},\quad \frac{dq}{dn_\psi},\quad \frac{dr}{dn_\psi},\quad \frac{dt}{dn_\psi},$$

the direction-variables of the domainal normals to a region $\psi(p, q, r, t)=0$. Thus we have

$$\Omega\theta_N\frac{dp}{dn_\theta}=a\theta_1+h\theta_2+g\theta_3+l\theta_4,$$

with corresponding values for the normal derivatives of q, r, t: also

$$\frac{\theta_1}{\theta_N}=A\frac{dp}{dn_\theta}+H\frac{dq}{dn_\theta}+G\frac{dr}{dn_\theta}+L\frac{dt}{dn_\theta},$$

with corresponding values for θ_2, θ_3, θ_4, while

$$\Omega\theta_N{}^2=\sum a\theta_1{}^2.$$

There are corresponding equations connected with the ϕ-region and with the χ-region for the similarly associated magnitudes of those regions.

388. We denote the second derivatives of p, q, r, t, along the curve by p_0'', q_0'', r_0'', t_0'', using p'', q'', r'', t'', to denote the second derivatives of the parameters along the domainal geodesic in the same initial direction p', q', r', t', that is, along the domainal geodesic which touches (and is touched by) the curve. As the curve lies wholly in the θ-region, we have

$$\theta_1p_0''+\theta_2q_0''+\theta_3r_0''+\theta_4t_0''+\sum\theta_{11}p'^2=0.$$

Consequently

$$\begin{aligned}\theta_1(p_0''-p'')+\theta_2(q_0''-q'')+\theta_3(r_0''-r'')+\theta_4(t_0''-t'')\\ =-\sum(\theta_{11}-\theta_1\Gamma_{11}-\theta_2\Delta_{11}-\theta_3\Theta_{11}-\theta_4\Phi_{11})p'^2\\ =-\sum\bar{\theta}_{11}p'^2,\end{aligned}$$

using the symbol $\bar{\theta}_{ij}$ to denote $\theta_{ij}-\theta_1\Gamma_{ij}-\theta_2\Delta_{ij}-\theta_3\Theta_{ij}-\theta_4\Phi_{ij}$. But, if $1/\gamma_\theta$ denotes the domainal flexure of the geodesic of the θ-region in the direction p', q', r', t', that is, of the geodesic in that region touching the curve, then

$$-\sum\bar{\theta}_{11}p'^2=\frac{\theta_N}{\gamma_\theta};$$

and therefore, if we take

$$P_0=p_0''-p'', \quad Q_0=q_0''-q'', \quad R_0=r_0''-r'', \quad T_0=t_0''-t'',$$

the foregoing relation becomes

$$\theta_1 P_0+\theta_2 Q_0+\theta_3 R_0+\theta_4 T_0=\frac{\theta_N}{\gamma_\theta}.$$

Similarly, with the corresponding significance of γ_ϕ for the ϕ-region and of γ_χ for the χ-region, we have

$$\phi_1 P_0+\phi_2 Q_0+\phi_3 R_0+\phi_4 T_0=\frac{\phi_n}{\gamma_\phi},$$

$$\chi_1 P_0+\chi_2 Q_0+\chi_3 R_0+\chi_4 T_0=\frac{\chi_\nu}{\gamma_\chi}.$$

But the arc lies in the domain, so that

$$\sum Ap'^2=1,$$

and therefore, differentiating along the arc, we have (as always)

$$u_1(p_0''+\sum \Gamma_{11}p'^2)+u_2(q_0''+\sum \Delta_{11}q'^2)+u_3(r_0''+\sum \Theta_{11}p'^2) \\ +u_4(t_0''+\sum \Phi_{11}p'^2)=0,$$

that is,

$$u_1 P_0+u_2 Q_0+u_3 R_0+u_4 T_0=0.$$

The four equations thus obtained, linear in P_0, Q_0, R_0, T_0, can be resolved as follows. The last equation, being

$$\sum Ap'P_0=0,$$

indicates that the quantities P_0, Q_0, R_0, T_0, can be regarded as direction-variables of a line in the domain, at right angles to the tangent to the curve. There are three non-complanar domainal directions at right angles to this tangent: for it lies in the tangent flat of each region and therefore is perpendicular to the three domainal normals of the respective regions; and consequently P_0, Q_0, R_0, T_0, determining a direction lying in the domain, must lie in the flat which has the three domainal normals for its guiding lines. Thus, with suitable quantities κ, λ, μ, we can take

$$P_0=\kappa\frac{dp}{dn_\theta}+\lambda\frac{dp}{dn_\phi}+\mu\frac{dp}{dn_\chi},$$

$$Q_0=\kappa\frac{dq}{dn_\theta}+\lambda\frac{dq}{dn_\phi}+\mu\frac{dq}{dn_\chi},$$

$$R_0=\kappa\frac{dr}{dn_\theta}+\lambda\frac{dr}{dn_\phi}+\mu\frac{dr}{dn_\chi},$$

$$T_0=\kappa\frac{dt}{dn_\theta}+\lambda\frac{dt}{dn_\phi}+\mu\frac{dt}{dn_\chi};$$

and the values of κ, λ, μ, now must satisfy the three earlier equations.

Let the values be substituted in the first equation. The coefficient of κ

$$=\sum \theta_1 \frac{dp}{dn_\theta}=\theta_N;$$

the coefficient of λ

$$=\sum \theta_1 \frac{dp}{dn_\phi}$$

$$=\theta_N \sum \left(A \frac{dp}{dn_\theta}\frac{dp}{dn_\phi}\right)=\theta_N \cos\omega_3;$$

the coefficient of μ

$$=\sum \theta_1 \frac{dp}{dn_\chi}$$

$$=\theta_N \sum \left(A \frac{dp}{dn_\theta}\frac{dp}{dn_\chi}\right)=\theta_N \cos\omega_2;$$

and therefore we have

$$\kappa+\lambda\cos\omega_3+\mu\cos\omega_2=\frac{1}{\gamma_\theta}.$$

Similarly, from the second equation, we have

$$\kappa\cos\omega_3+\lambda+\mu\cos\omega_1=\frac{1}{\gamma_\phi},$$

and from the third equation,

$$\kappa\cos\omega_2+\lambda\cos\omega_1+\mu=\frac{1}{\gamma_\chi}.$$

(When the values of P_0, Q_0, R_0, T_0, are substituted in the fourth equation, it is satisfied without leaving any residual condition: the relations

$$\sum u_1 \frac{dp}{dn_\theta}=0,\quad \sum u_1 \frac{dp}{dn_\phi}=0,\quad \sum u_1 \frac{dp}{dn_\chi}=0,$$

merely express the property that the domainal normals of the regions are at right angles to the curve, which lies in each of the regions.)

Thus the second arc-variations of p, q, r, t, taken along the arc of the curve, are given by the equations

$$\left.\begin{aligned}
p_0''-p''&=\kappa\frac{dp}{dn_\theta}+\lambda\frac{dp}{dn_\phi}+\mu\frac{dp}{dn_\chi}\\
q_0''-q''&=\kappa\frac{dq}{dn_\theta}+\lambda\frac{dq}{dn_\phi}+\mu\frac{dq}{dn_\chi}\\
r_0''-r''&=\kappa\frac{dr}{dn_\theta}+\lambda\frac{dr}{dn_\phi}+\mu\frac{dr}{dn_\chi}\\
t_0''-t''&=\kappa\frac{dt}{dn_\theta}+\lambda\frac{dt}{dn_\phi}+\mu\frac{dt}{dn_\chi}
\end{aligned}\right\},$$

where κ, λ, μ, are expressible in terms of the domainal flexure of the respective regional geodesic tangents of the curve by the equations

$$\left.\begin{aligned} \kappa+\lambda\cos\omega_3+\mu\cos\omega_2&=\frac{1}{\gamma_\theta} \\ \kappa\cos\omega_3+\lambda+\mu\cos\omega_1&=\frac{1}{\gamma_\phi} \\ \kappa\cos\omega_2+\lambda\cos\omega_1+\mu&=\frac{1}{\gamma_\chi} \end{aligned}\right\}.$$

Obviously, these equations are analogous to the intrinsic equations of a geodesic in a domainal region, and to the intrinsic equations of a geodesic on a domainal surface given as the intersection of two regions.

Circular curvature and domainal flexure of the curve.

389. The derivation, of the magnitude of the circular curvature of the curve and also of the spatial direction-cosines of its prime normal, is immediate. For the typical point-variable in the plenary space, we have

$$\begin{aligned} y_0''&=y_1p_0''+y_2q_0''+y_3r_0''+y_4t_0''+\sum y_{11}p'^2 \\ &=y_1(p_0''-p'')+y_2(q_0''-q'')+y_3(r_0''-r'')+y_4(t_0''-t'') \\ &\qquad\qquad+\sum\{(y_{11}-y_1\Gamma_{11}-y_2\Delta_{11}-y_3\Theta_{11}-y_4\Phi_{11})p'^2\} \\ &=y_1P_0+y_2Q_0+y_3R_0+y_4T_0+\sum\eta_{11}p'^2. \end{aligned}$$

Let Y_0 denote the typical spatial direction-cosine of the prime normal to the curve, and let ρ_0 denote its radius of circular curvature; then

$$Y_0=y_0''\rho_0.$$

Also, when the values of P_0, Q_0, R_0, T_0, are substituted, the coefficient of κ

$$\begin{aligned} &=y_1\frac{dp}{dn_\theta}+y_2\frac{dq}{dn_\theta}+y_3\frac{dr}{dn_\theta}+y_4\frac{dt}{dn_\theta} \\ &=\frac{dy}{dn_\theta}, \end{aligned}$$

being the typical spatial direction-cosine of the domainal normal to the θ-region; and similarly the coefficient of λ and the coefficient of μ respectively are

$$\frac{dy}{dn_\phi},\quad\frac{dy}{dn_\chi}.$$

Moreover,

$$\sum\eta_{11}p'^2=y''=\frac{Y}{\rho},$$

the customary equation connected with the circular curvature of the domainal geodesic in the direction p', q', r', t'; thus

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\kappa\frac{dy}{dn_\theta}+\lambda\frac{dy}{dn_\phi}+\mu\frac{dy}{dn_\chi},$$

holding for each of the spatial point-variables.

390. Next, an estimate must be made of the magnitude and the direction of the domainal flexure of the curve, being the arc-rate of its deviation from the domainal geodesic. This estimate can be framed in two ways.

In the first place, the deviation is estimated by the rate of variation between the direction of the curve and the direction of the domainal geodesic tangent. The typical direction-cosines of the tangents to the curve, at O and at a point distant δ from O along the arc of the curve (δ being small), are

$$y', \quad y'+y_0''\delta+\ldots,$$

while the corresponding quantities for the geodesic tangents, at O and at a point distant δ from O along its tangent, are

$$y', \quad y'+y''\delta+\ldots,$$

The two first tangents coincide; let $d\epsilon$ denote the small angle between the two second tangents; and let γ denote the radius of domainal flexure, so that, by definition, we take

$$\frac{1}{\gamma}=\operatorname*{Lim}_{\delta\to 0}\frac{d\epsilon}{\delta}.$$

In an orb of representation in the plenary homaloidal space, let l denote the typical spatial direction-cosine of the elementary arc $d\epsilon$, so that

$$\begin{aligned} l\,d\epsilon &= y_0''\delta+\ldots-(y''\delta+\ldots)\\ &=(y_0''-y'')\delta+\ldots, \end{aligned}$$

the unexpressed terms involving powers of δ higher than the first; thus

$$\frac{l}{\gamma}=y_0''-y''.$$

In the second place, the deviation of the curve from the domainal geodesic tangent is estimated by the deviation in distance between the two points, on the curve and on the geodesic respectively, at the small distance δ from O. The typical space-variables for these two points are

$$y+y'\delta+\tfrac{1}{2}y_0''\delta^2+\ldots, \quad y+y'\delta+\tfrac{1}{2}y''\delta^2+\ldots,$$

respectively; and therefore, if ∇ denote the actual distance between them, while l has its preceding significance,

$$l\nabla=\tfrac{1}{2}(y_0''-y'')\delta^2+\ldots,$$

so that

$$l \lim_{\delta \to 0} \left(\frac{2\nabla}{\delta^2}\right) = y_0'' - y''.$$

By this mode of estimating the deviation, the definition of the radius of flexure would be

$$\frac{1}{\gamma} = \lim_{\delta \to 0} \left(\frac{2\nabla}{\delta^2}\right);$$

and therefore we have

$$\frac{l}{\gamma} = y_0'' - y'',$$

the same result as before.

Now

$$y_0'' = \frac{Y_0}{\rho_0}, \quad y'' = \frac{Y}{\rho};$$

and therefore we have

$$\frac{l}{\gamma} = \frac{Y_0}{\rho_0} - \frac{Y}{\rho},$$

as the relation between the domainal flexure of the curve, and the circular curvatures of the curve and the domainal geodesic tangent. Also we have, by the earlier relation,

$$\frac{l}{\gamma} = \kappa \frac{dy}{dn_\theta} + \lambda \frac{dy}{dn_\phi} + \mu \frac{dy}{dn_\chi},$$

where the quantities κ, λ, μ, are expressed in terms of the domainal flexures of the regional geodesic tangents by

$$\left.\begin{aligned} \kappa + \lambda \cos \omega_3 + \mu \cos \omega_2 &= \frac{1}{\gamma_\theta} \\ \kappa \cos \omega_3 + \lambda \quad\quad\quad + \mu \cos \omega_1 &= \frac{1}{\gamma_\phi} \\ \kappa \cos \omega_2 + \lambda \cos \omega_1 + \mu \quad\quad\quad &= \frac{1}{\gamma_\chi} \end{aligned}\right\}.$$

Consequently, the typical direction-cosine l of the radius of domainal flexure of the curve is given by the equation

$$\begin{vmatrix} \dfrac{l}{\gamma}, & \dfrac{dy}{dn_\theta}, & \dfrac{dy}{dn_\phi}, & \dfrac{dy}{dn_\chi} \\ \dfrac{1}{\gamma_\theta}, & 1, & \cos \omega_3, & \cos \omega_2 \\ \dfrac{1}{\gamma_\phi}, & \cos \omega_3, & 1, & \cos \omega_1 \\ \dfrac{1}{\gamma_\chi}, & \cos \omega_2, & \cos \omega_1, & 1 \end{vmatrix} = 0.$$

The magnitude γ of the radius of domainal flexure can be regarded as given by any one of the equations

$$\frac{1}{\gamma^2}=\kappa^2+\lambda^2+\mu^2+2\lambda\mu\cos\omega_1+2\mu\kappa\cos\omega_2+2\kappa\lambda\cos\omega_3,$$

$$\frac{1}{\gamma^2}=\frac{\kappa}{\gamma_\theta}+\frac{\lambda}{\gamma_\phi}+\frac{\mu}{\gamma_\chi},$$

$$\begin{vmatrix} \dfrac{1}{\gamma^2}, & \dfrac{1}{\gamma_\theta}, & \dfrac{1}{\gamma_\phi}, & \dfrac{1}{\gamma_\chi} \\ \dfrac{1}{\gamma_\theta}, & 1, & \cos\omega_3, & \cos\omega_2 \\ \dfrac{1}{\gamma_\phi}, & \cos\omega_3, & 1, & \cos\omega_1 \\ \dfrac{1}{\gamma_\chi}, & \cos\omega_2, & \cos\omega_1, & 1 \end{vmatrix}=0.$$

$$\begin{aligned}\frac{\Xi}{\gamma^2}= & \frac{\sin^2\omega_1}{\gamma_\theta^{\,2}}+\frac{2}{\gamma_\phi\gamma_\chi}(\cos\omega_2\cos\omega_3-\cos\omega_1)\\ & +\frac{\sin^2\omega_2}{\gamma_\phi^{\,2}}+\frac{2}{\gamma_\chi\gamma_\theta}(\cos\omega_3\cos\omega_1-\cos\omega_2)\\ & +\frac{\sin^2\omega_3}{\gamma_\chi^{\,3}}+\frac{2}{\gamma_\theta\gamma_\phi}(\cos\omega_1\cos\omega_2-\cos\omega_3),\end{aligned}$$

where

$$\Xi=1-\cos^2\omega_1-\cos^2\omega_2-\cos^2\omega_3+2\cos\omega_1\cos\omega_2\cos\omega_3.$$

Perpendicular on tangent to the curve and the prime normal.

391. The identification of the direction of the prime normal of the curve with the limiting position of the perpendicular on the tangent to the curve from a neighbouring point on the curve, as that point is made to approach O, and the derivation of the foregoing typical relation between directions and curvatures, can be effected as follows.

The tangent line of the curve lies in the tangent block of the domain; and thus its equations can be taken in the form

$$\bar{y}-y=\bar{\lambda}y_1+\bar{\mu}y_2+\bar{\nu}y_3+\bar{\varpi}y_4,$$

typical of the set for all the point-coordinates in the plenary space, the parameters $\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$, $\bar{\varpi}$, being subject to the three conditions

$$\bar{\lambda}\theta_1+\bar{\mu}\theta_2+\bar{\nu}\theta_3+\bar{\varpi}\theta_4=0,$$

$$\bar{\lambda}\phi_1+\bar{\mu}\phi_2+\bar{\nu}\phi_3+\bar{\varpi}\phi_4=0,$$

$$\bar{\lambda}\chi_1+\bar{\mu}\chi_2+\bar{\nu}\chi_3+\bar{\varpi}\chi_4=0.$$

Let Π_0 be the length of the perpendicular, from a neighbouring point with a typical coordinate η, upon the tangent line to the curve; and let the typical direction-cosine of that perpendicular be $\overline{Y}_0$, so that, if the foregoing point, with the typical coordinate $\bar{y}$, be the foot of the perpendicular, we have

$$\begin{aligned}\overline{Y}_0\Pi_0 &= \eta - \bar{y}\\ &= \eta - (y + \bar{\lambda}y_1 + \bar{\mu}y_2 + \bar{\nu}y_3 + \bar{\varpi}y_4),\end{aligned}$$

as the typical equation. Because Π_0 is the perpendicular in question, the quantity

$$\sum \{\eta - (y + \bar{\lambda}y_1 + \bar{\mu}y_2 + \bar{\nu}y_3 + \bar{\varpi}y_4)\}^2$$

must be a minimum for all admissible values of $\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$, $\bar{\varpi}$, that is, for all values of $\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$, $\bar{\varpi}$, satisfying the three conditions. The four equations, critical for this minimum, are

$$\sum y_i(\eta - \bar{y}) = P\theta_i + Q\phi_i + R\psi_i,$$

for $i = 1, 2, 3, 4$, where P, Q, R, are multipliers undetermined in the construction of the critical equations.

These equations can be written in the form

$$\Pi_0 \sum y_i \overline{Y}_0 = P\theta_i + Q\phi_i + R\chi_i;$$

and it is easy to verify that the direction typified by $\overline{Y}_0$ lies in the flat which, within the domain, is orthogonal to the curve.

They also can be taken in the form

$$\begin{aligned}
A\bar{\lambda} + H\bar{\mu} + G\bar{\nu} + L\bar{\varpi} + P\theta_1 + Q\phi_1 + R\chi_1 &= \sum y_1(\eta - y),\\
H\bar{\lambda} + B\bar{\mu} + F\bar{\nu} + M\bar{\varpi} + P\theta_2 + Q\phi_2 + R\chi_2 &= \sum y_2(\eta - y),\\
G\bar{\lambda} + F\bar{\mu} + C\bar{\nu} + N\bar{\varpi} + P\theta_3 + Q\phi_3 + R\chi_3 &= \sum y_3(\eta - y),\\
L\bar{\lambda} + M\bar{\mu} + N\bar{\nu} + D\bar{\varpi} + P\theta_4 + Q\phi_4 + R\chi_4 &= \sum y_4(\eta - y).
\end{aligned}$$

Now

$$\sum y_1(\eta - y) = \sum y_1(y'\delta + \tfrac{1}{2}y_0''\delta^2),$$

accurately up to the second power of δ inclusive. Also

$$\sum y_1 y' = Ap' + Hq' + Gr' + Lt';$$

and, as

$$y_0'' = y_1 p_0'' + y_2 q_0'' + y_3 r_0'' + y_4 t_0'' + \sum y_{11} p'^2,$$

we have

$$\begin{aligned}
\sum y_1 y_0'' &= Ap_0'' + Hq_0'' + Gr_0'' + Lt_0'' + \sum [\sum y_1 y_{11}] p'^2\\
&= A(p_0'' + \sum \Gamma_{11} p'^2) + H(q_0'' + \sum \Delta_{11} p'^2)\\
&\quad + G(r_0'' + \sum \Theta_{11} p'^2) + L(t_0'' + \sum \Phi_{11} p'^2).
\end{aligned}$$

Hence

$$\begin{aligned}\sum y_1(\eta-y)=A\{p'\delta+\tfrac{1}{2}(p_0''+\sum\Gamma_{11}p'^2)\delta^2\}\\ +H\{q'\delta+\tfrac{1}{2}(q_0''+\sum\Delta_{11}p'^2)\delta^2\}\\ +G\{r'\delta+\tfrac{1}{2}(r_0''+\sum\Theta_{11}p'^2)\delta^2\}\\ +L\{t'\delta+\tfrac{1}{2}(t_0''+\sum\Phi_{11}p'^2)\delta^2\}.\end{aligned}$$

Similarly for the other quantities $\sum y_i(\eta-y)$, for $i=2, 3, 4$. Let

$$\begin{aligned}\lambda_0&=\bar\lambda-p'\delta-\tfrac{1}{2}(p_0''+\sum\Gamma_{11}p'^2)\delta^2=\bar\lambda-p'\delta-\tfrac{1}{2}P_0\delta^2,\\ \mu_0&=\bar\mu-q'\delta-\tfrac{1}{2}(q_0''+\sum\Delta_{11}p'^2)\delta^2=\bar\mu-q'\delta-\tfrac{1}{2}Q_0\delta^2,\\ \nu_0&=\bar\nu-r'\delta-\tfrac{1}{2}(r_0''+\sum\Theta_{11}p'^2)\delta^2=\bar\nu-r'\delta-\tfrac{1}{2}R_0\delta^2,\\ \varpi_0&=\bar\varpi-t'\delta-\tfrac{1}{2}(t_0''+\sum\Phi_{11}p'^2)\delta^2=\bar\varpi-t'\delta-\tfrac{1}{2}T_0\delta^2;\end{aligned}$$

then the equations are

$$\begin{aligned}A\lambda_0+H\mu_0+G\nu_0+L\omega_0&=-(P\theta_1+Q\phi_1+R\chi_1),\\ H\lambda_0+B\mu_0+F\nu_0+M\omega_0&=-(P\theta_2+Q\phi_2+R\chi_2),\\ G\lambda_0+F\mu_0+C\nu_0+N\omega_0&=-(P\theta_3+Q\phi_3+R\chi_3),\\ L\lambda_0+M\mu_0+N\nu_0+D\omega_0&=-(P\theta_4+Q\phi_4+R\chi_4).\end{aligned}$$

Consequently

$$\begin{aligned}\Omega\lambda_0&=-P(\sum a\theta_1)-Q(\sum a\phi_1)-R(\sum a\chi_1),\\ \Omega\mu_0&=-P(\sum h\theta_1)-Q(\sum h\phi_1)-R(\sum h\chi_1),\\ \Omega\nu_0&=-P(\sum g\theta_1)-Q(\sum g\phi_1)-R(\sum g\chi_1),\\ \Omega\varpi_0&=-P(\sum l\theta_1)-Q(\sum l\phi_1)-R(\sum l\chi_1).\end{aligned}$$

Thus there are two sets of expressions for the values of the quantities $\lambda_0, \mu_0, \nu_0, \varpi_0$.

From the former set, we have

$$\begin{aligned}\sum\theta_1\lambda_0&=\theta_1\lambda_0+\theta_2\mu_0+\theta_3\nu_0+\theta_4\varpi_0\\ &=\sum\theta_1\bar\lambda-(\sum\theta_1p')\delta-\tfrac{1}{2}\sum\{\theta_1(p_0''+\sum\Gamma_{11}p'^2)\}\delta^2\\ &=-\tfrac{1}{2}\sum\{\theta_1(p_0''+\sum\Gamma_{11}p'^2)\}\delta^2.\end{aligned}$$

But second differentiation of the equation $\theta(p, q, r, t)=0$ yields the relation

$$\sum\theta_1p_0''+\sum\theta_{11}p'^2=0;$$

and therefore

$$\begin{aligned}\sum\theta_1\lambda_0&=\tfrac{1}{2}\sum\{(\theta_{11}-\theta_1\Gamma_{11}-\theta_2\Delta_{11}-\theta_3\Theta_{11}-\theta_4\Phi_{11})p'^2\}\delta^2\\ &=\tfrac{1}{2}\delta^2\sum\bar\theta_{11}\delta^2\\ &=-\tfrac{1}{2}\frac{\theta_N}{\gamma_\theta}\delta^2;\end{aligned}$$

and similarly, from the same first set,

$$\sum \phi_1 \lambda_0 = -\tfrac{1}{2}\frac{\phi_n}{\gamma_\phi}\delta^2, \quad \sum \chi_1 \lambda_0 = -\tfrac{1}{2}\frac{\chi_\nu}{\gamma_\chi}\delta^2.$$

From the second set of expressions for λ_0, μ_0, ν_0, ϖ_0, we have

$$\begin{aligned} \Omega \sum \theta_1 \lambda_0 &= -P \sum a\theta_1^2 - Q \sum a\theta_1\phi_1 - R \sum a\theta_1\chi_1 \\ &= -\Omega(P\theta_N^2 + Q\theta_N\phi_n \cos\omega_3 + R\theta_N\chi_\nu \cos\omega_2), \end{aligned}$$

that is, equating the two values of $\sum \theta_1 \lambda_0$,

$$\tfrac{1}{2}\delta^2 \frac{1}{\gamma_\theta} = P\theta_N + Q\phi_n \cos\omega_3 + R\chi_\nu \cos\omega_2.$$

Similarly we find

$$\tfrac{1}{2}\delta^2 \frac{1}{\gamma_\phi} = Q\phi_n + R\chi_\nu \cos\omega_1 + P\theta_N \cos\omega_3,$$

$$\tfrac{1}{2}\delta^2 \frac{1}{\gamma_\chi} = R\chi_\nu + P\theta_N \cos\omega_2 + Q\phi_n \cos\omega_1.$$

It follows, from the equations which define κ, λ, μ, in terms of γ_θ, γ_ϕ, γ_χ, that

$$P\theta_N = \tfrac{1}{2}\delta^2\kappa, \quad Q\phi_n = \tfrac{1}{2}\delta^2\lambda, \quad R\chi_\nu = \tfrac{1}{2}\delta^2\mu;$$

and therefore, inserting these values of P, Q, R, in the expressions for λ_0, μ_0, ν_0, ϖ_0, we have

$$\begin{aligned} \lambda_0 &= -\tfrac{1}{2}\delta^2 \left\{ \kappa \frac{\sum a\theta_1}{\Omega\theta_N} + \lambda \frac{\sum a\phi_1}{\Omega\phi_n} + \mu \frac{\sum a\chi_1}{\Omega\chi_\nu} \right\} \\ &= -\tfrac{1}{2}\delta^2 \left(\kappa \frac{dp}{dn_\theta} + \lambda \frac{dp}{dn_\phi} + \mu \frac{dp}{dn_\chi} \right), \\ \mu_0 &= -\tfrac{1}{2}\delta^2 \left(\kappa \frac{dq}{dn_\theta} + \lambda \frac{dq}{dn_\phi} + \mu \frac{dq}{dn_\chi} \right), \\ \nu_0 &= -\tfrac{1}{2}\delta^2 \left(\kappa \frac{dr}{dn_\theta} + \lambda \frac{dr}{dn_\phi} + \mu \frac{dr}{dn_\chi} \right), \\ \varpi_0 &= -\tfrac{1}{2}\delta^2 \left(\kappa \frac{dt}{dn_\theta} + \lambda \frac{dt}{dn_\phi} + \mu \frac{dt}{dn_\chi} \right). \end{aligned}$$

Consequently

$$\begin{aligned} & y_1\bar{\lambda} + y_2\bar{\mu} + y_3\bar{\nu} + y_4\bar{\varpi} \\ &\quad = (\textstyle\sum y_1 p')\delta + \tfrac{1}{2}(\sum y_1 P_0)\delta^2 + \sum y_1\lambda_0 \\ &\quad = y'\delta + \tfrac{1}{2}(\textstyle\sum y_1 P_0)\delta^2 - \tfrac{1}{2}\left(\kappa \dfrac{dy}{dn_\theta} + \lambda \dfrac{dy}{dn_\phi} + \mu \dfrac{dy}{dn_\chi} \right)\delta^2. \end{aligned}$$

The typical equation for the length and the spatial direction-cosines of the perpendicular Π_0 is

$$\bar{Y}_0 \Pi_0 = \eta - (y + y_1\bar{\lambda} + y_2\bar{\mu} + y_3\bar{\nu} + y_4\bar{\varpi}).$$

But, along the curve

$$\begin{aligned}\eta - y &= y'\delta + \tfrac{1}{2}y_0''\delta^2 + \dots \\ &= y'\delta + \tfrac{1}{2}(y_0'' - y'')\delta^2 + \tfrac{1}{2}y''\delta^2 + \dots \\ &= y'\delta + \tfrac{1}{2}(\sum y_1 P_0)\delta^2 + \tfrac{1}{2}\frac{Y}{\rho}\delta^2 + \dots,\end{aligned}$$

the unexpressed terms containing powers of δ higher than the second; and therefore, up to the second power of δ inclusive,

$$\overline{Y}_0 \Pi_0 = \tfrac{1}{2}\frac{Y}{\rho}\delta^2 + \tfrac{1}{2}\left(\kappa\frac{dy}{dn_\theta} + \lambda\frac{dy}{dn_\phi} + \mu\frac{dy}{dn_\chi}\right)\delta^2.$$

The radius of circular curvature of the curve, denoted by ρ_0, is such that

$$\frac{1}{\rho_0} = \operatorname*{Lim}_{\delta \to 0}\left(\frac{2\Pi_0}{\delta^2}\right);$$

and therefore

$$\frac{\overline{Y}_0}{\rho_0} = \frac{Y}{\rho} + \kappa\frac{dy}{dn_\theta} + \lambda\frac{dy}{dn_\phi} + \mu\frac{dy}{dn_\chi}.$$

As this relation is typical of the whole set, we verify the typical relation

$$\overline{Y}_0 = Y_0;$$

and thus the two methods of determining the magnitude of the radius of curvature of the curve and its direction-cosines lead to the same result.

The quantities γ_θ, γ_ϕ, γ_χ, which occur implicitly through the quantities κ, λ, μ, in these equations, are the domainal flexures of the geodesics in the θ-region, the ϕ-region, and the χ-region, respectively, all in the direction p', q', r', t', touching the curve. The circular curvatures of these geodesics, and their respective sets of spatial direction-cosines, are given by the typical equations

$$\left.\begin{aligned}\frac{Y_\theta}{\rho_\theta} &= \frac{Y}{\rho} + \frac{1}{\gamma_\theta}\frac{dy}{dn_\theta} \\ \frac{Y_\phi}{\rho_\phi} &= \frac{Y}{\rho} + \frac{1}{\gamma_\phi}\frac{dy}{dn_\phi} \\ \frac{Y_\chi}{\rho_\chi} &= \frac{Y}{\rho} + \frac{1}{\gamma_\chi}\frac{dy}{dn_\chi}\end{aligned}\right\},$$

while

$$\begin{aligned}\frac{Y_0}{\rho_0} &= \frac{Y}{\rho} + \frac{l}{\gamma} \\ &= \frac{Y}{\rho} + \kappa\frac{dy}{dn_\theta} + \lambda\frac{dy}{dn_\phi} + \mu\frac{dy}{dn_\chi}.\end{aligned}$$

392. A geometrical construction can be made, indicating some of the properties of a rectilinear frame connected with these equations.

Let Ag_1, Ag_2, Ag_3, represent (in magnitude and direction) the radius of domainal flexure of the geodesics in the θ-region, the ϕ-region, and the χ-region, respectively, drawn in the same direction p', q', r', t', as the curve which is the intersection of those regions. Then from the equations

$$\frac{l}{\gamma}=\kappa\frac{dy}{dn_\theta}+\lambda\frac{dy}{dn_\phi}+\mu\frac{dy}{dn_\chi},$$

we have

$$\frac{1}{\gamma}\sum l\frac{dy}{dn_\theta}=\kappa+\lambda\cos\omega_3+\mu\cos\omega_2=\frac{1}{\gamma_\theta},$$

and similarly

$$\frac{1}{\gamma}\sum l\frac{dy}{dn_\phi}=\frac{1}{\gamma_\phi},\quad \frac{1}{\gamma}\sum l\frac{dy}{dn_\chi}=\frac{1}{\gamma_\chi}.$$

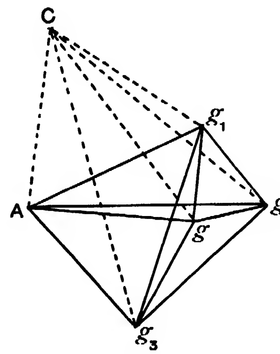

FIG. 37.

Let a perpendicular Ag be drawn in the flat $g_1g_2g_3A$ to the triangle $g_1g_2g_3$, meeting it in g; then Ag, in magnitude and direction, represents the radius of domainal flexure of the curve of intersection of the regions.

Again, let AC, in magnitude and direction, represent the radius of circular curvature of the domainal geodesic which touches this curve of intersection. Then the radius of circular curvature of the geodesic in the θ-region touching the curve is represented (in magnitude and direction) by the perpendicular Ac_1 drawn in the plane ACg_1 upon the line Cg_1; the radius of circular curvature of the tangential geodesic in the ϕ-region is similarly represented by the perpendicular Ac_2 drawn in the plane ACg_2 upon the line Cg_2; and the radius of circular curvature of the tangential geodesic in the χ-region is similarly represented by the perpendicular Ac_3 drawn in the plane ACg_3 upon the line Cg_3.

The radius of circular curvature of the curve of intersection of the three regions is represented, in magnitude and in direction, by the perpendicular Ac drawn in the plane ACg upon the line Cg.

Further, there are three domainal surfaces of intersection, of the three regions taken in pairs; and, in each of these surfaces, there is a superficial geodesic touching the curve of intersection, the domainal flexure and the circular curvature of which can similarly be represented in the diagram. Thus for the surface which is the intersection of the ϕ-region and the χ-region, the radius of domainal flexure of the geodesic tangential to the curve is represented by a perpendicular AG_1 drawn to the line g_2g_3, the line gG_1 being at right angles to g_2g_3; and the radius of circular curvature of that geodesic is represented by the perpendicular drawn from A upon the line CG_1.

Other inferences can be derived from the four-dimensional frame $CAg_1g_2g_3$ thus constructed, the tangent of the curve of intersection being orthogonal to the block of the frame. With the frame, a (four-dimensional) globe on AC as diameter can be constructed; all the foregoing centres of domainal flexure and centres of circular curvature lie in this globular range.

The analytical values of the circular curvature and the domainal flexure of the curve follow at once from the full equations. Thus (p. 620) the typical equations for the flexure are

$$\frac{l}{\gamma}=\kappa\frac{dy}{dn_\theta}+\lambda\frac{dy}{dn_\phi}+\mu\frac{dy}{dn_\chi},$$

where

$$\frac{1}{\gamma_\theta}=\kappa+\lambda\cos\omega_3+\mu\cos\omega_2,$$

$$\frac{1}{\gamma_\phi}=\kappa\cos\omega_3+\lambda+\mu\cos\omega_1,$$

$$\frac{1}{\gamma_\chi}=\kappa\cos\omega_2+\lambda\cos\omega_1+\mu,$$

and, for the circular curvature,

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma}.$$

As noted in the geometrical construction, the radius of domainal flexure, the radius of circular curvature of the domainal geodesic tangent, and the radius of circular curvature of the curve, lie in one plane. If ψ denote the inclination of these two radii of circular curvature, we have

$$\sum YY_0=\cos\psi,\quad \sum lY_0=\sin\psi,\quad Y_0=Y\cos\psi+l\sin\psi,$$

$$\rho\cos\psi=\gamma\sin\psi=\rho_0,$$

$$\frac{1}{{\rho_0}^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2}.$$

Further, we have

$$\frac{1}{\gamma^2}=\kappa^2+\lambda^2+\mu^2+2\lambda\mu\cos\omega_1+2\mu\kappa\cos\omega_2+2\kappa\lambda\cos\omega_3,$$

and therefore

$$\begin{aligned}\frac{\Xi}{\gamma^2}=&\frac{\sin^2\omega_1}{{\gamma_\theta}^2}+\frac{2}{\gamma_\phi\gamma_\chi}(\cos\omega_2\cos\omega_3-\cos\omega_1)\\ &+\frac{\sin^2\omega_2}{{\gamma_\phi}^2}+\frac{2}{\gamma_\chi\gamma_\theta}(\cos\omega_3\cos\omega_1-\cos\omega_2)\\ &+\frac{\sin^2\omega_3}{{\gamma_\chi}^2}+\frac{2}{\gamma_\theta\gamma_\phi}(\cos\omega_1\cos\omega_2-\cos\omega_3),\end{aligned}$$

where, as usual,

$$\Xi = 1 - \cos^2 \omega_1 - \cos^2 \omega_2 - \cos^2 \omega_3 + 2 \cos \omega_1 \cos \omega_2 \cos \omega_3.$$

Diverse forms can be given to the expression for $1/\gamma$, some of them being the equivalent of the properties of the tetrahedron $Ag_1g_2g_3$ in the diagram on p. 625, with Ag as the perpendicular from A on the face $g_1g_2g_3$: or (what is the equivalent) of a spherical triangle on a sphere with A as centre. And, in this same association, the domainal flexure of the curve can be brought into corresponding analytical relations with the domainal flexures of geodesics, tangential to the curve and drawn upon surfaces which are the pair-intersections of the three regions.

Ex. Shew that, if quantities $\bar{\kappa}$, $\bar{\lambda}$, $\bar{\mu}$, are defined by the equations

$$\begin{aligned}
-\frac{1}{\theta_N}\bar{\theta}_{ij} &= \bar{\kappa}_{ij} + \bar{\lambda}_{ij}\cos\omega_3 + \bar{\mu}_{ij}\cos\omega_2,\\
-\frac{1}{\phi_n}\bar{\phi}_{ij} &= \bar{\kappa}_{ij}\cos\omega_3 + \bar{\lambda}_{ij} + \bar{\mu}_{ij}\cos\omega_1,\\
-\frac{1}{\chi_\nu}\bar{\chi}_{ij} &= \bar{\kappa}_{ij}\cos\omega_2 + \bar{\lambda}_{ij}\cos\omega_1 + \bar{\mu}_{ij},
\end{aligned}$$

for all values of i and j, then

$$y_0'' = \frac{Y_0}{\rho_0} = \sum_i \sum_j \left(\eta_{ij} - \bar{\kappa}_{ij}\frac{dy}{dn_\theta} - \bar{\lambda}_{ij}\frac{dy}{dn_\phi} - \bar{\mu}_{ij}\frac{dy}{dn_\chi}\right) x_i' x_j',$$

with the usual conventions $x_1, x_2, x_3, x_4, = p, q, r, t$.

Binormal of the curve: the torsion.

393. In estimating the further curvatures of the curve, it proves convenient, in the tangent block of the domain, to change the leading lines, from the set of directions of the parametric curves at a point, to the set of directions connected with the curve and constituted by its tangent and the three domainal normals to the three regions the intersection of which is the curve. Thus there will be relations of the form

$$y_i = E_i y' + P_i \frac{dy}{dn_\theta} + Q_i \frac{dy}{dn_\phi} + R_i \frac{dy}{dn_\chi},$$

for the values $i = 1, 2, 3, 4$. To determine the coefficients E_i, P_i, Q_i, R_i, there are the relations

$$\begin{aligned}
u_i &= \sum y_i y' = E_i,\\
\frac{\theta_i}{\theta_N} &= \sum y_i \frac{dy}{dn_\theta} = P_i + Q_i\cos\omega_3 + R_i\cos\omega_2,\\
\frac{\phi_i}{\phi_n} &= \sum y_i \frac{dy}{dn_\phi} = P_i\cos\omega_3 + Q_i + R_i\cos\omega_1,\\
\frac{\chi_i}{\chi_\nu} &= \sum y_i \frac{dy}{dn_\chi} = P_i\cos\omega_2 + Q_i\cos\omega_1 + R_i;
\end{aligned}$$

and thus any expression, homogeneous and linear in the four quantities y_1, y_2, y_3, y_4, can be transformed into another expression, with modified coefficients, which is homogeneous and linear in the four quantities

$$y', \quad \frac{dy}{dn_\theta}, \quad \frac{dy}{dn_\phi}, \quad \frac{dy}{dn_\chi}.$$

A symbol $[L]$ will be used, generically, to denote an expression which, so far as concerns directional elements, involves only the members of this modified set of leading lines for the tangent block and which is homogeneous and linear in the four members. At a later stage, when the final expression has to become specific, the four coefficients in the final quantity $[L]$ are made determinate.

Some preliminary results are required for the evaluation of the typical direction-cosine of the binormal of the curve.

In the first place, because

$$\frac{Y}{\rho} = \sum \eta_{11} p'^2,$$

we have

$$\frac{d}{ds_0}\left(\frac{Y}{\rho}\right) - \frac{d}{ds}\left(\frac{Y}{\rho}\right) = 2\{\eta_1(p_0'' - p'') + \eta_2(q_0'' - q'') + \eta_3(r_0'' - r'') + \eta_4(t_0'' - t'')\}$$
$$= 2(\eta_1 P_0 + \eta_2 Q_0 + \eta_3 R_0 + \eta_4 T_0),$$

where P_0, Q_0, R_0, T_0, have the values obtained in § 388. Also

$$\frac{d}{ds}\left(\frac{Y}{\rho}\right) = Y\frac{d}{ds}\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\left(\frac{\lambda_3}{\sigma} - \frac{y'}{\rho}\right)$$
$$= Y\frac{d}{ds}\left(\frac{1}{\rho}\right) + [L],$$

because λ_3 and y' are linear in y_1, y_2, y_3, y_4; and therefore

$$\frac{d}{ds_0}\left(\frac{Y}{\rho}\right) = 2(\eta_1 P_0 + \eta_2 Q_0 + \eta_3 R_0 + \eta_4 T_0) + Y\frac{d}{ds}\left(\frac{1}{\rho}\right) + [L].$$

Next, the radius of domainal flexure, in direction and magnitude, is given by the typical equation (§ 390)

$$\frac{l}{\gamma} = \kappa\frac{dy}{dn_\theta} + \lambda\frac{dy}{dn_\phi} + \mu\frac{dy}{dn_\chi};$$

and therefore

$$\frac{d}{ds_0}\left(\frac{l}{\gamma}\right) = \kappa\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right) + \lambda\frac{d}{ds_0}\left(\frac{dy}{dn_\phi}\right) + \mu\frac{d}{ds_0}\left(\frac{dy}{dn_\chi}\right) + [L],$$

again introducing a generic set of terms $[L]$. But the arc-derivatives of $\frac{dy}{dn_\theta}$, $\frac{dy}{dn_\phi}$, $\frac{dy}{dn_\chi}$, which do not involve the direction-variables of the tangent of

the curve, are the same along the curve as along the domainal geodesic tangent and therefore, by the result in § 324,

$$\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right)+\frac{\theta_N'}{\theta_N}\frac{dy}{dn_\theta}=\eta_1\frac{dp}{dn_\theta}+\eta_2\frac{dq}{dn_\theta}+\eta_3\frac{dr}{dn_\theta}+\eta_4\frac{dt}{dn_\theta}$$
$$+\frac{1}{\Omega\theta_N}(a\between\bar{\theta}_1,\bar{\theta}_2,\bar{\theta}_3,\bar{\theta}_4\between y_1,y_2,y_3,y_4),$$

that is,

$$\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right)=\eta_1\frac{dp}{dn_\theta}+\eta_2\frac{dq}{dn_\theta}+\eta_3\frac{dr}{dn_\theta}+\eta_4\frac{dt}{dn_\theta}+[L].$$

Similarly for the other two corresponding arc-derivatives of direction-cosines of domainal normals to the regions. Hence

$$\begin{aligned}\frac{d}{ds_0}\left(\frac{l}{\gamma}\right)-[L]= &\ \eta_1\left(\kappa\frac{dp}{dn_\theta}+\lambda\frac{dp}{dn_\phi}+\mu\frac{dp}{dn_\chi}\right)\\ &+\eta_2\left(\kappa\frac{dq}{dn_\theta}+\lambda\frac{dq}{dn_\phi}+\mu\frac{dq}{dn_\chi}\right)\\ &+\eta_3\left(\kappa\frac{dr}{dn_\theta}+\lambda\frac{dr}{dn_\phi}+\mu\frac{dr}{dn_\chi}\right)\\ &+\eta_4\left(\kappa\frac{dt}{dn_\theta}+\lambda\frac{dt}{dn_\phi}+\mu\frac{dt}{dn_\chi}\right)\\ = &\ \eta_1P_0+\eta_2Q_0+\eta_3R_0+\eta_4T_0.\end{aligned}$$

To obtain the direction of the binormal and the magnitude of the torsion of the curve, we proceed from the equation

$$\frac{Y_0}{\rho_0}=\frac{Y}{\rho}+\frac{l}{\gamma}.$$

Differentiating along the curve, we have

$$\begin{aligned}\frac{d}{ds_0}\left(\frac{Y_0}{\rho_0}\right)&=Y_0\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right)+\frac{1}{\rho_0}\frac{dY_0}{ds_0}\\ &=\left(\frac{\rho_0}{\rho}Y+\frac{\rho_0}{\gamma}l\right)\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right)+\frac{1}{\rho_0}\left(\frac{\lambda_3}{\sigma_0}-\frac{y'}{\rho_0}\right)\\ &=\frac{\lambda_3}{\rho_0\sigma_0}+Y\left\{\frac{\rho_0}{\rho}\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right)\right\}+[L],\end{aligned}$$

once more introducing the symbol $[L]$ as denoting the aggregate of terms arising through l and y'. When the foregoing arc-derivatives of Y/ρ and l/γ are substituted, the differentiated relation acquires the form

$$\frac{\lambda_3}{\rho_0\sigma_0}=Y\left\{\frac{d}{ds}\left(\frac{1}{\rho}\right)-\frac{\rho_0}{\rho}\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right)\right\}+3(\eta_1P_0+\eta_2Q_0+\eta_3R_0+\eta_4T_0)+[L].$$

But (§ 297)

$$\eta_i = Yv_i + l_6\xi_i,$$

for $i=1, 2, 3, 4$, the values of $\xi_1, \xi_2, \xi_3, \xi_4$, being obtained in § 299; and therefore

$$\frac{\lambda_3}{\rho_0\sigma_0} = l_6 R + YS + [L],$$

where

$$R = 3(\xi_1 P_0 + \xi_2 Q_0 + \xi_3 R_0 + \xi_4 T_0),$$

$$S = \frac{d}{ds}\left(\frac{1}{\rho}\right) - \frac{\rho_0}{\rho}\frac{d}{ds_0}\left(\frac{1}{\rho_0}\right) + 3(v_1 P_0 + v_2 Q_0 + v_3 R_0 + v_4 S_0).$$

Ex. 1. Shew that

$$\tfrac{1}{2}\Omega^{\frac{1}{2}}\theta_N\phi_n\chi_\nu\Xi^{\frac{1}{2}}\left\{\frac{d}{ds_0}\left(\frac{1}{\rho}\right) - \frac{d}{ds}\left(\frac{1}{\rho}\right)\right\} = \begin{vmatrix} \frac{\theta_N}{\gamma_\theta}, & \theta_1, & \theta_2, & \theta_3, & \theta_4 \\ \frac{\phi_n}{\gamma_\phi}, & \phi_1, & \phi_2, & \phi_3, & \phi_4 \\ \frac{\chi_\nu}{\gamma_\chi}, & \chi_1, & \chi_2, & \chi_3, & \chi_4 \\ 0, & v_1, & v_2, & v_3, & v_4 \\ 0, & u_1, & u_2, & u_3, & u_4 \end{vmatrix}.$$

Ex. 2. Establish the relation

$$\begin{aligned}\frac{1}{\rho^2}\left(\frac{dY}{ds_0} - \frac{dY}{ds}\right) = 2l_6[\ & m_{23}(q'R_0 - r'Q_0) + m_{14}(p'T_0 - t'P_0) \\ & + m_{13}(p'R_0 - r'P_0) + m_{24}(q'T_0 - t'Q_0) \\ & + m_{12}(p'Q_0 - q'P_0) + m_{34}(r'T_0 - t'R_0)].\end{aligned}$$

394. We now make the generic quantity $[L]$ specific, by postulating a value in the form

$$Iy' + I_\theta\frac{dy}{dn_\theta} + I_\phi\frac{dy}{dn_\phi} + I_\chi\frac{dy}{dn_\chi},$$

and determining the coefficients so as to satisfy the modified equation

$$\frac{\lambda_3}{\rho_0\sigma_0} = l_6 R + YS + Iy' + I_\theta\frac{dy}{dn_\theta} + I_\phi\frac{dy}{dn_\phi} + I_\chi\frac{dy}{dn_\chi}.$$

Two relations can be obtained from the fact that the binormal of the curve is at right angles to the tangent to the curve, with the typical direction-cosine y', and is at right angles to the prime normal of the curve with the typical direction-cosine Y_0, so that

$$\sum \lambda_3 y' = 0, \quad \sum \lambda_3 Y_0 = 0.$$

Because

$$\sum y' l_6 = 0, \quad \sum y' Y = 0, \quad \sum y'\frac{dy}{dn_\theta} = 0, \quad \sum y'\frac{dy}{dn_\phi} = 0, \quad \sum y'\frac{dy}{dn_\chi} = 0,$$

the first of these relations requires the condition

$$I=0.$$

Again, we have

$$Y_0=\frac{\rho_0}{\rho}Y+\frac{\rho_0}{\gamma}l,$$

so that

$$\sum Y_0 l_\theta=0,\quad \sum Y_0 Y=\frac{\rho_0}{\rho},\quad \sum Y_0 y'=0,$$

while

$$\sum Y_0\frac{dy}{dn_\theta}=\rho_0\sum\frac{l}{\gamma}\frac{dy}{dn_\theta}$$

$$=\rho_0(\kappa+\lambda\cos\omega_3+\mu\cos\omega_2)=\frac{\rho_0}{\gamma_\theta};$$

and similarly

$$\sum Y_0\frac{dy}{dn_\phi}=\frac{\rho_0}{\gamma_\phi},\quad \sum Y_0\frac{dy}{dn_\chi}=\frac{\rho_0}{\gamma_\chi};$$

and therefore the second of the relations requires the condition

$$\frac{S}{\rho}+\frac{I_\theta}{\gamma_\theta}+\frac{I_\phi}{\gamma_\phi}+\frac{I_\chi}{\gamma_\chi}=0.$$

The quantities I_θ, I_ϕ, I_χ, can be determined, without reference to this condition, as follows. We have

$$\sum Y_0\frac{dy}{dn_\theta}=\frac{\rho_0}{\rho}\sum Y\frac{dy}{dn_\theta}+\frac{\rho_0}{\gamma}\sum l\frac{dy}{dn_\theta}=\frac{\rho_0}{\gamma_\theta};$$

and therefore

$$\sum Y_0\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right)+\sum\frac{dy}{dn_\theta}\left(\frac{\lambda_3}{\sigma_0}-\frac{y'}{\rho_0}\right)=\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma_\theta}\right).$$

Let the derived value of λ_3 be inserted in this equation; after some re-arrangement, it assumes the form

$$\frac{1}{\rho_0}\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma_\theta}\right)-(I_\theta+I_\phi\cos\omega_3+I_\chi\cos\omega_2)$$

$$=\sum\frac{Y_0}{\rho_0}\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right)$$

$$=\sum\frac{Y}{\rho}\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right)+\sum\frac{l}{\gamma}\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right).$$

To evaluate the right-hand side, we use the value of $\dfrac{d}{ds_0}\left(\dfrac{dy}{dn_\theta}\right)$ cited in § 393;

and, because of the relations $\sum Yy_i=0$ for all the values of $i=1, 2, 3, 4$, the first term

$$=\frac{1}{\rho}\left(v_1\frac{dp}{dn_\theta}+v_2\frac{dq}{dn_\theta}+v_3\frac{dr}{dn_\theta}+v_4\frac{dt}{dn_\theta}\right).$$

In the second term, let the value of $\frac{l}{\gamma}$, which is

$$\kappa\frac{dy}{dn_\theta}+\lambda\frac{dy}{dn_\phi}+\mu\frac{dy}{dn_\chi},$$

be inserted. The resulting coefficient of κ

$$=\sum\left\{\frac{dy}{dn_\theta}\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right)\right\}=0,$$

because $\sum\left(\frac{dy}{dn_\theta}\right)^2=1$; the resulting coefficient of λ

$$\begin{aligned}
&=\sum\frac{dy}{dn_\phi}\left\{\frac{d}{ds_0}\left(\frac{dy}{dn_\theta}\right)\right\}\\
&=-\frac{\theta_N'}{\theta_N}\cos\omega_3+\frac{1}{\Omega\theta_N}\left(a\between\bar\theta_1, \bar\theta_2, \bar\theta_3, \bar\theta_4\between\frac{\phi_1}{\phi_n}, \frac{\phi_2}{\phi_n}, \frac{\phi_3}{\phi_n}, \frac{\phi_4}{\phi_n}\right)\\
&=\frac{1}{\Omega\theta_N}\left(a\between\bar\theta_1, \bar\theta_2, \bar\theta_3, \bar\theta_4\between\frac{\phi_1}{\phi_n}-\frac{\theta_1}{\theta_N}\cos\omega_3, \frac{\phi_2}{\phi_n}-\frac{\theta_2}{\theta_N}\cos\omega_3,\right.\\
&\qquad\qquad\left.\frac{\phi_3}{\phi_n}-\frac{\theta_3}{\theta_N}\cos\omega_3, \frac{\phi_4}{\phi_n}-\frac{\theta_4}{\theta_N}\cos\omega_3\right)\\
&=\frac{1}{\Omega\theta_N}\left(a\between\bar\theta_i\between\frac{\phi_j}{\phi_n}-\frac{\theta_j}{\theta_N}\cos\omega_3\right),
\end{aligned}$$

using this notation for brevity; and the resulting coefficient of μ, similarly,

$$\begin{aligned}
&=\frac{1}{\Omega\theta_N}\left(a\between\bar\theta_1, \bar\theta_2, \bar\theta_3, \bar\theta_4\between\frac{\chi_1}{\chi_\nu}-\frac{\theta_1}{\theta_N}\cos\omega_2, \frac{\chi_2}{\chi_\nu}-\frac{\theta_2}{\theta_N}\cos\omega_2,\right.\\
&\qquad\qquad\left.\frac{\chi_3}{\chi_\nu}-\frac{\theta_3}{\theta_N}\cos\omega_2, \frac{\chi_4}{\chi_\nu}-\frac{\theta_4}{\theta_N}\cos\omega_2\right)\\
&=\frac{1}{\Omega\theta_N}\left(a\between\bar\theta_i\between\frac{\chi_j}{\chi_\nu}-\frac{\theta_j}{\theta_N}\cos\omega_2\right).
\end{aligned}$$

When all these results are collected, the resulting equation in the quantities I_θ, I_ϕ, I_χ, becomes

$$\begin{aligned}
&I_\theta+I_\phi\cos\omega_3+I_\chi\cos\omega_2\\
&\quad=\frac{1}{\rho_0}\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma_\theta}\right)-\frac{1}{\rho}\left(v_1\frac{dp}{dn_\theta}+v_2\frac{dq}{dn_\theta}+v_3\frac{dr}{dn_\theta}+v_4\frac{dt}{dn_\theta}\right)\\
&\quad-\frac{\lambda}{\Omega\theta_N}\left(a\between\bar\theta_i\between\frac{\phi_j}{\phi_n}-\frac{\theta_j}{\theta_N}\cos\omega_3\right)-\frac{\mu}{\Omega\theta_N}\left(a\between\bar\theta_i\between\frac{\chi_j}{\chi_\nu}-\frac{\theta_j}{\theta_N}\cos\omega_2\right).
\end{aligned}$$

Proceeding similarly with reference to $\frac{dy}{dn_\phi}$ and $\frac{dy}{dn_\chi}$ in place of $\frac{dy}{dn_\theta}$ initially, we obtain the two equations

$$\begin{aligned}
&I_\theta \cos\omega_3 + I_\phi + I_\chi \cos\omega_1 \\
&\quad = \frac{1}{\rho_0}\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma_\phi}\right) - \frac{1}{\rho}\left(v_1\frac{dp}{dn_\phi} + v_2\frac{dq}{dn_\phi} + v_3\frac{dr}{dn_\phi} + v_4\frac{dt}{dn_\phi}\right) \\
&\qquad - \frac{\kappa}{\Omega\phi_n}\left(a \between \bar{\phi}_i \between \frac{\theta_j}{\theta_N} - \frac{\phi_j}{\phi_n}\cos\omega_3\right) - \frac{\mu}{\Omega\phi_n}\left(a \between \bar{\phi}_i \between \frac{\chi_j}{\chi_\nu} - \frac{\phi_j}{\phi_n}\cos\omega_1\right),
\end{aligned}$$

$$\begin{aligned}
&I_\theta \cos\omega_2 + I_\phi \cos\omega_1 + I_\chi \\
&\quad = \frac{1}{\rho_0}\frac{d}{ds_0}\left(\frac{\rho_0}{\gamma_\chi}\right) - \frac{1}{\rho}\left(v_1\frac{dp}{dn_\chi} + v_2\frac{dq}{dn_\chi} + v_3\frac{dr}{dn_\chi} + v_4\frac{dt}{dn_\chi}\right) \\
&\qquad - \frac{\kappa}{\Omega\chi_\nu}\left(a \between \bar{\chi}_i \between \frac{\theta_j}{\theta_N} - \frac{\chi_j}{\chi_\nu}\cos\omega_2\right) - \frac{\lambda}{\Omega\chi_\nu}\left(a \between \bar{\chi}_i \between \frac{\phi_j}{\phi_n} - \frac{\chi_j}{\chi_\nu}\cos\omega_1\right).
\end{aligned}$$

Thus the quantities I_θ, I_ϕ, I_χ, can be regarded as known; and the equation for the typical direction-cosine λ_3 of the binormal now is

$$\frac{\lambda_3}{\rho_0\sigma_0} = l_6 R + YS + I_\theta\frac{dy}{dn_\theta} + I_\phi\frac{dy}{dn_\phi} + I_\chi\frac{dy}{dn_\chi}.$$

The magnitude of the torsion $1/\sigma_0$ of the curve manifestly is given by the equation

$$\frac{1}{\rho_0{}^2\sigma_0{}^2} = R^2 + S^2 + I_\theta{}^2 + I_\phi{}^2 + I_\chi{}^2 + 2I_\phi I_\chi\cos\omega_1 + 2I_\chi I_\theta\cos\omega_2 + 2I_\theta I_\phi\cos\omega_3.$$

The value of l_6 has been obtained in § 294; and thus all the quantities, which occur in the expression for λ_3, belong to the domainal geodesic touching the curve and to the regions giving rise to the curve.

Similar analysis would lead to the direction of the trinormal and the magnitude of the tilt: it will not be pursued. At this stage, a halt is made in these investigations in the intrinsic geometry of amplitudes in multiple space.

So my long labour ceases, not unfitly in the sigh of Dante as he passed from a loftier theme,

> O quanto è corto il dire, e come fioco
> al mio concetto! e questo, a quel ch'io vidi,
> è tanto che non basta a dicer poco.

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